Examples of Minimal Surfaces

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Contents

1 Enneper's Surface

Enneper's surface was discovered in 1863 by Alfred Enneper, who was 33 at the time. This was seven years after his Ph. D. under the supervision of Dirichlet at Göttingen, where Enneper lived his entire life, from student to Professor Extraordinarius.

Enneper's surface is defined in the entire complex plane, so it is an example of a complete minimal surface (no boundary). However, we are interested in considering portions of it, defined in a disk of radius R. Then it is bounded by "Enneper's wire",

$$\Gamma_R(\theta) = \begin{bmatrix} R\cos\theta - \frac{1}{3}R^3\cos 3\theta \\ -R\sin\theta + \frac{1}{3}R^3\sin 3\theta \\ R^2\cos 2\theta \end{bmatrix}$$

The same formula, with r in place of R, defines Enneper's surface in polar coordinates.

1.1 Weierstrass representation

To show that Enneper's surface as defined above is indeed a minimal surface, we show that it arises from the Weierstrass representation if we take $f(z) = 2$ and $g(z) = 2z$. This gives us

$$u_z = \begin{bmatrix} 1 - z^2 \\ i(1 + z^2) \\ 2z \end{bmatrix}$$

Integrating, we have

$$\begin{aligned} u &= \mathrm{Re} \begin{bmatrix} z - \frac{1}{3}z^3 \\ i(z + \frac{1}{3}z^3) \\ z^2 \end{bmatrix} \\ &= \begin{bmatrix} r\cos\theta - \frac{1}{3}r^3\cos 3\theta \\ -r\sin\theta + \frac{1}{3}r^3\sin 3\theta \\ r^2\cos 2\theta \end{bmatrix} \end{aligned}$$

1.2 Non-parametric form

According to Rado's theorem, as long as the Jordan curve Γ has a convex project on the xy plane, any disk-type minimal surface bounded by Γ is expressible in non-parametric form, i.e. $z = f(x, y)$. Since there is a maximum principle for the difference of two solutions of the non-parametric minimal surface equation, the solution for a given boundary is unique.

Let Γ_R be "Enneper's wire", defined above.

Lemma 1 *For $R \le \frac{1}{\sqrt{3}}$, the projection of Enneper's wire Γ_R on the xy plane is convex. Hence Γ_R bounds exactly one minimal surface.*

Proof. Let γ_ρ be the projection of Γ_ρ onto the xy plane. Then

$$\gamma_\rho(\theta) = \begin{bmatrix} \rho\cos\theta - \frac{1}{2}\rho^3\cos 3\theta \\ -\rho\sin\theta + \frac{1}{3}\rho^3\sin 3\theta \end{bmatrix}$$

History and references. See pp. 80-84 of [17]; see also [19] ? and [**?**] ?

1.3 total curvature of Enneper's wire

1.4 Self-intersection

1.5 First and second eigenvalues

1.6 Ruchert's uniqueness theorem

1.7 The second variation of D^2E

Consider the kernel equation for Enneper's surface in the disk of radius 1,

$$k_z u_z = 0$$

or in real form with $k = \psi u_\theta$,

$$u_\theta((\psi u_\theta)_r - (\psi u_r)_\theta) = 0.$$

We will show that $\psi = \sin(2\theta)$ solves this equation. On S^1 we have (even when $t \neq 0$)

$$\begin{aligned} k &= \psi u_\theta \\ &= \sin 2\theta \,\mathrm{Re}\,(izu_z) \\ &= \frac{1}{2}i(z^2 - \bar{z}^2)\,\mathrm{Re} \begin{bmatrix} i(z - z^3) \\ -(z + z^3) \\ 2iz^2 \end{bmatrix} \\ &= \frac{1}{4}i(z^2 - \bar{z}^2) \begin{bmatrix} -iz^3 + iz - i\bar{z} + i\bar{z}^3 \\ -z^3 - z - \bar{z} - \bar{z}^3 \\ 2iz^2 - 2i\bar{z}^2 \end{bmatrix} \\ &= \frac{1}{4} \begin{bmatrix} (z^2 - \bar{z}^2)(z^3 - z + \bar{z} - \bar{z}^3) \\ -i(z^2 - \bar{z}^2)(z^3 + z + \bar{z} + \bar{z}^3) \\ -(z^2 - \bar{z}^2)(2z^2 - 2\bar{z}^2) \end{bmatrix} \\ &= \frac{1}{4} \begin{bmatrix} z^5 - z^3 - \bar{z}^3 + \bar{z}^5 \\ -i(z^5 + z^3 - \bar{z}^3 - \bar{z}^5) \\ -2z^4 + 4 - 2\bar{z}^4 \end{bmatrix} \end{aligned}$$

This expression for k is harmonic in the entire plane since evidently $\Delta k = k_{z\bar{z}} = 0$. Differentiating with respect to z we have (even when $t \neq 0$)

$$k_z = \frac{1}{4} \begin{bmatrix} 5z^4 - 3z^2 \\ -5iz^4 - 3iz^2 \\ -8z^3 \end{bmatrix} \qquad (1)$$

Taking the dot product with u_z we have, when $t = 0$,

$$\begin{aligned} k_z u_z &= \frac{1}{4}\begin{bmatrix} 5z^4 - 3z^2 \\ -5iz^4 - 3iz^2 \\ -8z^3 \end{bmatrix} \cdot \begin{bmatrix} 1 - z^2 \\ i(1 + z^2) \\ 2z \end{bmatrix} \\ &= \frac{1}{4}\begin{bmatrix} 5z^4 - 3z^2 \\ 5z^4 + 3z^2 \\ -8z^3 \end{bmatrix} \cdot \begin{bmatrix} 1 - z^2 \\ 1 + z^2 \\ 2z \end{bmatrix} = 0 \end{aligned}$$

Similar calculations show that $\psi = a + b\cos\theta + c\sin\theta$ also yields a solution; this three-parameter family accounts for the conformal directions, and $\psi = 2\cos\theta$ represents a non-trivial kernel direction.

That k is the only kernel direction (orthogonal to the conformal directions) can be shown directly by writing a Fourier series for an unknown $\psi = \sum_{n=-\infty}^{\infty} a_n z^n$ on S^1 and showing ψ must have the form $a + b\cos\theta + c\sin\theta + d\sin 2\theta$, which is how we found $\psi = \sin 2\theta$ in the first place.

We also give a more informative proof that k is the only non-trivial kernel direction. The function $g(z)$ in the Weierstrass representation is the stereographic projection of the unit normal N, and for Enneper's surface $g(z) = z$. Hence, the Gaussian image of Enneper's surface in the unit disk is exactly the upper hemisphere. Hence the first eigenvalue of $D^2A(u)$ is 2, so the kernel of $D^2A(u)$ is one-dimensional, as the eigenspace of the least eigenvalue. But every member k of the kernel of $D^2E(u)$ gives rise to a member $\phi = k \cdot N$ of the kernel of $D^2A(u)$, and the map $k \mapsto \phi$ is one to one.

For $R < 1$, the Gaussian area of Enneper's surface over the disk of radius R is contained in a hemisphere, so the critical eigenvalue is more than 2 and the surface is a relative minimum of area. For $R > 1$, the Gaussian area contains a hemisphere, so the surface is not a relative minimum of area. Hence $R = 1$ is the only value for which the second variation has a kernel.

1.8 The third variation of Enneper's surface

We now calculate the third variation of Enneper's surface (defined in the unit disk). We consider a variation $u(t)$ defined on S^1 by

$$u = u_0(e^{i(\theta + t\psi + O(t^2))})$$

where $\psi(\theta) = \sin 2\theta$ and the subscript in u_0 indicates $t = 0$. Differentiating with respect to t we have

$$u_t = (\psi + O(t))u_\theta.$$

Thus $k = u_t$ lies in the kernel of $D^2E(u_0)$ when $t = 0$.

Lemma 2 *The third variation of Enneper's surface is zero. Specifically, with the variation $u(t)$ given above, we have*

$$\left.\frac{\partial^3 E}{\partial t^3}\right|_{t=0} = 0.$$

Proof. We have (as shown in Lectures on Minimal Surfaces, Chapter 10, following [24])

$$\left.\frac{\partial^3 E}{\partial t^3}\right|_{t=0} = 4\,\mathrm{Re}\int zk_z^2\psi\,dz + 4\,\mathrm{Re}\int z(\psi k_\theta)_z u_z \psi\,dz \tag{2}$$

By (1) we have $k_z = z^2 v$ for some holomorphic vector v. Then $k_z^2 = z^4 v^2$. Then on S^1 we have

$$\begin{aligned} zk_z^2\psi &= z^5 v^2 \frac{i}{2}(z^2 - \bar{z}^2) \\ &= z^3 v^2 \frac{i}{2}(z^4 - 1) \end{aligned}$$

The integrand is holomorphic in the unit disk, so by Cauchy's theorem, the first integral in (2) is zero.

We now work on the second integral. From (1) we have

$$zk_z = \frac{1}{4} 5z^5 - 3z^3 {-} 5iz^5 - 3iz^3 {-} 8z^4$$

Then

$$\begin{aligned}
k_\theta &= -\,\mathrm{Im}\,(zk_z) \;=\; -2i(zk_z + \bar{z}\overline{zk_z}) \\
&= \frac{1}{2}\begin{bmatrix} -5iz^5 + 3iz^3 - 3i\bar{z}^3 + 5i\bar{z}^5 \\ -5z^5 - 3z^3 - 3\bar{z}^3 - 5\bar{z}^5 \\ 8iz^4 - 8i\bar{z}^4 \end{bmatrix} \\
\psi k_\theta &= \frac{-i}{4}(z^2 - \bar{z}^2)\begin{bmatrix} -5iz^5 + 3iz^3 - 3i\bar{z}^3 + 5i\bar{z}^5 \\ -5z^5 - 3z^3 - 3\bar{z}^3 - 5\bar{z}^5 \\ 8iz^4 - 8i\bar{z}^4 \end{bmatrix} \\
&= \frac{1}{4}\begin{bmatrix} -5z^7 + 3z^5 + 5z^3 - 3z - 3\bar{z} + 5\bar{z}^3 + 3\bar{z}^5 - 5\bar{z}^7 \\ 5iz^7 + 3iz^5 - 5iz^3 - 3iz + 3i\bar{z} + 5i\bar{z}^3 - 3i\bar{z}^5 - 5i\bar{z}^7 \\ 8z^6 - 8z^2 - 8\bar{z}^2 + 8\bar{z}^6 \end{bmatrix} \\
(\psi k_\theta)_z &= \frac{1}{4}\begin{bmatrix} -35z^6 + 15z^4 + 15z^2 - 3 \\ 35iz^6 + 15iz^4 - 15iz^2 - 3i \\ 48z^5 - 16z \end{bmatrix} \\
(\psi k_\theta)_z u_z &= \frac{1}{4}\begin{bmatrix} -35z^6 + 15z^4 + 15z^2 - 3 \\ 35iz^6 + 15iz^4 - 15iz^2 - 3i \\ 48z^5 - 16z \end{bmatrix}\begin{bmatrix} 1 - z^2 \\ i(1 + z^2) \\ 2z \end{bmatrix} \\
&= -z^6 + z^2 \\
z((\psi k_\theta)_z u_z \psi &= z(-z^6 + z^2)\left(\frac{z^2 - \bar{z}^2}{2}\right) \\
&= \frac{1}{2}(-z^9 - z^5 - z)
\end{aligned}$$

Since this is analytic, its integral around S^1 is zero, so the second term in (2) is zero. That completes the proof of the lemma.

1.9 The fourth variation of Enneper's surface

We will compute the fourth variation of Enneper's surface along the path given by

$$u(t,\theta) = u_0(\theta + t\psi) \qquad \text{with } \psi = 2\sin 2\theta$$

We write $k = u_t = \psi u_\theta$. Since ψ does not depend on t, we have $k_t = \psi u_{\theta t} = \psi k_\theta$. The following formula for the fourth variation in a direction belonging to the kernel of the second variation is given in [4].

$$\begin{aligned} \left.\frac{\partial^4 E}{\partial t^4}\right|_{t=0} &= 8\,\mathrm{Re}\int z k_z k_{zt}\psi\, dz + 4\,\mathrm{Re}\int z k_{ttz} u_z \psi\, dz \\ &\quad + 12\,\mathrm{Re}\int z k_{zt} u_z \psi_t\, dz + 8\,\mathrm{Re}\int z k_z^2 \psi_t\, dz \end{aligned}$$

Since we have assumed $\psi_t = 0$ the last two terms can be dropped:

$$\left.\frac{\partial^4 E}{\partial t^4}\right|_{t=0} = 8\,\mathrm{Re}\int z k_z k_{zt}\psi\, dz + 4\,\mathrm{Re}\int z k_{ttz} u_z \psi\, dz$$

We have

$$\psi = \sin\theta = \frac{1}{2}(-iz^2 + iz^{-2}),$$

By (1), k_z is divisible by z^2. Hence $k_z\psi$ is holomorphic. Since the z-derivative of any harmonic function is holomorphic, and k_t is harmonic, so $k_{zt} = k_{tz}$ is holomorphic. Hence the first term also vanishes:

$$\left.\frac{\partial^4 E}{\partial t^4}\right|_{t=0} = 4\,\mathrm{Re}\int z k_{ttz} u_z \psi\, dz$$

Recall from (1) that (even when $t \neq 0$)

$$k_z = \frac{1}{4}\begin{bmatrix} 5z^4 - 3z^2 \\ -5iz^4 - 3iz^2 \\ -8z^3 \end{bmatrix}$$

To use this equation when $t \neq 0$ we should put $z = e^{i(\theta + t\psi)}$, so we have

$$z_t = i\psi z$$

Differentiating k_z with respect to t we obtain

$$\begin{aligned} k_{zt} &= \frac{1}{4}\frac{\partial}{\partial t}\begin{bmatrix} 5z^4 - 3z^2 \\ -5iz^4 - 3iz^2 \\ -8z^3 \end{bmatrix} \\ &= \frac{z_t}{4}\begin{bmatrix} 20z^3 - 6z \\ -20iz^3 - 6iz \\ -24z^2 \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
&= \frac{i}{4}\psi z \begin{bmatrix} 20z^3 - 6z \\ -20iz^3 - 6iz \\ -24z^2 \end{bmatrix} \\
&= \frac{i}{4}dt\frac{z^2 - \bar{z}^2}{2i} z \begin{bmatrix} 20z^3 - 6z \\ -20iz^3 - 6iz \\ -24z^2 \end{bmatrix} \\
&= \frac{1}{8} \begin{bmatrix} 20z^6 - 6z^4 - 20z^2 + 6 \\ -20iz^6 - 6iz^4 + 20iz^2 + 6i \\ -24z^4 + 24 \end{bmatrix}
\end{aligned}
$$

This came out holomorphic, as it had to, since it is also k_{tz} and k_t is harmonic. Now differentiate again with respect to t:

$$
\begin{aligned}
k_{ztt} &= \frac{z_t}{8} \begin{bmatrix} 120z^5 - 24z^3 - 40z \\ -120iz^5 - 24iz^3 + 40iz \\ -96z^3 \end{bmatrix} \\
&= \frac{i\psi z}{8} \begin{bmatrix} 120z^5 - 24z^3 - 40z \\ -120iz^5 - 24iz^3 + 40iz \\ -96z^3 \end{bmatrix} \\
&= \frac{i}{8}\frac{z^2 - \bar{z}^2}{2i} z \begin{bmatrix} 120z^5 - 24z^3 - 40z \\ -120iz^5 - 24iz^3 + 40iz \\ -96z^3 \end{bmatrix} \\
&= \frac{1}{16} \begin{bmatrix} 120z^8 - 24z^6 - 160z^4 + 24z^2 + 40 \\ -120iz^8 - 24iz^6 - 80iz^4 + 24iz^2 - 40i \\ -96z^6 + 96z^2 \end{bmatrix}
\end{aligned}
$$

For Enneper's surface we have

$$
zu_z = \begin{bmatrix} z - z^3 \\ iz + iz^3 \\ 2z^2 \end{bmatrix}
$$

Taking the dot product with the previous equation, we have

$$
\begin{aligned}
zk_{ttz}u_z &= \frac{1}{16} \begin{bmatrix} 120z^8 - 24z^6 - 160z^4 + 24z^2 + 40 \\ -120iz^8 - 24iz^6 - 80iz^4 + 24iz^2 - 40i \\ -96z^6 + 96z^2 \end{bmatrix} \cdot \begin{bmatrix} z - z^3 \\ iz + iz^3 \\ 2z^2 \end{bmatrix} \\
&= 5z + O(z^2)
\end{aligned}
$$

Multiplying by ψ we have

$$
\begin{aligned}
zk_{ttz}u_z\psi &= (5z + O(z^2))\frac{z^2 - z^{-2}}{2i} \\
&= 5iz^{-1} + O(1)
\end{aligned}
$$

Integrating this around S^1, the $O(1)$ part is holomorphic, so it integrates to 0, and we have

$$\begin{aligned} \left.\frac{\partial^4 E}{\partial t^4}\right|_{t=0} &= 4\,\mathrm{Re}\int z k_{ttz} u_z \psi\, dz \\ &= 4\,\mathrm{Re}\int \frac{5i}{z}\, dz \\ &= 4\,\mathrm{Re}\,\frac{5i}{2\pi i} \qquad \text{by Cauchy's residue theorem} \\ &= \frac{10}{\pi} \end{aligned}$$

We have proved

$$\left.\frac{\partial^4 E}{\partial t^4}\right|_{t=0} > 0 \tag{3}$$

1.10 Relative minimum for $R = 1$

We need the following theorem, which is discussed in [4].

Theorem 1 *Let u be a minimal surface of disk type bounded by a Jordan curve Γ. Suppose that $D^2E(u)$ has a one-dimensional kernel (aside from the conformal directions) and that for some one-parameter family $u(t)$ of harmonic surfaces bounded by Γ, with $u(0) = u$ and $u_t(0) = k$ in the kernel of $D^2E(u)$, the third and fourth derivatives of $E(u(t))$ with respect to t are respectively zero and positive. Then u is a relative minimum of Dirichlet's energy.*

Corollary 1 *Enneper's surface for $R = 1$ is a relative minimum of area.*

Proof. Let u be Enneper's surface for $R = 1$, and let $\psi = \sin 2\theta$. Let $u(t, \theta) = u(\theta + t\psi)$. We have calculated the required second, third, and fourth derivatives of $E(u(t))$ in the previous sections, and they meet the hypotheses of the theorem. That completes the proof.

1.11 Trifurcation and the cusp catastrophe

2 The catenoid

2.1 Weierstrass representation

2.2 Non-parametric form

2.3 Unique minimal surface of revolution

2.4 Existence between coaxial circles in parallel planes

2.5 Stability

2.6 The second variation

3 The helicoid

3.1 Weierstrass representation

3.2 Non-parametric form

3.3 Unique ruled minimal surface

3.4 Second variation of D^2A

3.5 The second, third, and fourth variations of D^2E

3.6 Trifurcation and the cusp catastrophe

4 Scherk's surface

4.1 Weierstrass representation

4.2 Non-parametric form

4.3 Use as a comparison surface

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