A Rigorous Theory of Infinite Limits

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Abstract

We give rigorous definitions and theorems supporting the use of symbols for (various kinds of) infinity and undefinedness in calculations involving limits. Such calculations are often made on paper and blackboards, but are usually barred admission to textbooks on the grounds of insufficient pedigree. Here we make them respectable, using the theory of filters.

1 Introduction

Every calculus textbook treats the topics *Limits at Infinity* and *Infinite Limits*. The textbooks carefully explain that the symbol '∞' means nothing in isolation; only certain phrases containing that symbol are defined. Nevertheless, when it comes to the practical calculation of limits, the use of infinity and worse, the use of zero in denominators, is quite common. The first author wrestled with this issue when writing the mathematical software *Mathpert* [1]. At first, he was determined to keep it pure: no such 'unrigorous' calculation would be allowed. However, the fact was that he was not able to make the program solve all the desired examples without introducing either logical concepts and steps not used in calculus classes (such as quantifier alternations) or 'illegal' uses of infinity. He therefore formulated a consistent set of rules for such calculations, justifiable in terms of the usual epsilon-delta definitions of limit, and proceeded to write the software. Curiously, he received no complaints about these calculations from calculus students or teachers. However, a few logicians complained, including the second author of this paper, who demanded a rigorous semantics.

The issue goes beyond logical nitpicking, in that it affects the way mathematics is taught and used. Mathematicians try to avoid calculating with infinity, because they think it is unjustified. When they can't get the answer any other way, though, they do calculate with infinity, and they get correct answers. They show their students how to get the answers to limit problems this way quickly, on the blackboard, and then how to get the same answer 'correctly', some longer and less comprehensible way that will be given full credit on an examination.

Consider the following calculation:

$$\lim_{x\to\infty} \sin(1/x) \quad = \quad \sin(\lim_{x\to\infty}(1/x))$$

$$
\begin{aligned}
&= \sin(1/\lim_{x\to\infty} x)\\
&= \sin(1/\infty)\\
&= \sin 0\\
&= 0
\end{aligned}
$$

While the answer is correct, many calculus instructors would find this solution wanting. Indeed, they would object to each of the first three steps. They would prefer to see a change of limit variable to $u = 1/x$, resulting in a calculation like this:

$$
\begin{aligned}
\lim_{x\to\infty} \sin(1/x) &= \lim_{u\to 0} \sin u\\
&= \sin 0\\
&= 0
\end{aligned}
$$

The issue here is not whether the second calculation is shorter, or more elegant, but whether the first calculation is correct or not. But to show that this issue is not irrelevant, let us give an example where the calculation involving manipulation of infinities seems harder to eliminate.

$$
\begin{aligned}
\lim_{x\to\infty} x-\sqrt{x} &= \lim_{x\to\infty} \frac{(x-\sqrt{x})(x+\sqrt{x})}{(x+\sqrt{x})} =\\
&= \lim_{x\to\infty} \frac{x^2-x}{x+\sqrt{x}}\\
&= \lim_{x\to\infty} \frac{x-1}{1+\dfrac{1}{\sqrt{x}}}\\
&= \frac{\lim_{x\to\infty} x-1}{\lim_{x\to\infty} 1+\dfrac{1}{\sqrt{x}}}\\
&= \frac{\lim_{x\to\infty} x - \lim_{x\to\infty} 1}{\lim_{x\to\infty} 1 + \lim_{x\to\infty} \dfrac{1}{\sqrt{x}}}\\
&= \frac{\infty - 1}{\lim_{x\to\infty} 1 + \dfrac{1}{\lim_{x\to\infty} \sqrt{x}}}\\
&= \frac{\infty}{1+\dfrac{1}{\sqrt{\lim_{x\to\infty} x}}}\\
&= \frac{\infty}{1+\dfrac{1}{\sqrt{\infty}}}\\
&= \frac{\infty}{1+\dfrac{1}{\infty}}\\
&= \frac{\infty}{1+0}
\end{aligned}
$$

$$= \quad \frac{\infty}{1} = \infty$$

Just to be explicit, we give the calculus teacher's reasons why this computation does not get full credit: At line 4, the limit cannot be pushed into the fraction without first proving that the limit in the denominator exists and is nonzero. At the next step, the limit cannot be pushed into the sum without first proving that the limits exist. At the next step, it is illegal to change $\lim_{x\to\infty} 1/\sqrt{x}$ to $1/\lim_{x\to\infty} \sqrt{x}$ since the latter limit has not been shown to be (and is not) different from zero. (We are only allowed to use the rule $\lim(1/u) = 1/\lim u$ if $\lim u$ exists and is nonzero.) From these inauspicious beginnings, the computation degenerates into sheer nonsense, with meaningless expressions like $\sqrt{\infty}$ and $1/\infty$. Miraculously, however, the right answer emerges, and that might earn partial credit in some calculus classes.

Of course, this limit can be evaluated without calculating with infinity. Indeed, there are at least three different ways to do that. One of them involves replacing the numerator and denominator with other expressions which are respectively smaller and larger, for large x, than the original numerator and denominator. We must then prove lemmas establishing these bounding inequalities. But the question arises: how did we know that we should try to prove the limit is ∞ and not 0? Quite probably, we made a calculation like the above first, to find out the answer, before attempting a more 'rigorous' proof. The second way of 'rigorously' evaluating the limit is to break the calculation into a number of subcalculations. For example, we first prove

$$\lim_{x\to\infty} \sqrt{x} = \infty$$

and then, using the theorem that the limit of $1/u(x)$ is 0 if the limit of $u(x)$ is ∞, we can conclude that

$$\lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$$

Proceeding in this way from the end of the calculation to the beginning, we can construct a proof that the answer is what the calculation says it is. But of course, we first make the calculation, in order to see what the sequence of lemmas should be. There is also a very clever third method, pointed out by a calculus teacher, which begins by factoring x into $\sqrt{x}\sqrt{x}$ and then factoring $x - \sqrt{x} = \sqrt{x}\,(\sqrt{x} - 1)$. But to finish the calculation this way, we still need to use $\infty \cdot \infty = \infty$. Of course, we can eliminate that illegal calculation by reference to a theorem, that if u and v both approach infinity as x approaches infinity, then so does uv. But again, at least mentally, we make the taboo calculation before constructing an acceptable proof.

Thus: any of the known methods to evaluate this limit requires us to to first make a taboo calculation. What are we to tell the students? Check the section on Limits at Infinity in the popular calculus textbooks to see how this matter is handled. In most cases the issue is delicately skirted by simply not giving in the text any examples of worked problems of the type discussed above, though problems like this do occur in the exercises. Worked examples are restricted to quotients of rational functions. In some cases, an example containing a square root does occur, but the steps involving calculations with infinity are condensed into one too-fast and unjustified step, so that no explicit infinities are printed, but the calculation is not, strictly speaking, justified.[1] The teaching assistant then shows the students the taboo calculations, on the blackboard where no textbook reviewers can criticize the lack of rigor. Our purpose in this paper is to legitimize this kind of computation. Perhaps the next generation of calculus books will be freed from the taboo.

There is, of course, a long history of efforts to justify and explain limit calculations which could be made correctly by practicing mathematicians. Walter Felscher has done a thorough job of explaining the origins of the epsilon-delta definition in his historical paper [2]. While epsilon and delta explain the meaning of the *concept* of limits, not every *calculation* involving infinity is explained thereby. We show in this paper that certain calculation steps involving infinity will lead to correct answers in the epsilon-delta sense.

2 Towards a semantics of infinity

The main idea of this paper is to use filters to explain infinite limits and limits at infinity. Our intention is that the paper should require no mathematical background other than freshman calculus. This aim conflicts with our desire to present this theory in its most general setting, which involves the concept of a topological space. The reader who does not know what a topological space is should ignore the relatively few references to the concept below, and just think of the set R of real numbers. Also, for the benefit of such readers, we define a set of reals B to be *open* if whenever $x \in B$, then there is an interval (a, b) with $a < x < b$ and $(a, b) \subset B$. Note that an interval (u, v) (which you may have learned to call an 'open interval') is an open set, but $[u, v]$ is not.

We will use *filters* as a key concept in our theory of limits. This concept will be defined below. It is not new with us, but has a long history, originating in topology, and more recently it has figured in logic and computer science. Specifically, filters were introduced by Cartan in topology (see [4], p. 6) and since the time of Bourbaki have been used to define accumulation points and limits in spaces which are not first countable (so sequences may not be adequate). See for example [4], p. 63, or [5], pp. 116-118. In the topological tradition, filters are used to define convergence, but the value of a limit is always a point of the space. Our new idea is to use filters for the *values* of limits as well.

Filters have been used for decades in theoretical computer science to define the semantics of programming languages, and they played a role in the logical investigations that established the independence of the axiom of choice and the continuum hypothesis. The traditional definition of a filter used all subsets of a space; these logical and semantical uses led to the consideration of filters using

[1] We omit specific citations here, as it is not our intention to criticize specific authors or publishers for this situation. Rather, it is our intention to supply in this paper the required mathematical remedy.

only open sets, or sets selected from some other Boolean subalgebra of the power set of the space. In this paper, we use open sets.

Definition. A *filter* on a topological space X is a collection F of open subsets of X such that
(i) $A \in F$ and $A \subset B$ and B is open implies $B \in F$.
(ii) F is closed under finite intersection.

This allows the empty set and set of all open subsets of X to be filters. This is different from the definitions in both [4] and [5]. We feel free to call both of these filters, noting that for example [4] allows the set of all subsets, but not the empty set, while [5] allows the empty set but not the set of all subsets. In allowing both, then, we are not flying in the face of well-established terminology.

If F is a collection of open subsets of X, the *filter generated by* F is defined to be the intersection of all filters containing F. This concept also has a definition 'from below' which will be given in the next section, which applies even when the sets in F are not necessarily open.

For applications to first-year calculus, we will primarily be interested in the case $X = \mathrm{R}$, the topological space of real numbers with the usual topology; but we will also consider more general spaces for some special situations and some generalizations. All the spaces which interest us do have a natural metric, however, so for convenience we will assume throughout that X is a metric space. For ease of reading we will write $|x-a|$ instead of $\rho(x,a)$, although our definitions require only a metric space or less. The reader who is bothered by this abuse of language may simply assume X is a normed linear space.

The *principal filter* generated by an element a of X is defined to be the collection of all open sets B such that $a \in B$. We denote this filter by $\overline{a}$.

The *punctured filter* generated by an element a of X is denoted by $[a]$, and defined to be the filter generated by all punctured neighborhoods of the form $\{x \in X : 0 < |x-a| < \epsilon\}$, for some ϵ.

$[\mathsf{Undefined}]$ is the filter consisting of exactly the one open set X, the whole space.

Specializing now to the case $X = \mathrm{R}$, we will make use of the following special examples of filters, which get special names:

$[a^+]$ is the filter generated by intervals of the form $(a, a+\epsilon)$ for $\epsilon > 0$.

$[a^-]$ is the filter generated by intervals of the form $(a-\epsilon, a)$ for $\epsilon > 0$.

$[\infty]$ is the filter generated by intervals of the form (c, ∞) for $c \in \mathrm{R}$.

$[-\infty]$ is the filter generated by intervals of the form $(-\infty, c)$ for $c \in \mathrm{R}$.

$[\pm\infty]$ is the filter generated by sets of the form $(-\infty, -c) \cup (c, \infty)$ for $c \in \mathrm{R}$. This is the same as the filter generated by sets of the form $(-\infty, a) \cup (b, \infty)$ for $a, b > 0$.

$[\mathbf{a}, \mathbf{b}]$ is the filter generated by all open sets containing the interval $[a, b]$. The name of this filter can be read aloud as $\mathsf{Oscillations}[a, b]$. The brackets here are somewhat ambiguous, in that they are already part of the notation for a closed interval, and we have also used brackets to indicate a filter, as in $[\infty]$. In print we use boldface to indicate that a filter is intended. In handwriting, context will suffice, since you cannot write an interval in the context where a filter belongs.

Similarly we have filters generated by all open sets containing any given interval. We use boldface, combined with the brackets that would be used to indicate the generating interval. Thus we have, for example:

$(\mathbf{a}, \mathbf{b})$
$(\mathbf{a}, \mathbf{b}]$
$[\mathbf{a}, \mathbf{b})$
$[\mathbf{a}, \infty)$
$(-\infty, \mathbf{a}]$
$(-\infty, \infty)$ is the filter containing only R, which is the same as [Undefined].
[Improper] is the filter consisting of all open sets.
[DomainError] is the empty filter, containing no open set.

Our idea is to use filters as the possible values of limit expressions, that is, we want to have something like[2]

$$\lim_{x \to [0^+]} 1/x = [\infty]$$

and

$$\lim_{x \to 0} \sin(1/x) = [-\mathbf{1}, \mathbf{1}]$$

The case of a finite limit is also covered:

$$\lim_{x \to 0} \sin x = [0]$$

We can see that we are making progress already: we can make distinctions that are felt to be intuitively correct, such as the following.

$$\lim_{x \to [0^+]} 1/x = [\infty]$$

and

$$\lim_{x \to [0^-]} 1/x = [-\infty]$$

while the best we can say for the two-sided limit is

$$\lim_{x \to 0} 1/x = [\pm\infty]$$

This does capture something of the asymptotic behavior of the function, though. For example, the best we can say of a somewhat wilder function is

$$\lim_{x \to 0} (1/x) \sin(1/x) = (-\infty, \infty)$$

[2]In these equations, we have filters on the right side, but on the left we have ordinary limits. This is not our final, rigorous form, which will be reached in the next sections. We are simply indicating the direction of our development here.

It is not equal to $[\pm\infty]$ since for small x, the values of the function do not confine themselves to arbitrary elements of the filter $[\pm\infty]$. But here we begin to anticipate the definitions in the next section. Our aim in this section is simply to develop the reader's intuition.

Here is another distinction we can make: in the case of a finite limit value c, should the filter value be the punctured filter $[c]$ or the principal filter $\overline{c}$? Indeed, it might be a one-sided filter $[c^+]$ or $[c^-]$. To assert

$$\lim_{x \to a} f(x) = [c]$$

is to claim, in addition to the usual meaning of the limit, that for small enough punctured intervals about a, $f(x)$ does not take the value c. But for example we cannot assert

$$\lim_{x \to 0} x \sin(1/x) = [0]$$

We must instead be content with

$$\lim_{x \to 0} x \sin(1/x) = \overline{0}$$

On the other hand we have

$$\lim_{x \to 0} x^2 = \left[0^+\right]$$

while we only have

$$\lim_{x \to 0} x^3 = [0]\,.$$

Principal filters also arise as limits of constant functions:

$$\lim_{x \to 0} 1 = \overline{1}.$$

We won't get the punctured filter [1], since the function values do not keep away from 1 on punctured neighborhoods of 0.

A finite limit statement that $f(x) \to c$ as $x \to a$ could refine, in our filter limit theory, to any one of four statements, where the limit value c is replaced by $\overline{c}$, $[c]$, $[c^+]$, or $[c^-]$. These distinctions will be quite useful in limit calculations, as will be shown in examples later in the paper.

Here is a question every calculus instructor has heard from a student: is it correct to say infinity is undefined? When the student writes

$$\lim_{x \to 0} 1/x^2 = \mathsf{undefined}$$

should the instructor mark this wrong on the grounds that the correct answer is ∞? If she does, the student is certain to argue that infinity is undefined and the answer should be accepted. Not wishing to be pushed into an argument over that somewhat hazy point, the instructor will probably accept the answer. Our formalism casts some light on the question. Indeed, the answer is $[\infty]$, not $[\mathsf{Undefined}]$. However, we will formulate a notion of 'refinement' for filters, and it will be true that $[\infty]$ refines $[\mathsf{Undefined}]$. So the hazy notion that both answers are correct, but that ∞ is the better answer because it gives more information, will be justified with full rigor.

In addition to the idea of using filters as the values of limit expressions, there is a second interesting idea[3] in the paper: we can use filters not only as the values of limits, but as the 'places' which a limit variable 'approaches'. That is, in the expression

$$\lim_{x \to a} f(x),$$

we can replace the real number a by a filter Q. The usual limit corresponds to the case $Q = [a]$. The one-sided limit from the right corresponds to the case $Q = [a^+]$. The one-sided limit from the left corresponds to the case $Q = [a^-]$. Limits at infinity correspond to the filters $[\infty]$ and $[-\infty]$.

The first step in the development of our theory will be to define the expression

$$\lim_{x \to Q} f(x) = F$$

where Q and F are both filters. The first theorems will show that the definition corresponds to the epsilon-delta definitions of the usual two-sided and one-sided limits, in case Q is of the forms $[a]$, $[a^+]$, or $[a^-]$, and F is of one of the forms $\overline{a}$, $[\infty]$, or $[-\infty]$. These definitions and theorems will be given in the next section.

Having made these definitions and proved these theorems, what use are they? Here is where we come to the main point: We can actually *calculate* with (symbols for) filters! We can define addition, multiplication, subtraction, division, exponentiation, and more general functions to operate on filters in ways that naturally extend the way they work on real numbers. These definitions allow us to calculate with symbols for filters, and all these calculations refer to precisely defined objects and operations, with provably correct results. If such a calculation results in the evaluation of a limit expression to a form not involving filters, that result is as correct as if it had been derived by an epsilon-delta argument. Besides, there is the possibility that by the use of filters, we can state a result more precisely than would have been possible with traditional symbols only. For example,

$$\begin{aligned}
\lim_{x \to 0} \frac{1}{2 + \sin(1/x)} &= \frac{1}{\lim_{x \to 0}(2 + \sin(1/x))} \\
&= \frac{1}{2 + \lim_{x \to 0} \sin(1/x)} \\
&= \frac{1}{2 + [\mathbf{-1}, \mathbf{1}]} \\
&= \frac{1}{[\mathbf{1}, \mathbf{3}]} \\
&= [\mathbf{1/3}, \mathbf{1}]
\end{aligned}$$

[3]This is not really a new idea, since as we have discussed the concept of filter is often used to explain limits in topology textbooks. But the focus there is on giving a definition that works in spaces that may not be separable, while we focus on the reals and the context of freshman calculus. So far as we know, filters have not been considered in this context before.

Isn't that a nice answer? It conveys much more information than the answer that would be expected on a calculus exam today: the limit is undefined.

3 Definitions

We collect here the basic definitions concerning filters, which are required to formulate our notion of *filter limit*. The precise definition of this notion is the aim of this section.

3.1 Filters

We have already defined *filter* in the previous section, but we repeat the definition here using more symbols. Let R be the set of real numbers, and $\mathcal{O}(\mathrm{R})$ the set of open subsets of R. A *filter* is a set $A \subseteq \mathcal{O}(\mathrm{R})$ that satisfies:

- $U \in A$ and $V \in \mathcal{O}(\mathrm{R})$ and $U \subseteq V$ implies $V \in A$
- $U \in A$ and $V \in A$ implies $U \cap V \in A$

If we also have:

- $\emptyset \notin A$

we call the filter *proper*.

A filter A is called *bounded* when it contains some finite interval:

$$(l, r) \in A$$

with $l < r$. A filter A is called *positive* when it contains the set of the positive real numbers:

$$(0, \infty) \in A$$

3.2 The filter generated by A

Let A be a set of (not necessarily open) subsets of R that satisfies the following property:

$$U \in A \;\&\; V \in A \Rightarrow \exists W \in A : W \subseteq U \cap V$$

which is implied by the simpler property of being closed under finite intersections:

$$U \in A \;\&\; V \in A \Rightarrow U \cap V \in A$$

Then the filter $\mathcal{X}(A)$, which is called the filter *generated* by A, is defined to be:

$$\mathcal{X}(A) = \{U \in \mathcal{O}(\mathrm{R}) \mid \exists V \in A : V \subseteq U\}$$

The set A is called its *basis*. In case A contains only open sets, $\mathcal{X}(A)$ is the intersection of all filters containing all members of A.

3.3 Lifting

We want to be able to make sense of expressions like $\sin[0^+]$. This should be a filter; in this example we want the answer to be $[0^+]$. In general we want to extend a real-valued function f to be defined on a filter A and have a filter value $\overline{f}(A)$. The first attempt might be

$$\overline{f}(A) = \{f(U) : U \in A\}$$

but there are three problems with this: f might be only partial, $f(U)$ might not be an open set, and the result needs to be a filter, but with this definition might not be. The following definition solves these three problems:

Let f be a (possibly partial) function from R to R, then we define the *lifting* $\overline{f}$, which is a function from the set of filters to the set of filters, to be:

$$\overline{f}(A) = \mathcal{X}(\{f[U] \mid U \in A \ \&\ U \subseteq \mathrm{dom}(f)\})$$

$$= \{V \in \mathcal{O}(\mathrm{R}) \mid \exists\, U \in A : U \subseteq \mathrm{dom}(f) \ \&\ f[U] \subseteq V\}$$

For f from R^n to R this definition generalizes to:

$$\overline{f}(A_1, \ldots, A_n) =$$

$$\mathcal{X}(\{f[U_1 \times \cdots \times U_n] \mid U_1 \in A_1 \ \&\ \ldots\ \&\ U_n \in A_n \ \&\ U_1 \times \cdots \times U_n \subseteq \mathrm{dom}(f)\})$$

Note that if c is a function of zero arguments, i.e. a constant, then the two definitions of $\overline{c}$ coincide.

After proving the lemmas in the next section, we may suppress the overline and just write $f(A)$ instead of $\overline{f}(A)$.

It is worth noting that addition, subtraction, multiplication, and division are special cases of binary f. Hence their liftings to filters are defined. We shall omit the overline, writing $A + B$ instead of $\overline{+}(A, B)$ or $A \overline{+} B$.

3.4 Filter Limits

We define a *limit to a filter* A, or *filter limit*, by

$$\operatorname*{Lim}_{x \to A} f(x) = \overline{f}(A)$$

Note that this is a *definition*, so it also holds for non-continuous f.

We shall show in the next section that this definition does indeed generalize the usual definition. In particular, we expect to get the usual notion of limit when $A = [a]$ and $f(A) = [c]$, where c is the value of the limit. One-sided limits will arise from $A = [a^+]$ or $A = [a^-]$. Limits at infinity arise when $A = [\infty]$ or $A = [-\infty]$. Infinite limits arise when the filter $f(A)$ is $[\infty]$ or $[-\infty]$. But we can also find filter values for many limits about which, in the usual notation, we can only say that they are undefined. For example, $\lim_{x\to 0} 1/x = [\pm\infty]$.

Note that the alternating quantifiers $\forall\epsilon\exists\delta$ do not appear in the definition of filter limit. Where did they go? We cannot expect to eliminate this essential feature of limits. The answer is, they have been hidden in the definition of lifting. For simplicity, suppose that f is everywhere defined, and consider the statement

$$\operatorname*{Lim}_{x \to A} f(x) = B$$

for example. That means, according to the definition above,

$$\overline{f}(A) = B$$

That means, for every $V \in B$, there is a U in A such that $f(U) \subseteq V$. We see that V corresponds to ϵ and U corresponds to δ.

Notation. According to the above we can write

$$\operatorname*{Lim}_{x \to [\infty]} f(x)$$

In this notation there are two indications that a filter limit is intended: the use of Lim instead of lim, and the use of $[\infty]$ instead of just ∞. It is not necessary to have both; we may for convenience drop one or the other and write

$$\operatorname*{Lim}_{x \to \infty} f(x)$$

or even

$$\lim_{x \to [\infty]} f(x).$$

There is only one possible way to make sense of the expression: this is a filter limit. If people should drop both and interpret

$$\lim_{x \to \infty} f(x)$$

as a filter limit, we are not responsible.

3.5 Refinement

There is one more fundamental notion, not required for the definition of filter limits, but required for stating the connection between filter limits and ordinary limits. Namely, a filter A can *refine* a filter B, notation $A \sqsubseteq B$.[4] This notion is defined by:

$$A \sqsubseteq B \Leftrightarrow A \supseteq B$$

(Note that:

$$[\mathsf{Improper}] \sqsubseteq A \sqsubseteq [\mathsf{Undefined}] \sqsubseteq [\mathsf{DomainError}]$$

for all proper non-empty filters A.) Intuitively, A refines B if A gives more information than B. For example, $[\infty]$ refines $[\pm\infty]$, and $[0^+]$ refines $[0]$.

[4] If one wants to type this symbol in ordinary text, one can use the approximate form `[=`, analogous to using `<=` for $\leq$.

4 Filter limits and ordinary limits

We shall state a number of simple lemmas concerning filter limits, and then state and prove three theorems connecting ordinary limits to their filter-limit versions. After that, we continue to prove simple lemmas leading to calculation rules for filter limits. All these theorems and lemmas are nearly immediate consequences of the definitions. We give the details in this section, to check that indeed the definitions have been correctly given. In a later section we shall state many more computation rules without proof; the proofs can be given following the example proofs in this section.

Lemma 1 *One-sided filters refine principal filters:* $[a^+] \sqsubseteq [a] \sqsubseteq \overline{a}$.

Proof If $U \in [a]$, for some ϵ we have that $(a-\epsilon, a) \cup (a, a+\epsilon) \subseteq U$, which implies that $(a, a+\epsilon) \subseteq U$, so $U \in [a^+]$. Hence $[a] \subseteq [a^+]$, which means that $[a^+] \sqsubseteq [a]$. The other refinement is proved similarly. □

Lemma 2 *Limits of constants are principal: let A be proper and nonempty. Then* $\mathrm{Lim}_{x \to A}\, b = \overline{b}$.

Remark. The condition is necessary: if A is either [Improper] or [DomainError], we have $\mathrm{Lim}_{x \to A}\, b = A$.

Proof Let $b(x)$ be the constant function on R that takes the value b everywhere (so $\mathrm{dom}(b) = \mathrm{R}$.) Writing out the definitions, we get: $\mathrm{Lim}_{x \to A}\, b = \mathrm{Lim}_{x \to A}\, b(x) = \overline{b}(A) = \{V \in \mathcal{O}(\mathrm{R}) \mid \exists U \in A : U \subseteq \mathrm{dom}(b) \;\&\; b[U] \subseteq V\} = \{V \in \mathcal{O}(\mathrm{R}) \mid \exists U \in A : \{b\} \subseteq V\} = \{V \in \mathcal{O}(\mathrm{R}) \mid \{b\} \subseteq V\} = \mathcal{X}(\{b\}) = \overline{b}$. □

Lemma 3 *Limits of the identity function:* $\mathrm{Lim}_{x \to A}\, x = A$.

Proof Let $i(x)$ be the identity function on R (so $\mathrm{dom}(i) = \mathrm{R}$.) Writing out the definitions, we get: $\mathrm{Lim}_{x \to A}\, x = \mathrm{Lim}_{x \to A}\, i(x) = \overline{i}(A) = \{V \in \mathcal{O}(\mathrm{R}) \mid \exists U \in A : U \subseteq \mathrm{dom}(i) \;\&\; i[U] \subseteq V\} = \{V \in \mathcal{O}(\mathrm{R}) \mid \exists U \in A : U \subseteq V\} = A$. The last equality follows because filters are closed under taking supersets. □

Lemma 4 *Lifting of compositions: If f is continuous on its domain and that domain is open, then:*

$$\overline{f \circ g} = \overline{f} \circ \overline{g}$$

Proof We must prove that for any filter A we have that $\overline{f \circ g}(A) = \overline{f}(\overline{g}(A))$. $\overline{f \circ g}(A) \sqsubseteq \overline{f}(\overline{g}(A))$ already follows without any conditions on f: suppose $W \in \overline{f}(\overline{g}(A))$ then there is some $V \in \overline{g}(A)$ with $V \subseteq \mathrm{dom}(f)$ and $f[V] \subseteq W$. Now because $V \in \overline{g}(A)$, there is some $U \in A$ with $U \subseteq \mathrm{dom}(g)$ and $g[U] \subseteq V$. Together this implies that $U \subseteq \mathrm{dom}(f \circ g)$ and $f[g[U]] \subseteq f[V] \subseteq W$, so $W \in \overline{f \circ g}(A)$.

To prove that $\overline{f \circ g}(A) \sqsupseteq \overline{f}(\overline{g}(A))$, suppose that $W \in \overline{f \circ g}(A)$. Then there is a $U \in A$ with $U \subseteq \mathrm{dom}(f \circ g)$ and $f[g[U]] \subseteq W$. Define $V = g[U]$, then $V \subseteq \mathrm{dom}(f)$ and $f[V] \subseteq W$ (however, V doesn't need to be open). Consider

$V' = f^{-1}[W]$, then $V' \supseteq V$ and because of the requirements V' is open. Now $U \subseteq \mathrm{dom}(f)$ and $g[U] \subseteq V \subseteq V'$ so $V' \in \overline{g}(A)$. Furthermore by definition $V' \subseteq \mathrm{dom}(g)$ and $f[V'] \subseteq W$, so $W \in \overline{f}(\overline{g}(A))$. □

Lemma 5 *Limits of compositions. If*

- *f is continuous on its domain and that domain is open, or*
- *f is continuous on some $X \in \mathrm{Lim}_{x \to A}\, g(x)$*

then:

$$\mathop{\mathrm{Lim}}_{x \to A} f(g(x)) = \overline{f}(\mathop{\mathrm{Lim}}_{x \to A} g(x))$$

Remark. Compare this lemma with the next one, where no conditions are needed. Some conditions *are* needed in this lemma, as the following example shows: Take $g(x) = x^2$, and $f(x) = \sqrt{x}$. Then the composition $f(g(x)) = |x|$. Thus the left side $\mathrm{Lim}_{x \to \overline{0}}\, f(g(x)) = \overline{0}$, but the right side evaluates to $[\mathsf{DomainError}]$, because $\mathrm{Lim}_{x \to \overline{0}}\, g(x) = \overline{0}$, not $[0^+]$ as one would have in case $[0^+]$ were under the limit sign in place of $\overline{0}$.

Proof By writing out the definition of Lim, we need to prove, $\overline{f \circ g}(A) = \overline{f}(\overline{g}(A))$ which is just the previous lemma. So if the first variant of the condition holds we are finished, and with the second variant of the condition only $\overline{f \circ g}(A) \sqsupseteq \overline{f}(\overline{g}(A))$ remains to be proved.

So suppose that for some $X \in \overline{g}(A)$ we have that f is continuous on X, and let be given a $W \in \overline{f \circ g}(A)$. Then (like before) we have a $U \in A$ with $U \subseteq \mathrm{dom}(f \circ g)$ and $f[g[U]] \subseteq W$. Because $X \in \overline{g}(A)$ we also have a $U' \in A$ with $U' \subseteq \mathrm{dom}(g)$ and $g[U'] \subseteq X$. If we take $U' = U \cap U'$, then $U' \in A$, $U' \subseteq \mathrm{dom}(f \circ g) \subseteq \mathrm{dom}(g)$, $f[g[U']] \subseteq W$ and $g[U'] \subseteq X$. Now define $V = \{x \in X \mid f(x) \in W\}$. Then because X is open (being an element of the filter $\overline{g}(A)$) and because of the continuity of f on X we find that V is open. Also we have $g[U'] \subseteq V$ from which we find that $V \in \overline{g}(A)$. Finally $V \subseteq \mathrm{dom}(f)$ and $f[V] \subseteq W$, so we get that $W \in \overline{f}(\overline{g}(A))$. □

Lemma 6 *Limits of compositions of functions of several variables.*

$$\mathop{\mathrm{Lim}}_{x \to A} f(g_1(x), \ldots, g_n(x)) \sqsubseteq \overline{f}(\mathop{\mathrm{Lim}}_{x \to A} g_1(x), \ldots, \mathop{\mathrm{Lim}}_{x \to A} g_n(x)).$$

Remark. We do not in general have equality instead of refinement, in contrast to the one-variable situation treated in the previous lemma. For example:

$$\mathop{\mathrm{Lim}}_{x \to [\infty]} (x - x) = \mathop{\mathrm{Lim}}_{x \to [\infty]} 0 = \overline{0}$$

while:

$$(\mathop{\mathrm{Lim}}_{x \to [\infty]} x) - (\mathop{\mathrm{Lim}}_{x \to [\infty]} x) = [\infty] - [\infty] = [\mathsf{Undefined}]$$

This is consistent with the lemma, because $\overline{0} \sqsubseteq [\mathsf{Undefined}]$.

It is quite hard in general to say something about the left hand side. For instance when considering $f(x_1, x_2) = x_1/x_2$ then in all three of the cases:

- $g_1(x) = x$, $g_2(x) = x$
- $g_1(x) = 2x - 1$, $g_2(x) = x$
- $g_1(x) = \frac{1}{2-x}$, $g_2(x) = x$

we have $\mathrm{Lim}_{x\to[1]}\, g_1(x) = \mathrm{Lim}_{x\to[1]}\, g_2(x) = [1]$, but $\mathrm{Lim}_{x\to[1]}\, f(g_1(x), g_2(x))$ is respectively $\overline{1}$, $[1]$ and $[1^+]$.) Lemma 15 below presents a result in the special case of $1/g(x)$.

Proof Define $h(x) = f(g_1(x), \ldots, g_n(x))$. Suppose $W \in \overline{f}(\overline{g_1}(A), \ldots, \overline{g_n}(A))$. That means that there are $V_i \in \overline{g_i}(A)$ with $V_1 \times \cdots \times V_n \subseteq \mathrm{dom}(f)$ and $f[V_1 \times \cdots \times V_n] \subseteq W$. This implies that there are $U_i \in A$ with $U_i \subseteq \mathrm{dom}(g_i)$ and $g_i[U_i] \subseteq V_i$. Now define $U = U_1 \cap \cdots \cap U_n$, then $U \in A$, $U \subseteq \mathrm{dom}(g_i)$ and $g_i[U] \subseteq V_i$. This then implies that $U \subseteq \mathrm{dom}(h)$ and $h[U] \subseteq W$. Therefore $W \in \overline{h}(A)$.

This proves that $\overline{h}(A) \sqsubseteq \overline{f}(\overline{g_1}(A), \ldots, \overline{g_n}(A))$, which amounts to the required statement. $\square$

Lemma 7 *Function application and refinement.*

$$A_1 \sqsubseteq B_1 \;\&\; \ldots \;\&\; A_n \sqsubseteq B_n \;\Rightarrow\; \overline{f}(A_1, \ldots, A_n) \sqsubseteq \overline{f}(B_1, \ldots, B_n).$$

Proof Let V be given with $V \in \overline{f}(B_1, \ldots, B_n)$. Then there are $U_i \in B_i$ with $U_1 \times \cdots \times U_n \subseteq \mathrm{dom}(f)$ and $f[U_1 \times \cdots \times U_n] \subseteq W$. Because $A_i \sqsubseteq B_i$, also $U_i \in A_i$, which implies that also $V \in \overline{f}(A_1, \ldots, A_n)$. $\square$

Lemma 8 *Lifting and continuity.*

$$f \text{ continuous in } (a_1, \ldots, a_n) \;\Leftrightarrow\; \overline{f}(\overline{a_1}, \ldots, \overline{a_n}) = \overline{f(a_1, \ldots, a_n)}.$$

Remark. This lemma is useful to give actual equality instead of merely refinement as in Lemma 6.

Proof $(\Rightarrow, \sqsubseteq)$ If $V \in \overline{f(a_1 \ldots, a_n)}$, this means that V is a neighborhood of $f(a_1, \ldots, a_n)$. Because f is continuous in $(a_1, \ldots, a_n)$, there are neighborhoods U_i of a_i (so $U_i \in \overline{a_i}$) with $U_1 \times \cdots \times U_n \subseteq \mathrm{dom}(f)$ and $f[U_1 \times \cdots \times U_n] \subseteq V$. This means that $V \in \overline{f}(\overline{a_1}, \ldots, \overline{a_n})$.

$(\Rightarrow, \sqsupseteq)$ If $V \in \overline{f}(\overline{a_1}, \ldots, \overline{a_n})$, then there are $U_i \in \overline{a_i}$ (so $a_i \in U_i$) with $U_1 \times \cdots \times U_n \subseteq \mathrm{dom}(f)$ and $f[U_1 \times \cdots \times U_n] \subseteq V$. This implies that $f(a_1, \ldots, a_n) \in V$. Furthermore V is open (because $\overline{f}(\overline{a_1}, \ldots, \overline{a_n})$ is a filter), and therefore $V \in \overline{f(a_1 \ldots, a_n)}$.

$(\Leftarrow)$ Let V be some neighborhood of $f(a_1, \ldots, a_n)$: we have to prove that there are U_i which are neighborhoods of a_i such that $U_1 \times \cdots \times U_n \subseteq \mathrm{dom}(f)$ and $f[U_1 \times \cdots \times U_n] \subseteq V$. Now by definition $V \in \overline{f(a_1 \ldots, a_n)}$ and because $\overline{f(a_1 \ldots, a_n)} = \overline{f}(\overline{a_1}, \ldots, \overline{a_n})$ we have that also $V \in \overline{f}(\overline{a_1}, \ldots, \overline{a_n})$, which gives us $U_i \in \overline{a_i}$ (which are by definition neighborhoods of the a_i) with $U_1 \times \cdots \times U_n \subseteq \mathrm{dom}(f)$ and $f[U_1 \times \cdots \times U_n] \subseteq V$. $\square$

Theorem 1 *Filter limits generalize the usual definition of limits:*

$$\lim_{x \to a} f(x) = b \Leftrightarrow \operatorname*{Lim}_{x \to [a]} f(x) \sqsubseteq \overline{b}.$$

$$\lim_{x \to [a^+]} f(x) = b \Leftrightarrow \operatorname*{Lim}_{x \to [a^+]} f(x) \sqsubseteq \overline{b}.$$

$$\lim_{x \to \infty} f(x) = b \Leftrightarrow \operatorname*{Lim}_{x \to [\infty]} f(x) \sqsubseteq \overline{b}.$$

Proof We prove the first equivalence in detail. The other two are proved similarly.

($\Rightarrow$) Let be given that $\lim_{x \to a} f(x) = b$: we have to show that $\overline{f}([a]) \sqsubseteq \overline{b}$. So suppose that $V \in \overline{b}$, which means that for some $\epsilon > 0$ we have that $(b-\epsilon, b+\epsilon) \subseteq V$. Because of the limit, there is a $\delta > 0$ such that for all $x \in (a-\delta, a) \cup (a, a+\delta)$ (call this set U) we have that $f(x)$ is defined and $f(x) \in (b - \epsilon, b + \epsilon)$, or, in other words, $U \subseteq \mathrm{dom}(f)$ and $f[U] \subseteq (b - \epsilon, b + \epsilon)$. Clearly U satsifies $U \in [a]$ and $f[U] \subseteq V$, and therefore $V \in \overline{f}([a])$.

($\Leftarrow$) Now assume that $\overline{f}([a]) \sqsubseteq \overline{b}$. For a given $\epsilon > 0$, let $V = (b - \epsilon, b + \epsilon)$. Then $V \in \overline{b}$, so $V \in \overline{f}([a])$, and therefore there is some $U \in [a]$ with $U \subseteq \mathrm{dom}(f)$ and $f[U] \subseteq V$. Because $U \in [a]$, for some $\delta > 0$ we have that $(a-\delta, a) \cup (a, a+\delta) \subseteq U$. Now this implies that if $x \in (a - \delta, a) \cup (a, a + \delta)$ then $f(x)$ is defined and $f(x) \in (b - \epsilon, b + \epsilon)$. Because for each $\epsilon > 0$ there is a $\delta > 0$ with this property, $\lim_{x \to a} f(x) = b$. □

We now turn to establishing the first of many computation rules about filter limits, namely that $\overline{a}/[\infty] = [0^+]$.

Lemma 9 *$\overline{a}$ is bounded.*

Proof For arbitrary $\epsilon > 0$, we have $(a - \epsilon, a + \epsilon) \in \overline{a}$. This shows that $\overline{a}$ is bounded with $l = a - \epsilon$ and $r = a + \epsilon$. □

Lemma 10 *$a > 0 \Rightarrow \overline{a}$ positive.*

Proof $(0, \infty) \in \overline{a}$ because $(0, 2a) \subseteq (0, \infty)$ and $(0, 2a) = (a - \epsilon, a + \epsilon)$ for $\epsilon = a$. □

Lemma 11 *$A \sqsubseteq B$ and B bounded $\Rightarrow A$ bounded.*

Proof When $(l, r) \in B$ then from $A \sqsubseteq B$ it follows that also $(l, r) \in A$. □

Lemma 12 *$A \sqsubseteq B$ and B positive $\Rightarrow A$ positive.*

Proof When $(0, \infty) \in B$ then from $A \sqsubseteq B$ it follows that also $(0, \infty) \in A$. □

Lemma 13 *A proper and bounded and positive $\Rightarrow A/[\infty] = [0^+]$. In particular $\overline{a}/[\infty] = [0^+]$.*

Proof Given a proper, bounded, positive A, we must show that $A/[\infty] = [0^+]$. Because A is bounded and positive, for some $r > 0$ we have that $(0, r) \in A$. By Lemmas 9 and 10, $\overline{a}$ satisfies the hypotheses of being bounded and positive.

($\sqsubseteq$) Suppose $V \in [0^+]$. Then there is an $\epsilon > 0$ with $(0, \epsilon) \subseteq V$. Define $U_1 = (0, r)$ and $U_2 = (r/\epsilon, \infty)$, then $U_1 \in A$ and $U_2 \in [\infty]$. Furthermore for $0 < x_1 < r$ and $x_2 > r/\epsilon$ we have that $0 < x_1/x_2 < \epsilon$, so $\{x_1/x_2 \mid x_1 \in U_1 \ \& \ x_2 \in U_2\} \subseteq V$. From this it follows that $V \in A/[\infty]$.

($\sqsupseteq$) Suppose $V \in A/[\infty]$, then there are $U_1 \in A$ and $U_2 \in [\infty]$ with $0 \notin U_1$ and $\{x_1/x_2 \mid x_1 \in U_1 \ \& \ x_2 \in U_2\} \subseteq V$. Because A is proper there is an x with $0 < x < r$ and $x \in U_1$. Also, because $U_2 \in [\infty]$ there is an ω with $(\omega, \infty) \subseteq U_2$. Together this implies that $(0, x/\omega) \subseteq V$, and therefore that $V \in [0^+]$. □

Lemma 14 *Let f be continuous at a and strictly monotonically increasing in a neighborhood of a. Then the following rules apply for calculating limits:*

$$\operatorname*{Lim}_{x \to [a]} f(x) = [f(a)]$$

$$\operatorname*{Lim}_{x \to [a^-]} f(x) = \left[f(a)^-\right]$$

$$\operatorname*{Lim}_{x \to [a^+]} f(x) = \left[f(a)^+\right]$$

Proof We take the third assertion first. Let A be $[a^+]$. We must prove $\overline{f}(A) = [f(a)^+]$. Let $W \in \overline{f}(A)$. Then for some $c > a$, we have $f((a, c)) \subseteq W$. Decrease c if necessary so that f is strictly monotonic and continuous on (a, c). Since f is continuous and monotonic, we have $f((a, c)) = (f(a), f(c))$, and $f(c) > f(a)$. Hence $(f(a), f(c)) \subseteq W$ and so $W \in [f(a)^+]$. Conversely, suppose $W \in [f(a)^+]$. Then for some $C > f(a)$, we have $(f(a), C) \subseteq W$. Decrease C if necessary so that f is invertible on $(f(a), C)$ and strictly monotonic on (a, c), where $f(c) = C$ and $a < c$. Then $f((a, c)) = (f(a), C)$, so $f((a, c)) \subseteq W$. Hence $W \in \overline{f}(A)$, completing the proof of the third assertion. The first two assertions are proved similarly. □

Lemma 15

$$\operatorname*{Lim}_{x \to A} 1/g(x) = 1/\operatorname*{Lim}_{x \to A} g(x).$$

Proof The expression on the right is the lifting of the reciprocal function, applied to $\operatorname{Lim}_{x \to A} g(x)$. This observation makes this a special case of Lemma 5, since the reciprocal function satisfies the hypothesis required in that lemma, namely it is continuous on its domain and that domain is open. □

The previous lemma is an example of a more general situation. Suppose we have a function $f(x, y)$ of two (or more) variables. Then we write $f(c, A)$ where A is a filter, and c is a real number. In the previous lemma, f was division. What does $f(c, A)$ mean? Answer: if $g(y) = f(c, y)$ (with c fixed), then g has a lifting $\overline{g_c}$, and $f(c, A) = \overline{g_c}(A)$.

Lemma 16 $1/[\infty] = [0^+]$

Proof Let $W \in 1/[\infty]$. Here $1/[\infty]$ is the lifting of the reciprocal function $g(x) = 1/x$, so this means that for some $U \in [\infty]$, we have $g(U) \subseteq W$. If $U \in [\infty]$, then U has a subset of the form (c, ∞) with $c > 0$, and $g((c,\infty)) \subseteq g(U) \subseteq W$. That is, $(0, 1/c) \subseteq W$. But $(0, 1/c) \in [0^+]$, so $W \in [0^+]$. That proves $1/[\infty] \subseteq [0^+]$. Conversely, let $W \in [0^+]$. Then for some $c > 0$ we have $(0, c) \subseteq W$. Let $g(x) = 1/x$ and $U = (1/c, \infty)$. Then $g(U) = (0, c)$, so $W \in 1/[\infty]$. Hence $[0^+] \subseteq 1/[\infty]$. Hence $1/[\infty] = [0^+]$. □

5 Calculating with Filter Limits

5.1 An example calculation justified

Here is the informal calculation to be made precise:

$$\lim_{x\to\infty} \sin(1/x) = \sin(\lim_{x\to\infty}(1/x)) = \sin(1/(\lim_{x\to\infty} x))$$
$$= \sin(1/\infty) = \sin(0) = 0$$

The following is the filter-limit version of this calculation. The small numbers are references to lemmas in the previous section, which justify the steps.

$$\operatorname*{Lim}_{x\to[\infty]} \sin(1/x) \stackrel{5}{=} \overline{\sin}(\operatorname*{Lim}_{x\to[\infty]}(1/x)) \stackrel{15}{=} \overline{\sin}(1/(\operatorname*{Lim}_{x\to[\infty]} x))$$
$$\stackrel{3}{=} \overline{\sin}(1/[\infty]) \stackrel{16}{=} \overline{\sin}([0^+]) \stackrel{14}{=} [0^+]$$

This calculation gives a filter answer. If we want an answer in conventional terms, we can use Lemma 1 and Theorem 1 to conclude that:

$$\lim_{x\to\infty} \sin(1/x) = 0$$

Notation. We could, according to previously stated conventions, have omitted the brackets on the ∞ symbols under the limit sign, and we could have omitted the overlines on sin.

5.2 Calculation rules for filters

The plan is to give rules that enable us to calculate limits using (symbols for) filters. We have already shown how to extend the arithmetic operations, and functions in general to filters, using the notion of 'lifting'. The following calculation rules can be verified in the same way as the lemmas in the previous section.

$$\frac{\overline{1}}{[0]} = \frac{[1]}{[0]} = [\pm\infty]$$
$$\frac{[1]}{\overline{0}} = \frac{\overline{1}}{\overline{0}} = [\mathsf{DomainError}]$$

$$
\begin{aligned}
\frac{\overline{1}}{[\pm\infty]} &= [0] \\
\frac{\overline{1}}{[\infty]} &= \left[0^+\right] \\
\frac{\overline{1}}{[0^+]} &= [\infty] \\
\frac{\overline{1}}{[0^-]} &= [-\infty] \\
[\infty] + [a] &= [\infty] \\
[\infty] + \overline{a} &= [\infty] \\
[\infty] + \left[a^+\right] &= [\infty] \\
[a]\,[\infty] &= [\infty] \qquad \text{if } a > 0 \\
[\infty]\,[\infty] &= [\infty] \\
[\infty]\,[-\infty] &= [-\infty]
\end{aligned}
$$

What is zero times infinity? Well, we have

$$\left[0^+\right][\infty] = (\mathbf{0}, \infty)$$

$$\overline{0}\,[\infty] = [\mathsf{Undefined}]$$

$$[0]\,[\infty] = (-\infty, \mathbf{0}) \vee (\mathbf{0}, \infty)$$

This last answer uses the 'join' operation on filters, which we have not yet defined, but the filter in question here contains just two open sets: R and the union of the two intervals $(-\infty, 0)$ and $(0, \infty)$.

Using the notion of lifting of a function to filters, we can verify the following rules:

$$
\begin{aligned}
\ln[\infty] &= [\infty] \\
\ln\left[0^+\right] &= [-\infty] \\
e^{[-\infty]} &= \left[0^+\right] \\
e^{[\infty]} &= [\infty] \\
\sin([\mathbf{a}, \mathbf{b}]) &= [-\mathbf{1}, \mathbf{1}] \qquad \text{if } |b - a| \geq 2\pi \\
\sin([\infty]) &= [-\mathbf{1}, \mathbf{1}]
\end{aligned}
$$

Every mathematician has used these rules for years, but this is the first time they have been given a precise semantics. Now, for example, we can give a completely rigorous version of the calculation in the introduction:

$$
\begin{aligned}
\operatorname*{Lim}_{x\to\infty} x - \sqrt{x} &= \operatorname*{Lim}_{x\to\infty} \frac{(x - \sqrt{x})(x + \sqrt{x})}{(x + \sqrt{x})} \\
&= \operatorname*{Lim}_{x\to\infty} \frac{x^2 - x}{x + \sqrt{x}} \\
&= \operatorname*{Lim}_{x\to\infty} \frac{x - 1}{1 + \frac{1}{\sqrt{x}}}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\operatorname*{Lim}_{x\to\infty} x - 1}{\operatorname*{Lim}_{x\to\infty} 1 + \frac{1}{\sqrt{x}}} \\
&= \frac{\operatorname*{Lim}_{x\to\infty} x - \operatorname*{Lim}_{x\to\infty} 1}{\operatorname*{Lim}_{x\to\infty} 1 + \operatorname*{Lim}_{x\to\infty} \frac{1}{\sqrt{x}}} \\
&= \frac{[\infty] - \bar{1}}{\operatorname*{Lim}_{x\to\infty} 1 + \frac{1}{\operatorname*{Lim}_{x\to\infty} \sqrt{x}}} \\
&= \frac{[\infty]}{\bar{1} + \frac{1}{\sqrt{\operatorname*{Lim}_{x\to\infty} x}}} \\
&= \frac{[\infty]}{\bar{1} + \frac{1}{\sqrt{[\infty]}}} \\
&= \frac{[\infty]}{\bar{1} + \frac{1}{[\infty]}} \\
&= \frac{[\infty]}{\bar{1} + \bar{0}} \\
&= \frac{[\infty]}{\bar{1}} = [\infty]
\end{aligned}
$$

What happens when the function is not defined in a neighborhood of the limit point? For example, what is the limit of $\sqrt{x}$ as $x \to 0$? As it turns out, that depends on whether you are French or not. In France, this limit is 0; but in the U.S., Russia, Japan, and Germany, it is considered undefined.[5] Let us see what the filter theory of limits has to say. We have

$$\operatorname*{Lim}_{x\to[0]} \sqrt{x} = [\mathsf{DomainError}]$$

but

$$\operatorname*{Lim}_{x\to[0^+]} \sqrt{x} = [0^+]$$

On the other hand, if we replace the underlying space R by the domain of $\sqrt{x}$, namely $X = [0, \infty)$, and use the filter theory of limits on the space X, we would then find that the limit is defined and takes the value 0, corresponding to the French notion.

5.3 Filter limits and the asymptotic behavior of functions

We have several times emphasized that filter limits encapsulate more information than ordinary limits. That theme leads to a long list of filter-limit rules for

[5]The French consider only values from the domain of f as determining the limit. See for example the textbook [3], Chapter 5, Section 3. We also checked two other French textbooks.

the various functions of analysis. Each rule encodes a fact about the asymptotic behavior of a function near some special point. We will illustrate the process of formulating these rules, rather than give a six-page list of them. Consider the limit $(\ln(1+t))/t \to 1$ as $t \to 0$. This corresponds to four filter-limit rules, as follows:

$$\operatorname*{Lim}_{t\to[0]} \frac{\ln(1+t)}{t} = [1]$$

$$\operatorname*{Lim}_{t\to[0^-]} \frac{\ln(1+t)}{t} = [1^+]$$

$$\operatorname*{Lim}_{t\to[0^+]} \frac{\ln(1+t)}{t} = [1^-]$$

$$\operatorname*{Lim}_{t\to\overline{0}} \frac{\ln(1+t)}{t} = [\mathsf{DomainError}]$$

The last of these four is of purely academic interest, given here only for completeness. The other three correspond to the usual two-sided and one-sided limits. By definition of Lim, the filter limits on the left are defined (filter limits are *always* defined, as filters!) By Theorem 1, since the ordinary limit is defined and has the value 1, the value must be a filter refining the principal filter $\overline{1}$. Which filter might that be? Intuitively, that will depend on the asymptotic behavior of the function in the vicinity of the limit point, in this case 0. For example, in the case of the limit from the left, the function is decreasing to a limit of 1, so the filter value is $[1^+]$. In general, when considering the limit of $f(x)$ as $x \to a$, if the ordinary limit from the left is defined (and equals c, say), and the function is continuous in some interval $(a-\epsilon, a)$, then there are just three possibilities:

- $f(x) > c$ in some interval of the form $(a-\epsilon, a)$, and the value of the filter limit is $[c^+]$
- $f(x) < c$ in some interval of the form $(a-\epsilon, a)$, and the value of the filter limit is $[c^-]$
- $f(x) = c$ infinitely often as $x \to a$, and the value of the filter limit is $\overline{c}$

There are three similar possibilities for the right-hand limit. For the two-sided limits, there also arises the fourth possibility that the value of the filter limit is $[c]$, in case the function is strictly monotonic on a punctured neighborhood of a. These considerations show that the filter limit value contains, intuitively speaking, more information about the asymptotic behavior of the function than does the usual limit value.

We used this approach to formulate filter-limit versions of the usual rules for 'special limits' such as

$$(\sin t)/t \to 1 \qquad \text{as} \qquad t \to 0$$
$$(1-\cos t)/t^2 \to \frac{1}{2} \qquad \text{as} \qquad t \to 0$$

$$\begin{array}{llll} (1+t)^{1/t} \to e & \text{as} & t \to 0 & \\ (e^t - 1)/t \to 1 & \text{as} & t \to 0 & \\ (t^n \ln|t| \to 0 & \text{as} & t \to [0^+] & (\text{for } n > 0) \end{array}$$

Note that the filter answer in the last example depends on whether n is even or odd.

There are a number of rules concerning limits of powers of t as $t \to 0$ or $t \to \infty$, all of which can be converted to filter limit rules in the same way.

The behavior of oscillatory functions gives rise to some more interesting filter-limit rules, such as these:

$$\begin{aligned} \operatorname*{Lim}_{t\to 0} \cos\frac{1}{t} &= [-\mathbf{1}, \mathbf{1}] \\ \operatorname*{Lim}_{t\to\infty} \cos t &= [-\mathbf{1}, \mathbf{1}] \\ \operatorname*{Lim}_{t\to\infty} t\sin t &= (-\infty, \infty) \\ \operatorname*{Lim}_{t\to\infty} \tan t &= [\mathsf{DomainError}] \end{aligned}$$

We do not get $(-\infty, \infty)$ for this last limit since the tangent is undefined at a sequence of points converging to infinity.

5.4 Calculating with infinity

In addition to the rules for filter limits, there are also calculation rules involving filters only. We gave some special cases above, but now we formulate these rules more generally using the concepts of a 'positive filter' and a 'bounded filter'. (These concepts have been defined above.) We give several of these rules in an informal form, and for each form, we give one or more precise translations into rules about filters:

- $\pm\infty/\mathsf{positive} = \pm\infty$

$$A \text{ positive } \& \ A \text{ bounded } \Rightarrow [\infty]/A = [\infty]$$

$$A \text{ positive } \& \ A \text{ bounded } \Rightarrow [-\infty]/A = [-\infty]$$

$$A \text{ positive } \& \ A \text{ bounded } \Rightarrow [\pm\infty]/A = [\pm\infty]$$

- $\mathsf{finite}/\pm\infty = 0$

$$A \text{ bounded } \Rightarrow A/[\infty] \sqsubseteq \bar{0}$$

$$A \text{ bounded } \Rightarrow A/[-\infty] \sqsubseteq \bar{0}$$

$$A \text{ bounded } \Rightarrow A/[\pm\infty] \sqsubseteq \bar{0}$$

- $\mathsf{positive} \cdot \pm\infty = \pm\infty$

$$A \text{ positive } \& \ A \text{ bounded } \Rightarrow A \cdot [\infty] = [\infty]$$

$$A \text{ positive } \& \ A \text{ bounded } \Rightarrow A \cdot [-\infty] = [-\infty]$$

$$A \text{ positive } \& \ A \text{ bounded } \Rightarrow A \cdot [\pm\infty] = [\pm\infty]$$

- $\pm\infty \cdot \infty = \pm\infty$
$$[\infty] \cdot [\infty] = [\infty]$$
$$[-\infty] \cdot [\infty] = [-\infty]$$
$$[\pm\infty] \cdot [\infty] = [\pm\infty]$$

- $\pm\infty + \mathsf{finite} = \pm\infty$
$$A \text{ } \textit{bounded} \Rightarrow [\infty] + A = [\infty]$$
$$A \text{ } \textit{bounded} \Rightarrow [-\infty] + A = [-\infty]$$
$$A \text{ } \textit{bounded} \Rightarrow [\pm\infty] + A = [\pm\infty]$$

- $\infty + \infty = \infty$
$$[\infty] + [\infty] = [\infty]$$

- $u^\infty = \infty$ if $u > 1$
$$u > 1 \ \& \ A \sqsubseteq \overline{u} \Rightarrow A^{[\infty]} = [\infty]$$

- $u^\infty = 0$ if $0 < u < 1$
$$0 < u < 1 \ \& \ A \sqsubseteq \overline{u} \Rightarrow A^{[\infty]} = [0^+]$$

- $u^{-\infty} = 0$ if $u > 1$
$$u > 1 \ \& \ A \sqsubseteq \overline{u} \Rightarrow A^{[-\infty]} = [0^+]$$

- $u^{-\infty} = \infty$ if $0 < u < 1$
$$0 < u < 1 \ \& \ A \sqsubseteq \overline{u} \Rightarrow A^{[-\infty]} = [\infty]$$

- $\infty^n = \infty$ if $n > 0$
$$n > 0 \Rightarrow [\infty]^n = [\infty]$$

- $\infty - \infty = \mathsf{undefined}$
$$[\infty] - [\infty] = [\mathsf{Undefined}]$$

  Note that generally limits which informally are ‘undefined’ come out either to more precise filter answers or to [DomainError]. This is one of the few cases in which we get the filter [Undefined].

5.5 Functions at Infinity

The following rules enable one to evaluate limits by passing the Lim symbol through functions, then evaluating the inner limit to a filter, and then evaluating the function, as in $\mathrm{Lim}_{x\to 0}\ln(1/x) = \ln \mathrm{Lim}_{x\to 0} 1/x = \ln[\infty] = [\infty]$.

- $\ln\infty = \log\infty = \infty$
$$\ln[\infty] = \log[\infty] = [\infty]$$
- $\sqrt{\infty} = \infty$
$$\sqrt{[\infty]} = [\infty]$$
- $\sqrt[n]{\infty} = \infty$
$$\sqrt[n]{[\infty]} = [\infty]$$
- $\arctan \pm\infty = \pm\pi/2$
$$\arctan[\infty] = \left[\pi/2^-\right]$$
$$\arctan[-\infty] = \left[-\pi/2^+\right]$$
$$\arctan[\pm\infty] = \left[-\pi/2^+\right] \vee \left[\pi/2^-\right]$$
- $\operatorname{arccot}\infty = 0$
$$\operatorname{arccot}[\infty] = \left[0^+\right]$$
- $\operatorname{arccot} -\infty = \pi$
$$\operatorname{arccot}[-\infty] = \left[\pi^-\right]$$
- $\operatorname{arcsec} \pm\infty = \pi/2$
$$\operatorname{arcsec}[\infty] = \left[\frac{\pi}{2}^-\right]$$
$$\operatorname{arcsec}[-\infty] = \left[\frac{\pi}{2}^+\right]$$
$$\operatorname{arcsec}[\pm\infty] = \left[\frac{\pi}{2}\right]$$
- $\operatorname{arccsc} \pm\infty = 0$
$$\operatorname{arccsc}[\infty] = \left[0^+\right]$$
$$\operatorname{arccsc}[-\infty] = \left[0^-\right]$$
$$\operatorname{arccsc}[\pm\infty] = [0]$$

- Trig limits at ∞

$$\sin[\infty] = [-\mathbf{1}, \mathbf{1}]$$
$$\cos[\infty] = [-\mathbf{1}, \mathbf{1}]$$
$$\tan[\infty] = [\mathsf{DomainError}]$$
$$\cot[\infty] = [\mathsf{DomainError}]$$
$$\sec[\infty] = 1/[-1, 1] = (-\infty, -\mathbf{1}) \vee (\mathbf{1}, \infty)$$
$$\csc[\infty] = 1/[-1, 1] = (-\infty, -\mathbf{1}) \vee (\mathbf{1}, \infty)$$

  The 'join' operation used here will be defined in the next section.

- $\cosh \pm\infty = \infty$

$$\cosh[\infty] = \cosh[-\infty] = \cosh[\pm\infty] = \infty$$

- $\sinh \pm\infty = \pm\infty$

$$\sinh[\infty] = [\infty]$$
$$\sinh[-\infty] = [-\infty]$$
$$\sinh[\pm\infty] = [\pm\infty]$$

- $\tanh \pm\infty = \pm 1$

$$\tanh[\infty] = \left[1^-\right]$$
$$\tanh[-\infty] = \left[-1^+\right]$$
$$\tanh[\pm\infty] = \left[-1^+\right] \vee \left[1^-\right]$$

- $\ln 0 = -\infty$

$$\ln\left[0^+\right] = [-\infty]$$
$$\ln\left[0^-\right] = \ln[0] = \ln\overline{0} = [\mathsf{DomainError}]$$

6 Refinement

The general theme of this paper is that by using calculation rules for filter limits and filters, we can evaluate limits, arriving at rigorously justified answers. However, there is a subtlety in this process which is worth an extended discussion. Namely, the rule given in Lemma 6, for passing Lim through a function of two (or more) variables, does not preserve filter equality, but only has filter refinement on the right. For example, we do not in general have

$$\operatorname*{Lim}_{x\to 0} \frac{u}{v} = \frac{\mathrm{Lim}_{x\to 0}\, u}{\mathrm{Lim}_{x\to 0}\, v}$$

Instead, we only have

$$\operatorname*{Lim}_{x\to 0} \frac{u}{v} \sqsubseteq \frac{\mathrm{Lim}_{x\to 0}\, u}{\mathrm{Lim}_{x\to 0}\, v}$$

We barely avoided difficulties with this phenomenon when evaluating the example limit

$$\operatorname*{Lim}_{x\to\infty} \sin(1/x) = \sin \operatorname*{Lim}_{x\to\infty} 1/x$$

If we simply push the Lim through the fraction to get

$$\frac{\operatorname{Lim}_{x\to\infty} 1}{\operatorname{Lim}_{x\to\infty} x}$$

then we will lose equality, and find out in the end that the answer is some filter that refines $[0^+]$, rather than getting the exact answer $[0^+]$. We were able to avoid this difficulty by regarding $1/x$ as the reciprocal function (of one argument) instead of a fraction (function of two arguments), and applying Lemma 5, which allows us to pass Lim through a function of one argument.

But what if the numerator had not been a constant, but simply something whose limit is 1? Suppose $\operatorname{Lim}_{x\to\infty} f(x) = [1]$ and consider the problem

$$\operatorname*{Lim}_{x\to\infty} \sin \frac{f(x)}{x}$$

The first step is legal:

$$\operatorname*{Lim}_{x\to\infty} \sin \frac{f(x)}{x} = \sin \operatorname*{Lim}_{x\to\infty} \frac{f(x)}{x}$$

but we cannot conclude that this is equal to

$$\sin \frac{\operatorname{Lim}_{x\to\infty} f(x)}{\operatorname{Lim}_{x\to\infty} x},$$

only that the answer to the original problem refines this expression. Well, we can correctly evaluate this latter expression to $[1] \,/\, [\infty] = [0^+]$, so we know that the answer refines $[0^+]$. Therefore, if all we want is the answer to the ordinary limit, we are done: the answer is 0.

This situation is perfectly general. To evaluate a limit, we execute the following plan:

1. Change lim to Lim.
2. Evaluate the filter limit, using the rules given in this paper.
3. Project the filter answer back to a number. That's the answer.

The last step, 'projection', works as follows:

$[\infty]$	projects to	∞
$[-\infty]$	projects to	$-\infty$
$[\pm\infty]$	projects to	undefined
$\overline{c}$	projects to	c
$[c^+]$	projects to	c
$[c^-]$	projects to	c
$[c]$	projects to	c

Projection has the property that if filter A refines filter B, and filter B does not project to 'undefined', then A and B have the same projection. According to Theorem 1, the filter answer that we get by our calculation rules must refine the true filter answer. Therefore, the calculated answer and the true answer have the same projection. Refinement makes no difference if we are only interested in the answer to an ordinary limit. We can detour through filter calculations, and when we return to the ordinary answer, it is correct!

However, as we have seen, sometimes the filter limit answer is more informative than the ordinary answer, and the mathematical question arises, whether we can always calculate the correct filter answer, or whether we must settle for 'only a refinement'. The results in this section are directed to this question. By way of motivation, consider the above example again. We calculate with filters, and determine that the correct answer refines $[0^+]$. Well, what filter can possibly refine $[0^+]$? If we assume that f is continuous on some interval (a, ∞), we can show that the filter answer is a connected filter, in a sense to be defined below. We can also characterize the connected filters that refine a principal filter $\overline{c}$: they are just the filters $[c^+]$, $[c^-]$, and $\overline{c}$. Hence, if we calculate $[0^+]$, so we know the answer refines $[0^+]$ (and hence refines $\overline{0}$), then the answer must be one of $[0^+]$, $[0^-]$, and $\overline{0}$. But it must also refine $[0^+]$, the calculated answer. The only possibility is $[0^+]$, so that is the answer. The characterization of connected filters (proved below) thus enables us to get the exact filter answer whenever we can calculate the answer to the ordinary limit.

When considering two-sided limits, we note that $[c]$ is not a connected filter, but by (essentially) considering the two-sided limits as the 'join' of two one-sided limits, we are able to apply the result on connected filters in a similar way to this situation.

6.1 Connected Filters

Suppose we have calculated that a certain limit refines $\overline{0}$. There are many strange and wonderful filters that refine $\overline{0}$, for example the filter generated by all intervals of the form (p, q) where the interval (p, q) does not contain the reciprocal of any integer, but does contain 0. This filter, and many others, will never arise as the answer to a limit problem in freshman calculus. If the function whose limit is being calculated is continuous on each side of the limit point (that is, in some open one-sided neighborhood on each side), then we will show that the one-sided filter limit is one of a small number of possibilities, and not some wild filter like the one above. To that end, we define the concepts of 'connected filter' and 'interval filter', and then prove a theorem connecting them. This is the filter analogue of the theorem that the connected subsets of R are just the intervals.

A filter A is called *connected* when:

$$\forall U \in A \, \exists V \in A : V \subseteq U \;\&\; V \textit{ is connected in } \mathrm{R}$$

(Note that $\emptyset$ is a connected set, so [Improper] is a connected filter.)

An *interval filter* is a filter that has one of the following forms:

$$\begin{gathered} \mathcal{X}(\{I\}) \\ [a^+] \\ [a^-] \\ [\infty] \\ [-\infty] \\ [\mathsf{DomainError}] \\ [\mathsf{Improper}] \end{gathered}$$

where I is an non-empty interval (open, half-open or closed, and possibly infinite), and where a is a finite real number.

Note that we have:

$$\begin{aligned} \overline{a} &= \mathcal{X}(\{[a,a]\}) \\ [\mathsf{Undefined}] &= \mathcal{X}(\{(-\infty,\infty)\}) \end{aligned}$$

so $\overline{a}$ and $[\mathsf{Undefined}]$ are interval filters too.

A function f is called *continuous on A* when:

$$\exists U \in A : U \subseteq \mathrm{dom}(f) \ \&\ f \ \textit{continuous on}\ U$$

(Note that 'continuous on $\overline{a}$' is not equivalent to 'continuous in a'.)

6.2 Join and meet

When we analyze a two-sided limit into two one-sided limits, and then want to put the results back together, we need the concept of the 'join' of two filters, which we write $A \vee B$. For example, $[0^+] \vee [0^-] = [0]$, as is proved in Lemma 19 below.

For filters A and B we define:

$$A \vee B = A \cap B$$

There is a dual operation 'meet'. Together join and meet turn the set of filters into a lattice. We have no direct use for meet, or for the lattice structure on the set of filters, but for completeness we give the definition.

$$A \wedge B = \{U \cap V \mid U \in A \ \&\ V \in B\}$$

Lemma 17

$$\overline{f}(A \vee B) = \overline{f}(A) \vee \overline{f}(B)$$

Proof ($\sqsubseteq$) Suppose $V \in \overline{f}(A) \vee \overline{f}(B)$, then both $V \in \overline{f}(A)$ and $V \in \overline{f}(B)$, so there are $U \in A$ and $U' \in B$ such that $U, U' \subseteq \mathrm{dom}(f)$ and $f[U], f[U'] \subseteq V$. Then $U \cup U' \in A \vee B$, and furthermore $U \cup U' \subseteq \mathrm{dom}(f)$ and $f[U \cup U'] \subseteq V$, so $V \in \overline{f}(A \vee B)$.

($\sqsupseteq$) If $V \in \overline{f}(A \vee B)$, there is a $U \in A \vee B$ (and so $U \in A$ and $U \in B$) with $U \subseteq \mathrm{dom}(f)$ and $f[U] \subseteq V$. This same U shows that $V \in \overline{f}(A)$ and $V \in \overline{f}(A)$, which means that $V \in \overline{f}(A) \vee \overline{f}(B)$. □

The following lemma is not used, but may be of some interest, since it is not quite dual to the previous lemma: we get only refinement, not equality.[6]

Lemma 18

$$\overline{f}(A \wedge B) \sqsubseteq \overline{f}(A) \wedge \overline{f}(B)$$

Proof Let $V \in \overline{f}(A) \wedge \overline{f}(B)$, so there are $V' \in \overline{f}(A)$ and $V' \in \overline{f}(B)$ with $V = V' \cap V'$. Then there are $U' \in A$ and $U' \in B$ with $U', U' \subseteq \mathrm{dom}(f)$ and $f[U'] \subseteq V'$, $f[U'] \subseteq V'$. ¿From this $U' \cap U' \in A \wedge B$, $U' \cap U' \subseteq \mathrm{dom}(f)$ and $f[U' \cap U'] \subseteq V' \cap V'$ and so $V' \cap V' \in \overline{f}(A \wedge B)$. □

Lemma 19

$$[a] = \left[a^-\right] \vee \left[a^+\right]$$

Proof A set contains a set of the form $(a - \epsilon, a) \cup (a, a + \epsilon)$ iff it contains both a set of the form $(a - \epsilon, a)$ and one of the form $(a, a + \epsilon)$ (take the minimum of those two epsilons). □

6.3 Interval filters and connected filters

Theorem 2 *Let A be a filter on* R. *Then A is connected if and only if A is an interval filter.*

Proof We may assume that A is proper and non-empty, because the only filters that don't satisfy that assumption are [Improper] and [DomainError] and both are connected and an interval filter. Also, it's easy to verify by inspection that every interval filter is connected. So all that is needed is to show that if a proper non-empty A is connected, then it is an interval filter.

Define $L = \{l \in \mathrm{R} \cup \{-\infty\} | (l, \infty) \in A\}$ and $R = \{r \in \mathrm{R} \cup \{\infty\} | (-\infty, r) \in A\}$. L will be closed to the left (if $l \in L$ and $l' < l$ then also $l' \in L$), so L has to have the form $\{-\infty\}$, $[-\infty, a)$, $[-\infty, a]$ or $[-\infty, \infty)$. Similarly R will be of the form $(-\infty, \infty]$, $[b, \infty]$, $(b, \infty]$ or $\{\infty\}$. Now from the fact that A is connected one can deduce that A consists of exactly those open U for which $(l, r) \subseteq U$ for some $l \in L$ and $r \in R$. This means that there is at most one connected A for a given L and R. Because for each combination of L and R that can occur – the constraint $L \cap R = \emptyset$ has to hold, because else A is improper – there exists an appropriate interval filter (which, as already has been noted, is connected), this means that A has to be that interval filter. □

Corollary 1 *(Characterization of connected filters)*

$$A \sqsubseteq \overline{a} \;\&\; A \textit{ proper and connected} \Leftrightarrow A \in \{\overline{a}, \left[a^+\right], \left[a^-\right]\}$$

$$A \sqsubseteq [\infty] \;\&\; A \textit{ proper and connected} \Leftrightarrow A = [\infty]$$

[6]This is analogous to the fact that, while the image of a union is the union of the images, the same is not true for intersections: the intersection of the images contains, but might be larger than, the image of the intersections.

Proof This follows from the theorem by considering the various possibilities for an interval filter. $\square$

Theorem 3 *If A is a connected filter and f is continuous on A then $\overline{f}(A)$ is connected.*

Proof Suppose that $V \in \overline{f}(A)$: we'll have to prove that there is some $V' \in \overline{f}(A)$ with $V' \subseteq V$ and V' connected. By the definition of $\overline{f}(A)$ there is a $U \in A$ with $U \subseteq \mathrm{dom}(f)$ and $f[U] \subseteq V$. Because f is continuous on A we can assume that f is continuous on U (making U smaller if necessary), and because A is connected we furthermore can assume that U is connected (again, making U smaller if necessary: this won't destroy the other facts about U.) Then because U is connected and f is continuous on U we find that $f[U]$ will be connected as well. Let V' be the component of V that contains $f[U]$, then V' will be open (the components of an open set are open), $f[U] \subseteq V'$, so $V' \in \overline{f}(A)$ and V' will be connected (because it's a component). So V' has the required properties. $\square$

Lemma 20

$$f \textit{ continuous on } \overline{a} \Rightarrow \exists\, b \in \mathrm{R} : \overline{f}(\overline{a}) = \overline{b}$$

$$f \textit{ continuous on } \overline{a} \Rightarrow \exists\, b \in \mathrm{R} : \operatorname*{Lim}_{x \to \overline{a}} f(x) = \overline{b}$$

Proof The second statement is by the definition of Lim the same as the first, so it suffices to prove the first statement.

Because f is continuous on $\overline{a}$, f is continuous on some neighborhood U of a. In particular, because $a \in U$ this implies that f is continuous in a, which implies that f is defined in a. Take $b = f(a)$. Because f is continuous in a we may apply Lemma 8, which gives that $\overline{f}(\overline{a}) = \overline{f(a)} = \overline{b}$. $\square$

Lemma 21 *If $A \in \{[a^+], [a^-], [\infty], [-\infty]\}$ then:*

$$f \textit{ continuous on } A \Rightarrow \overline{f}(A) \textit{ proper non-empty interval filter}$$

$$f \textit{ continuous on } A \Rightarrow \operatorname*{Lim}_{x \to A} f(x) \textit{ proper non-empty interval filter}$$

Proof Again, the second statement is the same as the first.

The A's are interval filters, hence by Theorem 2 connected. By Theorem 3 this implies that $\overline{f}(A)$ is connected, and so again by Theorem 2 this means that $\overline{f}(A)$ is an interval filter. It remains to show that $\overline{f}(A)$ is proper and non-empty.

By the definition of $\overline{f}(A)$ any $V \in \overline{f}(A)$ contains some $f[U]$ with $U \in A$ and $U \subseteq \mathrm{dom}(f)$. Because A is not improper, such a U will not be empty, therefore $f[U]$ won't be empty, and hence V won't be empty. Hence $\overline{f}(A)$ won't be improper. Because f is continuous on A, f will be continuous on some $U \in A$, which implies $U \subseteq \mathrm{dom}(f)$. Then because $f[U] \subseteq \mathrm{R}$ we find $\mathrm{R} \in \overline{f}(A)$ and so $\overline{f}(A)$ won't be empty. $\square$

Now we are in position to characterize definitively the continuous images of connected filters. By Corollary 1 we know what the (relatively few) possibilities are for connected filters. By Theorem 3 we know that continuous functions preserve connected filters. It is then an easy matter to spell out what the possible refinements are of images of the specific connected filters.

Theorem 4 *If* $A \in \{[a^+], [a^-], [\infty], [-\infty]\}$ *then:*

$$\overline{f}(A) \sqsubseteq \overline{b} \;\&\; f \textit{ continuous on its domain} \Rightarrow \overline{f}(A) \in \{[b^+], [b^-], \overline{b}\}$$

$$\operatorname*{Lim}_{x\to A} f(x) \sqsubseteq \overline{b} \;\&\; f \textit{ continuous on its domain} \Rightarrow \operatorname*{Lim}_{x\to A} f(x) \in \{[b^+], [b^-], \overline{b}\}$$

Proof Again, the second statement is the same as the first.

We first deduce that f is continuous on A: $\overline{f}(A)$ can't be empty (because $[\mathsf{DomainError}] \not\sqsubseteq \overline{b}$), so for some $U \in A$ we have that $U \subseteq \mathrm{dom}(f)$. Because f is continuous on its domain, it follows that f is continuous on U and so f is continuous on A.

Now from Lemma 21 we then get that $\overline{f}(A)$ is proper and connected and from this and $\overline{f}(A) \sqsubseteq \overline{b}$ it follows from the Corollary that $\overline{f}(A) \in \{[b^+], [b^-], \overline{b}\}$. □

Theorem 5

$$\overline{f}([a]) \sqsubseteq \overline{b} \;\&\; f \textit{ continuous on its domain} \Rightarrow \overline{f}([a]) \in \{[b^+], [b^-], [b], \overline{b}\}$$

$$\operatorname*{Lim}_{x\to [a]} f(x) \sqsubseteq \overline{b} \;\&\; f \textit{ continuous on its domain} \Rightarrow \operatorname*{Lim}_{x\to [a]} f(x) \in \{[b^+], [b^-], [b], \overline{b}\}$$

Proof Again, the second statement is the same as the first.

By Lemma 19 we know $[a] = [a^-] \vee [a^+]$ and so by Lemma 17 we can calculate $\overline{f}([a]) = \overline{f}([a^-]) \vee \overline{f}([a^+])$. Because $\overline{f}([a]) \sqsubseteq \overline{b}$ we also have $\overline{f}([a^-]) \sqsubseteq \overline{b}$ and $\overline{f}([a^+]) \sqsubseteq \overline{b}$. So by the previous lemma we get $\overline{f}([a^+]), \overline{f}([a^-]) \in \{[b^+], [b^-], \overline{b}\}$. By considering the various combinations of possibilities for these two filters we find that $\overline{f}([a^-]) \vee \overline{f}([a^+]) \in \{[b^+], [b^-], [b], \overline{b}\}$, which proves the statement. □

Theorem 6 *If* $A \in \{[a], [a^+], [a^-], [\infty], [-\infty]\}$ *then:*

$$\overline{f}(A) \sqsubseteq [\infty] \;\&\; f \textit{ continuous on its domain} \Rightarrow \overline{f}(A) = [\infty]$$

$$\operatorname*{Lim}_{x\to A} f(x) \sqsubseteq [\infty] \;\&\; f \textit{ continuous on its domain} \Rightarrow \operatorname*{Lim}_{x\to A} f(x) = [\infty]$$

Proof For $A \in \{[a^+], [a^-], [\infty], [-\infty]\}$ this is proved just like Theorem 4. Similarly for $A = \overline{a}$ this is proved just like Theorem 5 (using that $[\infty] \vee [\infty] = [\infty]$.) □

7 Limits in complex analysis

If the underlying space X is taken to be the complex plane C instead of R, there are a number of interesting filters that have no counterpart over R, and which are used in classical complex analysis. First, of course, there is the filter $[\mathsf{ComplexInfinity}]$, generated by the exteriors of closed disks centered at origin. The next question, one which bothers every beginning student of complex analysis, is this: what is the relation between the two infinities of real analysis and the single infinity of complex analysis? Is there one infinity, or are there three infinities, or what? Note that the filters [∞] and [$-\infty$] are not filters over the complex numbers, since the intervals which compose them are not open in the complex plane. However, there is a 'directional infinity' at each polar angle θ in the complex plane. The filter [DirInfinity] [θ] is generated by open sets which are ϵ-neighborhoods of a ray from $Re^{i\theta}$ to infinity at polar angle θ. Specifically, it is generated by sets of the following form (for θ fixed, and R and ϵ varying):

$$U(R, \theta, \epsilon) = \{z : \exists r \geq R(|z - re^{i\theta}| < \epsilon\}.$$

The directional infinities at angles 0 and π correspond to [∞] and [$-\infty$], but over the complex plane there are an infinity of other such infinities.[7] The question of the relationship of the directional infinities to [ComplexInfinity] can now be precisely and mathematically resolved: each of these infinities refines [ComplexInfinity].

The filter [DirInfinity] [θ] corresponds to limits taken along infinite rays. There are similar filters corresponding to limits taken along a fixed direction to a finite point. In general, the discussion of complex filters is simplified (as is complex analysis generally) by considering the Riemann sphere instead of the complex plane. Given a complex number a, a suitable rotation of the Riemann sphere will carry the compactification point ∞ (a member of the Riemann sphere) onto a, and the filter [ComplexInfinity] will be carried onto the punctured (complex) filter [a], generated by punctured disks about a. The filter [DirInfinity] [θ] is carried onto a similar filter corresponding to radial limits as z approaches a from a fixed direction. When a is a complex number, we denote by $a[\theta]$ the filter corresponding to limits along the line parametrized by $a + te^i\theta$.

There are also filters in between [DirInfinity] [θ] and [ComplexInfinity], corresponding to limit approaches constrained to a range of angles rather than a single angle. For example, a limit as z approaches infinity through the upper half plane, or the first quadrant. Rotating the Riemann sphere, we arrive at similar filters corresponding to limits as z approaches a from inside a specified angle. Limits of this kind play an important role in boundary-value estimates. For example, consider Poisson's problem, which is to find a harmonic function with specified boundary values φ on the unit circle. The solution f is defined by a certain integral, and is easily proved harmonic on the open unit disk. One then has to prove that when a lies on the unit circle, $f(z)$ converges to $\varphi(a)$ as z approaches a from within the unit disk. But to do this, limits through a restricted angular approach are used.

Just as in the real case, the case of a limit value c can correspond to various possible filters. Recall that in the reals we have four possibilities: [a], $\overline{a}$, [$a+^{+}$], and [a^{-}]. In the complex case there are infinitely more possibilities. We can have any of the various filters discussed above as a possible limit value, enabling the limit value to convey quite sophisticated information about the way in which the function approaches the limit. Of course, this same aim is served by big-Oh

[7] If X is a subspace of Y, there is a sense in which filters on X can be enlarged to 'corresponding' filters on Y. The relations between real and complex infinity are an example.

notation, in an even more detailed way, but the use of various types of filters as limit values may also be useful.

8 Limits in other spaces

Limits of sequences and series are often considered, in which one wants to take a limit through integral values, such as in

$$\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \tau$$

(where F_n is the n-th Fibonacci number and τ is the golden ratio.) This corresponds to the case in which the domain space X is the natural numbers N, and the range space Y is the reals. So, we use filters on N under the limit sign, and filters on R for the possible values of such limits. Thus limits of sequences fall under our theory nicely.

Uniform convergence can also be treated, by taking the range space (for example) to be $C([0,1])$, the space of continuous real-valued functions on $[0,1]$. When discussing uniform convergence of a sequence, we would take the domain space to be N. The limit value will refine the punctured filter $[g]$ whenever the limiting sequence does not actually contain the limit function g as a member (from some point on). But one might consider the filters $[g^+]$ and $[g^-]$ generated by 'half-neighborhoods' of functions greater than g (or less than g). These can serve as limit values when the convergence is 'from above' or 'from below'.

If one wants to consider uniform limits of complex-valued functions, then there are filters in this function space generalizing the various filters we considered over the complex numbers.

References

[1] Beeson, M., Design Principles of Mathpert: Software to support education in algebra and calculus, in: Kajler, N. (ed.) *Human Interfaces to Symbolic Computation*, Springer-Verlag, Berlin/ Heidelberg/ New York (1996).

[2] Felscher, Walter., Bolzano, Cauchy, Epsilon, Delta, *American Mathematical Monthly* **107**, No. 9, pp. 844-862, November 2000.

[3] Guinin, D., and Joppin, B., with Aubonnet, B., Bloch, P., Bonvalot, G., Bourhis-Laine, N., Dodard, A., and Houdart, I., *Mathematiques, Classes de Première S et E, Analyse-Algebre*, ABC Editions, Montreuil (1991).

[4] Kuratowski, K., *Topology, Volume 1*, Academic Press, New York/ London (1966).

[5] Sims, B., *Fundamentals of Topology*, MacMillan, New York (1976).