ASSIGNMENT 12: ARITHMETIZATION OF SYNTAX

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1. Compute the Gödel number of the formula $2+2=4$. In the lecture notes this formula is completely written out in official syntax. You can write the Gödel number in base 256, separating the digits base 256 by space.

2. Show that the function $\mathrm{Num}(x)$ that produces the Gödel number of $\bar{x}$ is primitive recursive. Hints: You have to prove that concatenation is primitive recursive. Show that

$$Concat(x,y) = (x << length(y)|y)$$

where $x << z$ is x left-shifted by z, i.e. $2^z \cdot x$, and $u|v$ is bitwise or. Show that bitwise or is primitive recursive.

3. Compute a formula of **PA** that represents the function Num. (It need not be the one that would be produced by applying the method used to prove every primitive recursive function is representable.)

4. If the Gödel numbers of A and B are respectively a and b, what is the Gödel number of $A \wedge B$? What is the Gödel number of $\forall x_3 A$? Again you may write the answer base 256.

5. Suppose that $\forall x\, A(x)$ is provable in **PA**. Then for each integer n, we can find a proof of the instance $A(\bar{n})$ by substituting $\bar{n}$ for x at the end of a fixed proof of $\forall x\, A(x)$. Find a primitive recursive function f that bounds the lengths of these proofs. More precisely, given a Gödel number k of a proof of $\forall x\, A(x)$, construct a primitive recursive $f(n)$ such that for all n, $A(\bar{n})$ has a proof with a Gödel number less than $f(n)$.

6. Show that there is a primitive recursive function V such that for closed terms t,

$$V(\ulcorner t \urcorner) = Val(t),$$

where $Val(t)$ is the value of t, as defined in lecture.