Euclid in the light of modern geometry: the case of the missing axioms

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In all the world there is nothing so curious and so interesting and so beautiful as truth.

–Hercule Poirot

Formalizing Euclid

In 2019, I formalized the proofs in Euclid Book I, and my two co-authors (Wiedijk and Narboux) translated those proofs into the well-known proof-checkers Coq and HOL Light.

- We were the "last elephant" in a long parade of critics and "fixers" of Euclid.
- To formalize Euclid, we had to deal with the problems pointed out by the previous elephants.
- We ended up proving 235 propositions, including Euclid's 48.
- That includes several needed as "Book Zero" (before Prop. I.1)

Dimensionality in Euclid

Is Euclid Book I about plane geometry?

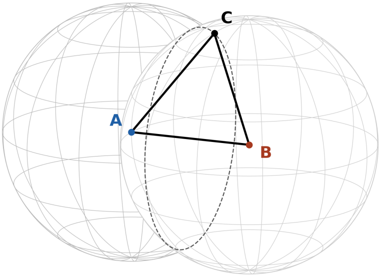
- Yes, all the diagrams are planar.
- No, there is nothing to require all points to lie in the same plane.

Book I has a model in which

- Points are points and lines are lines.
- But circles are spheres.

Nobody noticed this before (as far as I know). It would have been hard to be certain without computer checking.

Prop. I.1 in the 3D model

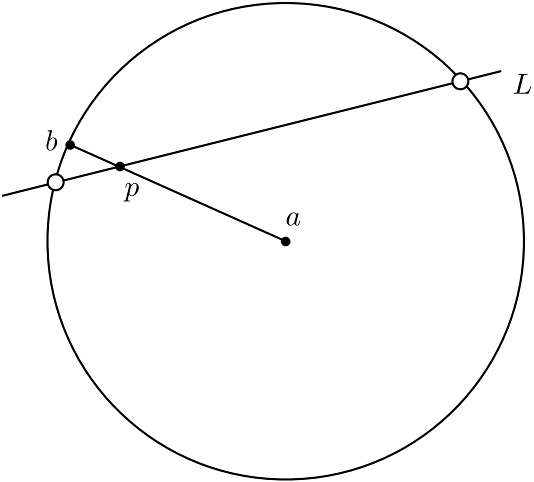


- Euclid's proof of I.1 needs the two circles to meet.
- In the model there are many meeting points–a whole circle of them.

Continuity–the first “missing axiom”

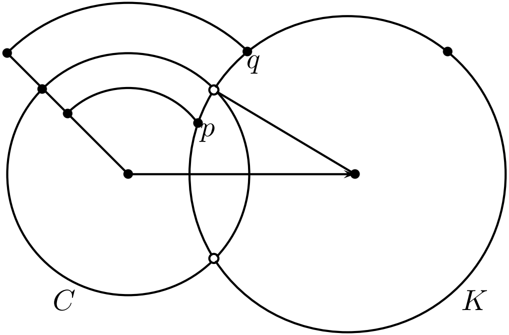
- There was a long parade of people pointing out the unjustified step in Prop. I.1, starting with (or before) Zeno of Sidon, about two centuries after Euclid.
- Those critics were pointing out, and perhaps trying to repair by a new posulate, a gap in inference.
- During many centuries, nobody questioned the continuum itself, or considered that a line might have gaps.
- That changed in 1872 with Dedekind.
- By 1899 geometry was no longer about “the” continuum. Other models were known, and some of them had “gaps.”
- It had also become possible to consider the independence of an axiom from others, once we had models of non-Euclidean geometry.

Line-circle continuity



Used in Prop. I.12 to drop a perpendicular.
The only use in Book I.

Circle-circle continuity



p is inside K, q is outside K,
as witnessed by the points on the left.

Used in Prop. I.1 and I.12 to construct triangles.
The only two uses in Book I.

Continuity added, that problem in Euclid is fixed!

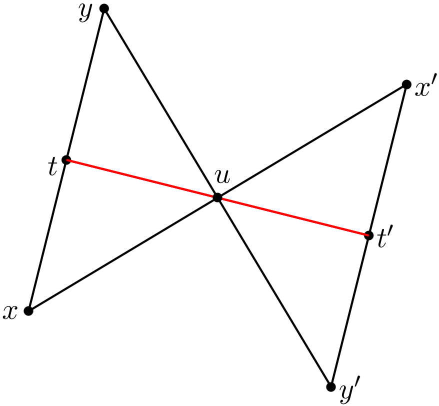
- So, we just add line-circle and circle-circle
- Then Euclid is freed of that defect.
- But "modern geometry" has more to say on the subject.

H. N. Gupta

- 1965 Ph. D. thesis under Tarski contained many fundamental results.
- Dropped perpendiculars can be constructed *without using circles*.
- So the use of line-circle in constructing them (I.12) can just be eliminated.
- The proof was published in SST (1984!) So the theorem got the German name, *Lotsatz*.
- I spoke with Gupta in September 2014. I asked why he never published his results. "Nobody ever asked me to."
- Wos and I computer-checked the proof in Otter.
- It has been checked in Coq too, by Boutry and Narboux.

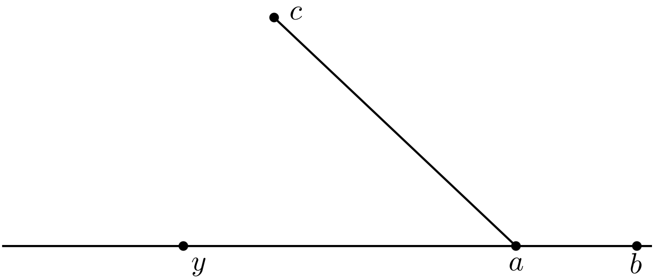
Gupta studied reflections and symmetries

He began by proving betweenness is preserved under reflection in a point. With that he proved lemmas like this one:



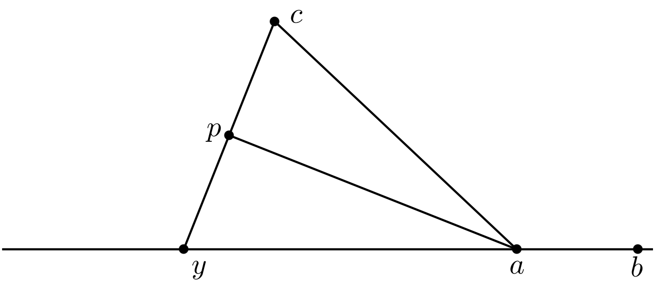
Suppose the black lines are all straight, and the midpoints of the black lines are as shown. Then the red line is straight and u is its midpoint.

Gupta's Perpendicular Construction, Step 1



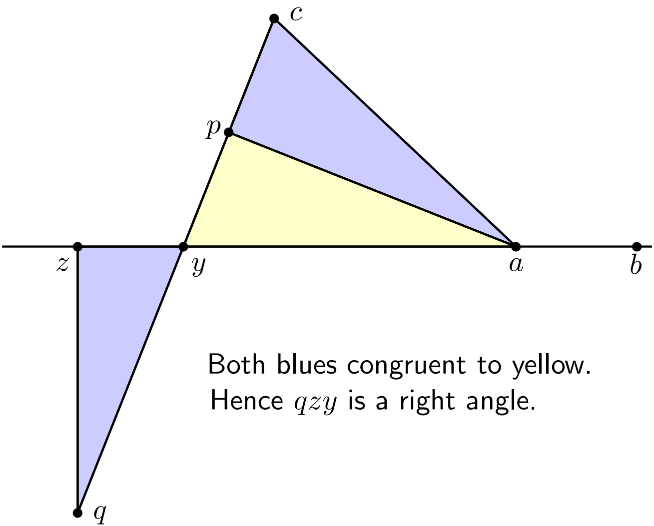
$$ay = ac$$

Gupta's Perpendicular Construction, Step 2



Bisect the isosceles triangle: $pc = py$.
Gupta showed how to do that with two uses of inner Pasch.

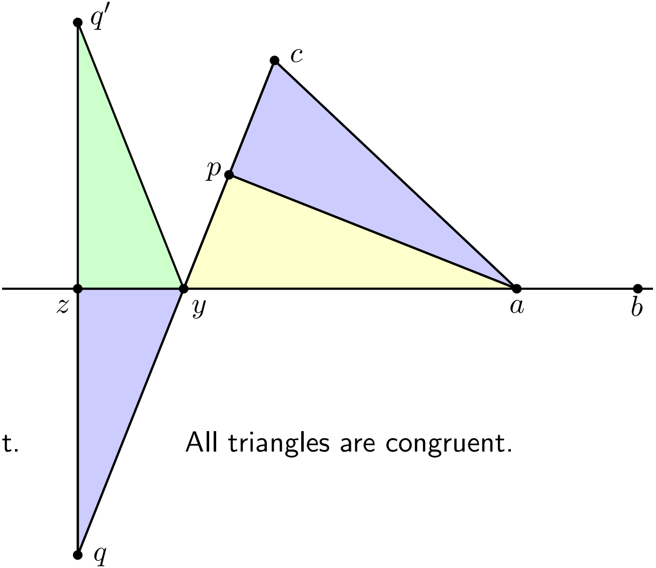
Gupta's Perpendicular Construction, Step 3



$yq = ac.$
$zy = py.$
$zq' = zq.$

Both blues congruent to yellow.
Hence qzy is a right angle.

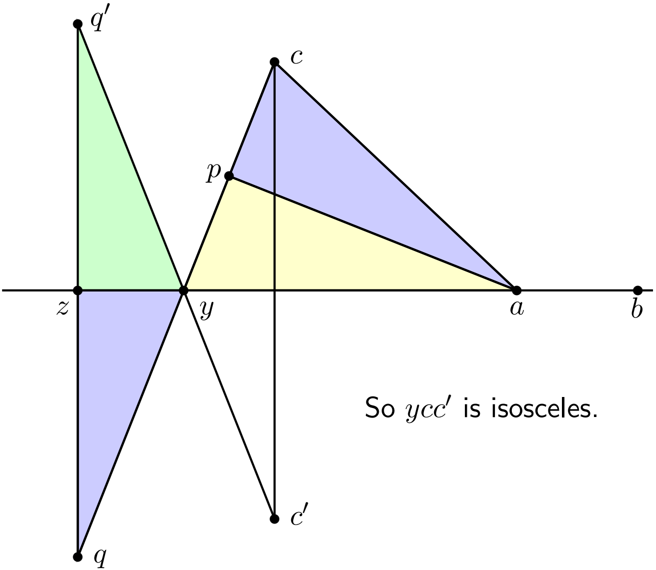
Gupta's Perpendicular Construction, Step 4



$zq' = zq$.
yzq' is right.

All triangles are congruent.

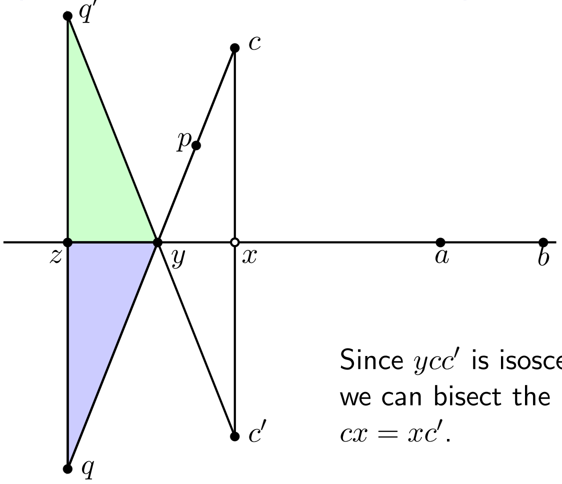
Gupta's Perpendicular Construction, Step 5



$yc' = yc.$

So ycc' is isosceles.

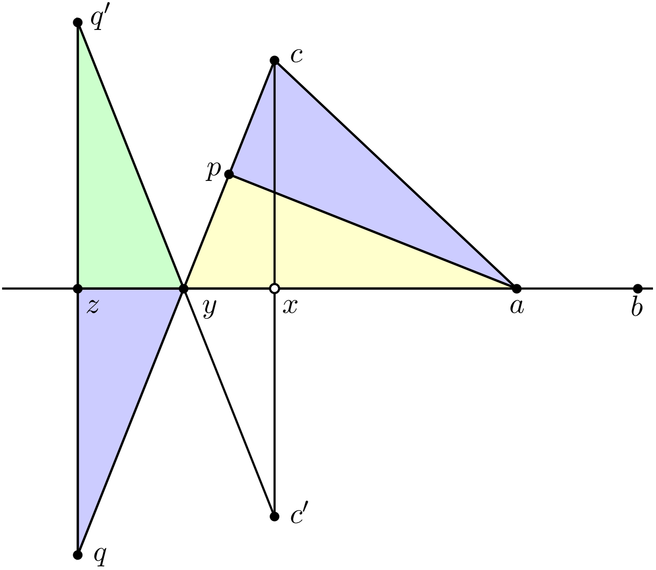
Gupta's Perpendicular Construction, Step 6



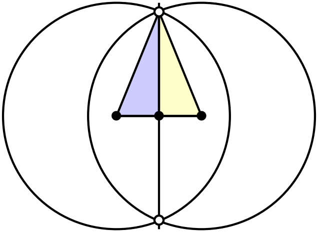
Since ycc' is isosceles.
we can bisect the base cc':
$cx = xc'$.

To finish, we need $\mathbf{B}(z, y, x)$, which I will not prove today. See p. 58 of Gupta's thesis. It uses the lemma I showed you earlier.

Gupta's Perpendicular Construction



Circle-circle implies line-circle

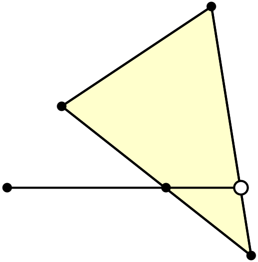


- Given one circle and the line, reflect the center in the line.
- You need Gupta's perpendicular to do that.
- Draw the second circle, using the same radius.
- Draw the two triangles with vertex at one of the circle intersection points.
- The two triangles are congruent by SSS.
- The angles that look right *are* right, by definition.
- The given line and the line between intersection points are both perpendicular to the line between centers at the midpoint.
- So they coincide.

Line-circle implies circle-circle

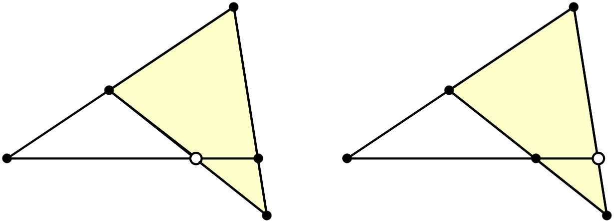
- The proof uses the “radical axis” of circles C and K.
- Time does not permit me to show you the construction.
- The proof has been computer-checked by Boutra *et. al*.
- The result is that line-circle and circle-circle are equivalent, so you can add either one and derive the other.

Pasch 1882



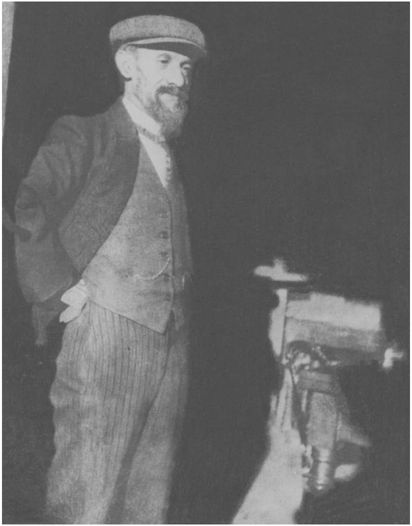
- If a line enters a triangle, it must exit.
- Fails in three dimensions
- In two dimensions, is it about continuity or betweenness?
- Pasch flagged it as a "missing axiom" in 1882.
- Roberval did so about two centuries earlier, according to de Risi.

Peano 1889–Inner Pasch and Outer Pasch



- Purely planar figures, but they work in any plane
- So these are not about dimensionality
- Probably inner Pasch plus dimension $= 2$ implies Pasch

Peano and his printing press



Peano and his followers wrote everything in symbols. They had trouble with editors and printers, so Peano started his own journal and printed it himself!

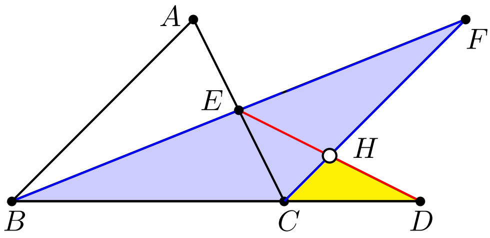
Here is how he wrote inner Pasch:

§ 11. Assioma XIV.

1. $a, b, c \in \mathbf{1} \,.\, a, b, c -\in \mathrm{Cl} \,.\, d \in bc \,.\, f \in ac \,:\, \supset \,.\!\cdot\!.\, e \in ad \,.\, e \in bf \,:\, -=_e \Lambda.$

He did not include a diagram!

Inner Pasch in the Exterior Angle Theorem, Prop. I.16



- E the midpoint of AC and then $FE = BE$
- angle $AEB = CEF$
- angle $BAE = ECF$. We want $BAE < ECD$.
- $ECF < ECD$ because the part is less than the whole
- But we need point H to see that it *is* a part;
- or in our system, to use the definition of angle $<$.

We still don't know for sure if we need Pasch

- The independence of (any form of) Pasch is open.
- That is, from the rest of Tarski's axioms.
- Sczerba proved the independence if we take the parallel postulate in the form "Every triangle is inscribed in a circle."
- He used the axiom of choice to do that, but if you just use the real algebraic numbers instead of $\mathbb{R}$, it works for the first-order Tarski axioms. Sczerba had second-order continuity.
- Boutry has studied 24 forms of the parallel postulate and grouped them by equivalence-without-Pasch, with computer-checked proofs.

Tarski, Szmielew, Gupta, Euclid, and Pasch

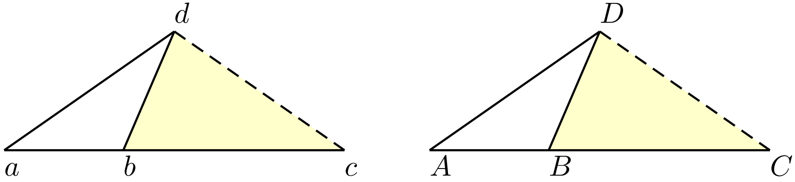
- Gupta proved inner and outer Pasch equivalent
- So which one would be an axiom and which a theorem? (Tarski hated redundant axioms.)
- Tarski favored outer, Szmielew favored inner.
- Euclid's proofs use both inner and outer Pasch.
- The equivalence proof is *way* harder than Euclid.
- I added both axioms to formalize Euclid.
- The task was to formalize Euclid, not to formalize Gupta.
- Gupta's proofs were checked in Otter when I worked with Larry Wos, and checked again in Coq by Narboux.

Side-angle-side (SAS)

Euclid's Prop. I.3 is "proved by superposition."

- Proclus reported "unease" with the method, but did not reject it.
- Superposition was rejected at least by 1557 by Pelatarius.
- The simplest fix would be to assume SAS as an axiom.
- That seems heavy-handed, so there is a long history of attempts to find the simplest axiom to replace SAS.
- Pasch (1882) and Hilbert (1899) had versions discussed in Heath. They are interesting, but time is limited.
- Tarski's five-segment axiom A5 is a points-only version.

Tarski's five-line axiom A5



If the five solid lines are pairwise congruent,
then the dashed lines are congruent.

- This expresses SAS for the colored triangles.
- I used A5 in formalizing Euclid.
- It worked well; I did not have to add many steps to Euclid's proofs, and in some cases the proof became shorter!

Reducing angles to points

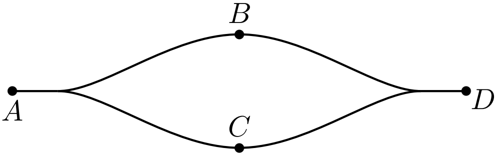
That the notions of "angle" and "congruence of angles" are eliminated in favor of congruence of lines is an important difference between Tarski and Hilbert. Hilbert was not interesting in minimizing the axioms; Tarski was. The definition of angle congruence has this history:

- Veronese 1891 (see p. 228 Heath)
- Mollerup 1904 (with reference to Veronese) has the diagram for A5 but not the axiom.
- Tarski 1927, 1965+ (explicit formulation of A5)

Tarski hated using defined concepts in axioms. Hence he did not want to just take SAS as an axiom. The 5-line axiom only uses primitive relations.

Two lines cannot enclose space (connectivity)

This can't happen:



Do we need this for an axiom or not?
Opinion has varied a lot during that long parade of elephants.
But answering today:

- Yes, we can't formalize Book I without it!
- No, because Gupta gave a proof of it.
- Yes anyway, because we're formalizing Euclid, not Gupta.

Where do we need connectivity?

- To prove the uniqueness of the midpoint. (Euclid never proved that!)
- The basic property of collinearity: if $a \neq b$ and $b \neq c$ then

$$L(a,b,c) \to L(a,b,d) \to L(b,c,d).$$

- Euclid just used this without justification when it looked true in the diagram.
- It is used in almost every one of our formal proofs.
- Also to derive "outer connectivity": Two lines cannot have a segment in common. As Zeno of Sidon pointed out, Prop. I.1 needs that!

Formalizing Book I, summary

We added:

- line-circle and circle-circle
- inner and outer Pasch
- 5-segment axiom (to get SAS)
- connectivity

Result: Book I checks and has a three-dimensional model in which "circles are spheres."

Comment: Gupta proved inner and outer Pasch equivalent, and his work on perpendiculars can be used to show line-circle and circle-circle are equivalent, and he proved connectivity.

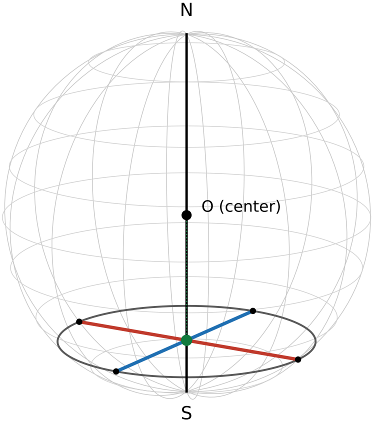
Wow! all in one thesis. His results, like Euclid's, had a long journey, carried to several distant countries by different people, before reaching worldwide publication.

Moving on to Books III and IV

- Book II is another story, not to be told today.
- Book III is about properties of circles.
- Book IV is about constructing circles and inscribing polygons.

Prop. III.4 fails if "circles are spheres"

Prop. III.4: if two chords bisect each other, the intersection point is the center.



Prop. III.10 is Tarski's dimension-two axiom

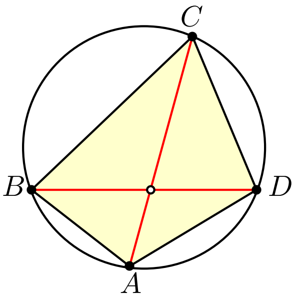
- You can't have three points equidistant from the same two points.
- Or put as Euclid did: two circles cannot meet in three points.
- So in Book III, space is two-dimensional.
- But Euclid has no axiom to guarantee that!
- We need to find the missing axiom.

Euclid's only hint

- A circle is defined to be a *planar figure* such that *etc*.
- The "planar" part of the definition is never used in Euclid.
- But we look to it to guess the right "missing axiom".

cyclicquad

`cyclicquad`: *the diagonals of a cyclic quadrilateral cross*



- *meet* means there is a a point collinear with both lines.
- *cross* means there is a point actually on both lines, i.e. between the endpoints.
- By hypothesis, BC does not cross AD.
- The conclusion is that AC crosses BD.

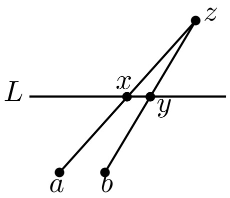
`cyclicquad` is not the only possible candidate for the missing axiom!

There are several other fundamental expressions of two-dimensionality:

- `circlePasch`: Pasch holds for triangles inscribed in a circle.
- Pasch (in context of inner Pasch)
- `twosides`: if P and Q are not on opposite sides of AB, they are on the same side.
- Euclid III.10, which is Tarski's dimension axiom: two circles cannot meet in more than two points.
- We will take these up one by one and show in the end that they are all equivalent, to each other and to `cyclicquad`.

Sides of a line

- Euclid mentioned "same side" and "opposite side", but did not define those concepts, relying on the diagram.
- Prop. I.12 constructs "erected perpendiculars" to a line, on the opposite side from a given point not on the line.
- We used Tarski's definition of "same side": a and b on the same side of L, as witnessed by x, y, z.



Plane Separation and `twosides`

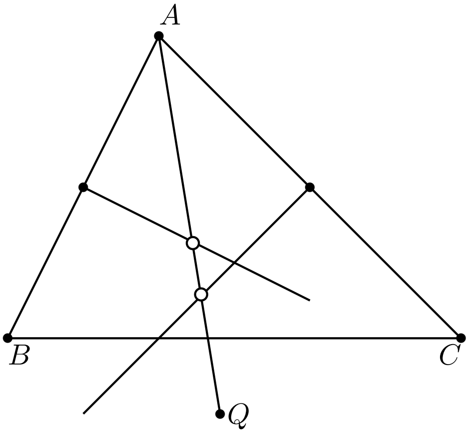
- With Tarski's definition, Gupta proved `planeseparation`: If p and q are on the same side of line L, and p and r are on opposite sides of L, then q and r are also on opposite sides of L.
- It works fine in the 3D model.
- Contrast it with `twosides`: *p and q are either on opposite sides or the same side of line L.*
- That says space is two-dimensional.
- So it is a candidate for the "missing axiom."
- Euclid never mentioned anything like any of these.
- As far as I know, nobody before Pasch ever even thought of defining "same side."

Pasch and `cyclicPasch`

Recall `cyclicPasch` is Pasch for triangles inscribed in a circle. These two are equivalent once we have the theorem that every triangle can be circumscribed. That is Euclid's proposition IV.5.

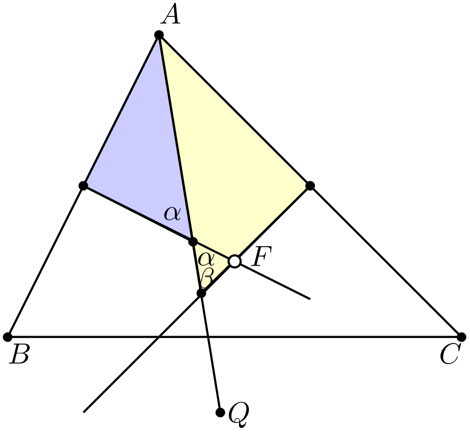
- It is in Book IV, but that is only because Euclid put all the propositions that construct circles in Book IV.
- It can be proved with the methods of Book I. In fact it is known to be equivalent to the parallel postulate.
- Prop. IV.5 has nothing to do with dimensionality.
- We have to prove it out of Euclid's order, but since Euclid's only hint about two-dimensionality was that circles are planar, it's not surprising that we have to construct circles to get planarity.
- Euclid's proof is not correct, and this has long been known (see Heath)
- But also the corrections suggested in Heath are also not correct!

Proof of Prop. IV.5, first part



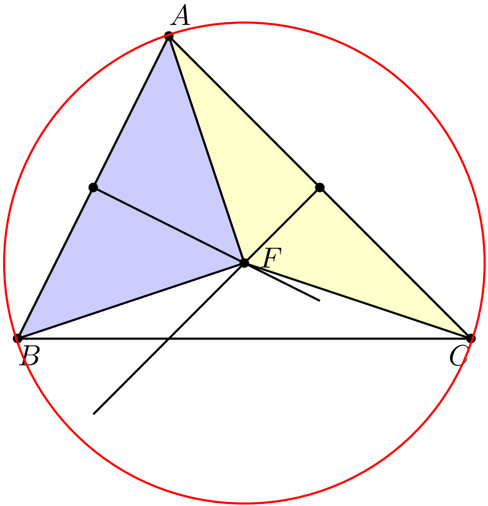
- Bisect angle A by AQ.
- Erect perpendicular bisectors to AB and AC on the same side as Q.
- We want to show they meet (Euclid assumed it without proof.)
- They at least meet AQ, by the parallel postulate, using transversals AB and AC.
- If those meeting points are equal, we're done, so
- the meeting points are distinct, and AQ is a transversal of the perpendicular bisectors.

Proof of Prop. IV.5, second part



- AQ is a transversal of the perpendicular bisectors.
- $\alpha < \frac{\pi}{2}$ as the blue triangle is right.
- $\beta < \frac{\pi}{2}$ as the yellow triangle is right.
- The corresponding interior angles are less than two right angles.
- So the bisectors meet at F as desired, by the parallel postulate.

Proof of Prop. IV.5, Euclid's ending

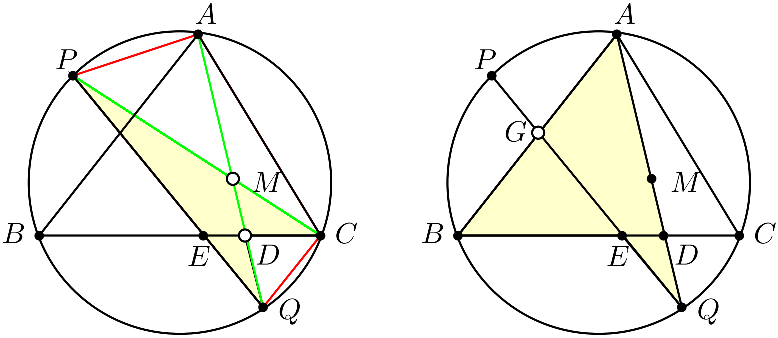


- Now that the perpendicular bisectors meet at F:
- $FA = FB$ by the blue congruent triangles
- $FB = FC$ by the yellow congruent triangles.
- So a circle with center F contains A, B, and C.

`cyclicquad` $\to$ `circlePasch`

Case 1, AC does not cross PQ.

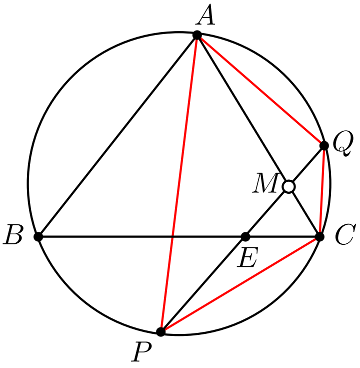
Then M exists by `cyclicquad`; D exists by inner Pasch; G exists by outer Pasch.



cyclicquad → circlePasch

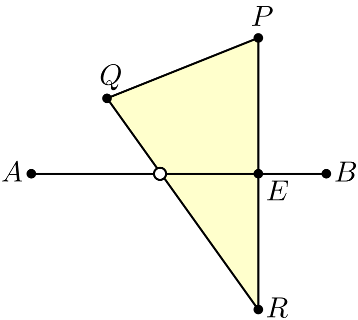
Case 2, M does cross PQ.

Then we're already done. No need for inner or outer Pasch.

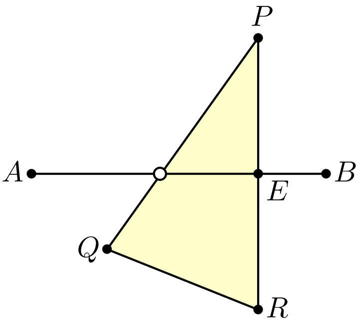


Pasch $\rightarrow$ `twosides`

Given P, Q not collinear with AB,
drop perpendicular PE, make $ER = PE$, and use Pasch:



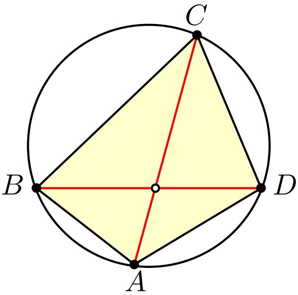
same side



opposite sides

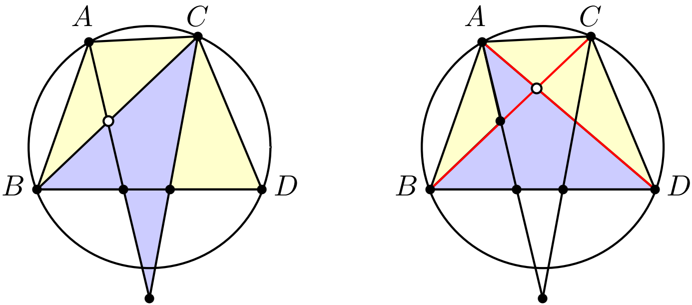
twosides $\to$ cyclicquad

Let $ABCD$ be a cyclic quadrilateral. Then
A and C are either on the same side or opposite sides of BD.
If they are on opposite sides, we're done immediately:



twosides → cyclicquad

Let $ABCD$ be a cyclic quadrilateral.
Suppose A and C are on the same side of BD.



Apply outer Pasch twice.

Putting it together

- `cyclicquad` → `circlePasch`
- `circlePasch` → Pasch (using IV.5)
- Pasch → `twosides`
- `twosides` → `cyclicquad`

So they are all equivalent.

Summary

- Book I has a 3 dimensional model where circles are spheres!
- But Book III is two-dimensional
- various 19th century ways of expressing two dimensionality are all equivalent using the methods of Book I.

In all the world there is nothing so curious
and so interesting and so beautiful as truth.
And sometimes–so surprising!