# Higher Variations of Dirichet's Integral and Area

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March 12, 2015

In these notes we take up some calculations of the second, third, and fourth variations of area and Dirichlet's energy. These are useful in investigating whether a particular minimal surface (for which the second variation is zero) does or does not furnish a relative minimum of area. With respect to Dirichlet's energy, we follow Tromba [27], who worked out these formulas in the case of variations in forced Jacobi directions. We work them out more generally for variations in the kernel of the second variation. To be clear: the purpose of these notes is just to check my understanding of Tromba's methods.

## 1 Notation

Conceptually, we follow Tromba in studying spaces of harmonic surfaces (some of which are minimal). To be completely precise one has to distinguish between $X : [0, 2\pi] \to \mathbf{R}$, and the function $\hat{X}$ defined on $S^1$ by $\hat{X}(e^{i\theta}) = X(\theta)$, and the harmonic extension of $\hat{X}$ to the unit disk, also denoted by $\hat{X}$. We will often omit the hat, which simplifies the appearance of formulas and does not lead to confusion.

When we write a tangent vector in the form $k = \phi X_\theta$, technically $\phi : [0, 2\pi] \to \mathbf{R}$ and $k : B \to \mathbf{R}^3$ is the harmonic extension of $\phi(\theta)\hat{X}(e^{i\theta})$. Thus $k_r$ refers to the radial derivative of a harmonic function defined in the disk. When we write $(\phi X_\theta)_r$ we mean the radial derivative of the harmonic extension of the tangent vector $k = \phi X_\theta$. These slight abuses of notation are convenient and simplify many formulas, but one must remember for example not to use the product rule on an expression like $(\phi X_\theta)_r$, which would make no sense since $\phi$ is not defined except on $S^1$.

## 2 The second variation of $E$

The second variation $D^2E(X)$ is a bilinear form on the space of "tangent vectors", which are sufficiently smooth harmonic vectors defined in the closed unit disk (or equivalently, on $S^1$, since the harmonic extension to the disk is unique). The "kernel" is the subspace of tangent vectors $k$ such that $D^2E(X)[k, h] = 0$ for all tangent vectors $h$. This is given by the "kernel equation" $k_z X_z = 0$.

(We use subscripts for differentiation.) This equation is due to Tromba and is derived in several of his papers and books, and also for example in [4].

By definition

$$E(X) = \frac{1}{2}\int_B |\nabla X|^2\,dxdy = \frac{1}{2}\int_0^{2\pi} X\cdot X_r\,d\theta$$

Consider a variation $X(t,\theta) = X_0(\theta + t\phi)$ where $\phi$ may depend on $t$. Then $k = X_t = \phi X_\theta$ and

$$\begin{aligned} E_t(X) &= \frac{\partial E(X)}{\partial t} \\ &= \frac{1}{2}\int (X_{rt}X + X_r X_t)\,d\theta \\ &= \frac{1}{2}\int (X_t X_r + X_r X_t)\,d\theta \\ &= \int kX_r\,d\theta \\ &= \int \phi X_\theta X_r\,d\theta \end{aligned}$$

Evaluating at $t=0$ we get $DE(X)[k]$, which by the fundamental lemma of the calculus of variations is 0 for all $\phi$ if and only if $X_r X_\theta$ is identically zero, i.e. $X$ is minimal. We can write this as a complex integral using $dz = iz\,d\theta$ and $X_\theta = \text{Re}\,(izX_z)$ and $X_r = \text{Re}\,(zX_z)$.We get

$$\begin{aligned} E_t(X) &= \frac{1}{4}\int \phi(izX_z - i\bar{z}\overline{X_z})(zX_z + \bar{z}\overline{X_z})\,d\theta \\ &= \frac{1}{2}\int \text{Re}\,(iz^2X_z^2)\,d\theta \\ &= \frac{1}{2}\text{Re}\int iz^2X_z^2\,d\theta \\ &= \text{Re}\int_{S^1}\phi zX_z^2\,dz \end{aligned}$$

When $t=0$ we can write this as

$$DE(X)[k] = \text{Re}\int_{S^1}\phi zX_z^2\,dz \tag{1}$$

The minimal surface equation is thus $X_z^2 = 0$, which can be seen either directly from $X_rX_\theta = 0$ or by the fundamental lemma of the calculus of variations.

Differentiating again with respect to $t$ we have

$$E_{tt}(X) = 4\,\text{Re}\int zX_{tz}X_z\phi\,dz + 2\,\text{Re}\int zX_z^2\phi_t\,dz \tag{2}$$

Just to review: when $t=0$, the second term vanishes when $t=0$ because $X_z^2=0$ is the minimal surface equation, and the first term vanishes when $t=0$ since $X_{tz} = k_z$ and $k_zX_z = 0$ is the kernel equation.

Converting the kernel equation $k_z X_z = 0$ to a form not involving complex numbers, using $k = \phi X_\theta$ and defining $\tilde{k} = \phi X_r$, we find

$$\begin{aligned} D^2E(X)[k,k] &= \int k(k_r - \tilde{k}_\theta)\,d\theta \\ &= \int k k_r\,d\theta - \int k(\phi X_r)_\theta\,d\theta \end{aligned}$$

More details can be found in [4]

# 3 Another formula for $D^2E(X)$

In this section we present a formula for the second variation of Dirichlet's energy that involves the geodesic curvature of the boundary. The formula in question is due to Böhme and Tromba (independently) and can be found on p. 538 of [10].

We begin with a formula for the geodesic curvature.

**Lemma 1** *Let $u$ be a minimal surface defined in the unit disk. Let $r$ and $\theta$ be polar coordinates. Then the geodesic curvature along curves given by constant $r$ (with positive sign corresponding to curvature vector pointing inwards) is given by*

$$\kappa_g = \frac{X_\theta X_{\theta r}}{X_\theta^2}$$

*Proof.* The proof can be found on p. 157 (formula 49.9a) of [18], in the more general setting of orthogonal coordinates (i.e. $g_{12} = g_{21} = 0$). It is a straightforward calculation (using the Christoffel symbols). The lemma can also be proved using Minding's formula but care must be taken since Minding's formula requires isothermal coordinates, and polar coordinates are not isothermal.

**Theorem 1 (Böhme, Tromba)** *Let $X$ be a minimal surface bounded by the $C^{6,\alpha}$ Jordan curve $\Gamma$, and let $k$ be any tangent vector to $X$. Then*

$$D^2E(X)[k,k] = \int_B |\nabla k|^2\,dx\,dy - \int_{S^1} \kappa_g |k|^2\,d\theta$$

*where $\kappa_g$ is the signed geodesic curvature of $\Gamma$ (relative to $u$), with the sign positive for curvature vector pointing to the interior of $u$.*

*Remark.* Tromba (*op. cit.*) has a positive sign on the second term, corresponding to taking the sign of $\kappa_g$ to be positive for curvature vector pointing to the exterior.

*Proof.* Let $k = \phi X_\theta$, and let $\tilde{k} = \phi X_r$. As discussed above, we have

$$\begin{aligned} D^2E(X)[k,k] &= \int k(k_r - \tilde{k}_\theta)\,d\theta \\ &= \int k k_r\,d\theta - \int k(\phi X_r)_\theta\,d\theta \end{aligned}$$

Integrating $\int_B |\nabla k|^2\,dx\,dy$ by parts, we obtain the first term above, since $\Delta k = 0$. Putting that in, we have

$$\begin{aligned} D^2E(X)[k,k] &= \int_B |\nabla k|^2\,dx\,dy - \int k(\phi X_r)_\theta\,d\theta \\ &= \int_B |\nabla k|^2\,dx\,dy - \int k\phi_\theta X_r + k\phi X_{r\theta}\,d\theta \end{aligned}$$

Since $k = \phi X_\theta$ is a tangent vector, $kX_r = 0$, so we have

$$\begin{aligned} D^2E(X)[k,k] &= \int_B |\nabla k|^2\,dx\,dy - \int k\phi X_{r\theta}\,d\theta \\ &= \int_B |\nabla k|^2\,dx\,dy - \int k\phi X_{r\theta}\,d\theta \end{aligned}$$

Putting in $k = \phi X_\theta$ on the right, we have

$$\begin{aligned} D^2E(X)[k,k] &= \int_B |\nabla k|^2\,dx\,dy - \int \phi^2 X_\theta^2 \frac{X_\theta X_{\theta r}}{X_\theta^2}\,d\theta \\ &= \int_B |\nabla k|^2\,dx\,dy - \int |k|^2 \frac{X_\theta X_{\theta r}}{X_\theta^2}\,d\theta \end{aligned}$$

By Lemma 1, the fraction in the second integrand is $\kappa_g$, with the sign positive for curvature vector pointing to the interior. That completes the proof of the theorem.

*Example.* In order to check that all the signs and constants are correct, we calculate all the terms for a trivial example. Take $X = (x, y, 0)$ to be a flat disk, and $k$ to be a conformal direction, for example $k = \mathrm{Re}\,(izX_z)$, so $D^2E(X)[k] = 0$. Then $X_z = (1/2, -i/2, 0)$ so $izX_z = (iz/2, z/2, 0) = \frac{1}{2}(ix - y, x + iy, 0)$ and $k = \frac{1}{2}(-y, x, 0)$, so $|k|^2 = r^2/4$. We have $\kappa_g = 1$ and $|X_\theta| = r^2$, so $\int_{S_1} \kappa_g |k|^2\,d\theta = 2\pi/4 = \pi/2$. We have $\nabla k_1 = -\frac{1}{2}\nabla y = -\frac{1}{2}(0,1,0)$. We have $\nabla k_2 = \frac{1}{2}\nabla x = \frac{1}{2}(1,0,0)$. So $|\nabla k|^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. Integrating over the unit disk we find $\int_B |\nabla k|^2\,dxdy = \pi/2$. So the formula checks out correctly.

*Remark.* The term $\int_{S^1} |k|\kappa_g\,d\theta$ can be written as $\int \phi^2 \kappa_g W\,d\theta$ where $k = \phi X_\theta$ on $S^1$.

## 4 The third variation of $E$

We now follow Tromba [27] in computing the third variation of $E$, when $X$ depends on a (single) parameter $t$. When $t = 0$ we write $X_0$ for $X$. When we calculated the second variation, we allowed any dependence of the surface $X$ on a parameter $t$, subject only to the condition that when $t = 0$, we had $X_t = k = \phi X_\theta$ for some scalar function $\phi$, i.e., the variation is tangential to the boundary curve when $t = 0$. As it turns out, the third variation is *intrinsic*, in the sense that it also depends only on the tangent vectors when $t = 0$. We consider in more detail what it means for the variation to be intrinsic. We consider variations of the form

$$X(t,\theta) = X_0(\theta + t\phi)$$

where $\phi$ can *a priori* depend on $t$. For the third variation to be intrinsic means that terms in $\phi_t$ and $\phi_{tt}$ drop out when the variation is in a kernel direction, i.e. $k_z X_z = 0$ when $t = 0$, where $k = X_t$. If we have a proof in advance that the third variation is intrinsic, then it suffices to consider variations in which $\phi$ does not depend on $t$ at all. But since the *fourth* variation is not intrinsic, we do not assume that.

We suppose henceforth that $k = X_t$ belongs to the kernel of $D^2E(X_0)$ when $t = 0$. But we do not yet set $t$ to zero. Differentiating (2) again with respect to $t$, we have

$$\begin{aligned} \frac{\partial^3 E}{\partial t^3} &= 4\,\mathrm{Re}\int zX_{tz}^2\phi\,dz + 4\,\mathrm{Re}\int zX_{ttz}X_z\phi\,dz \\ &\quad +8\,\mathrm{Re}\int zX_{tz}X_z\phi_t\,dz + 2\,\mathrm{Re}\int zX_z^2\phi_{tt}\,dz \end{aligned} \tag{3}$$

**Lemma 2** *Suppose $X$ is a family of minimal surfaces depending on a parameter $t$ such that $k = X_t(0)$ is in the kernel of $D^2E(X(0))$ and $X_t = \phi X_\theta$. Then the third variation is given by*

$$\left.\frac{\partial^3 E}{\partial t^3}\right|_{t=0} = 4\,\mathrm{Re}\int zk_z^2\phi\,dz + 4\,\mathrm{Re}\int zX_{ttz}X_z\phi\,dz$$

*When $\phi$ is assumed to be independent of $t$, we have the following form of the third variation, also independent of $t$:*

$$\left.\frac{\partial^3 E}{\partial t^3}\right|_{t=0} = 4\,\mathrm{Re}\int zk_z^2\phi\,dz + 4\,\mathrm{Re}\int z(\phi k_\theta)_z X_z\phi\,dz$$

*Remark.* Compare [27], p. 16, where these formulas are essentially proved under the additional assumption that $k$ is a forced Jacobi field. In that case, $zX_z\phi$ is analytic, so the second term does not appear, and the first term takes another form.

*Proof.* We set $t = 0$ in (3). The last term vanishes because $X_z^2 = 0$ is the minimal surface equation. The next-to-last term vanishes because when $t = 0$, we have $X_{tz}X_z = k_zX_z = 0$, because that is the kernel equation. That leaves

$$\left.\frac{\partial^3 E}{\partial t^3}\right|_{t=0} = 4\,\mathrm{Re}\int zk_z^2\phi\,dz + 4\,\mathrm{Re}\int zX_{ttz}X_z\phi\,dz. \tag{4}$$

Now

$$\begin{aligned} X_t &= \frac{d}{dt}X(\theta + t\phi) \\ &= (\phi + t\phi_t)X_\theta \\ X_{tt} &= (\phi_t + t\phi_{tt})X_\theta + (\phi + t\phi_t)X_{\theta t} \end{aligned}$$

When $t=0$ we have

$$X_{tt} \quad = \quad \phi_t X_\theta + \phi k_\theta$$

Now it is convenient to assume $\phi_t = 0$, in which case we have

$$X_{tt} = \phi k_\theta.$$

We want to differentiate with respect to $z$, which means to differentiate the harmonic extension with respect to $z$. Differentiating with respect to $z$ we have

$$X_{ttz} \quad = \quad (\phi k_\theta)_z$$

Taking the dot product with $u_z$ we have

$$X_{ttz} X_z \quad = \quad (\phi k_\theta)_z X_z$$

Multiplying by $z\phi$ we have

$$z X_{ttz} X_z \phi \quad = \quad z(\phi k_\theta)_z X_z \phi$$

Putting this into (4) we have the formula of the lemma. That completes the proof.

## 5 The third variation is intrinsic

In the calculation above we assumed $\phi_t = 0$, i.e. we considered the third variation only for variations $X(t,\theta) = X_0(\theta + t\phi)$ where $\phi$ does not depend on $t$. There are abstract reasons why, if $\phi$ is allowed to depend on $t$, then we get the same answer for the third variation, i.e. the terms in the $t$ derivatives of $\phi$ contribute nothing. See [10], p. 533, Theorem 3. In this section we verify this result by direct calculation.

In the calculation of the third variation in the preceding section, we do not assume $\phi_t = 0$, then we get an extra term

$$4\,\mathrm{Re} \int z(\phi_t X_\theta)_z X_z \phi \, dz$$

or in the notation of [27],

$$4\,\mathrm{Re} \int w(\phi_t \hat{Z}_\theta)_w \hat{Z}_w \phi \, dw$$

Of course this vanishes if $k$ is a forced Jacobi direction since then $zX_z\phi$ is holomorphic. But if we only assume $k$ is a kernel direction then why does it vanish? The proof uses the kernel equation in the form

$$X_\theta(k_r - \tilde{k}_\theta) = 0 \qquad \text{where } k = \phi X_\theta \text{ and } \tilde{k} = \phi X_r.$$

Here is the proof:

$$\begin{aligned}
\mathrm{Re}\int z(\phi_t X_\theta)_z X_z \phi\, dz &= \mathrm{Re}\int (\phi_t X_\theta)_z (X_r - iX_\theta)\phi\, dz \\
&= \mathrm{Re}\int (\phi_t X_\theta)_z (\tilde{k} - ik)\, dz \\
&= \mathrm{Re}\int ((\phi_t X_\theta)_r - i(\phi_t X_\theta)_\theta)(\tilde{k} - ik)\bar{z}\, dz \\
&= \mathrm{Re}\int ((\phi_t X_\theta)_r - i(\phi_t X_\theta)_\theta)(\tilde{k} - ik) i\, d\theta \\
&= \mathrm{Re}\int ((\phi_t X_\theta)_r - i(\phi_t X_\theta)_\theta)(\tilde{k} - ik) i\, d\theta \\
&= \int (\phi_t X_\theta)_r k\, d\theta + \int (\phi_t X_\theta)_\theta \tilde{k}\, d\theta \\
&= \int (\phi_t X_\theta) k_r\, d\theta - \int (\phi_t X_\theta)\tilde{k}_\theta\, d\theta \\
&= \int \phi_t X_\theta (k_r - \tilde{k}_\theta)\, d\theta
\end{aligned}$$

But the kernel equation can be written $X_\theta(k_r - \tilde{k}_\theta) = 0$. Hence the integrand is zero.

# 6 The fourth variation of $E$

We now compute the fourth variation along a variation in a kernel direction; that is, we assume $k$, the value of $X_t$ when $t = 0$, is in the kernel of $D^2E(X)$. Tromba computes the fourth variation when $k$ is a forced Jacobi direction; here we do not assume that. The fourth variation, unlike the second and third, is not intrinsic. That is, when the variation is given by $X(t,\theta) = X_0(\theta + t\phi)$, the derivative $\phi_t$ and even higher derivatives of $\phi$ with respect to $t$ may be involved in the fourth variation of $E(u)$; it will not depend only on $X_0$ and $k$.

**Lemma 3** *Assume that $k$ is in the kernel of $D^2E(X)$, and moreover the third variation in direction $k$ is also zero. Then (writing $k = X_t$ also when $t \neq 0$) we have*

$$\begin{aligned}
\left.\frac{\partial^4 E}{\partial t^4}\right|_{t=0} &= 8\,\mathrm{Re}\int z k_z k_{zt}\phi\, dz + 4\,\mathrm{Re}\int z k_{ttz} X_z \phi\, dz \\
&\quad + 12\,\mathrm{Re}\int z k_{zt} X_z \phi_t\, dz + 8\,\mathrm{Re}\int z k_z^2 \phi_t\, dz
\end{aligned}$$

*Remark.* Compare with the formula in [27], p. 18, Prop. 2.3, which is valid when $k$ is a forced Jacobi field. Our second term does not occur there, because it vanishes when $k$ is a forced Jacobi field, and the terms in $\phi_t$ can be expressed in terms of $X_{zz}$ in that case.

*Proof.* Differentiating (3) with respect to $t$ we have

$$\begin{aligned}\frac{\partial^4 E}{\partial t^4} &= 4\,\mathrm{Re}\int z(X_{tz}^2\phi)_t\,dz + 4\,\mathrm{Re}\int z(X_{ttz}X_z\phi)_t\,dz \\ &\quad +8\,\mathrm{Re}\int z(X_{tz}X_z\phi_t)_t\,dz + 2\,\mathrm{Re}\int z(X_z^2\phi_{tt})_t\,dz\end{aligned}$$

Applying the product rule for differentiation, and writing $k$ for $X_t$ (even when $t \neq 0$) we have

$$\begin{aligned}\frac{\partial^4 E}{\partial t^4} &= 4\,\mathrm{Re}\int z2k_zk_{zt}\phi + zk_z^2\phi_t\,dz \\ &\quad +4\,\mathrm{Re}\int zk_{ttz}X_z\phi + zk_{tz}X_{tz}\phi + zk_{tz}X_z\phi_t\,dz \\ &\quad +8\,\mathrm{Re}\int zk_{zt}X_z\phi_t + zk_z^2\phi_t + k_zX_z\phi_{tt}\,dz \\ &\quad +2\,\mathrm{Re}\int z2X_zk_z\phi_{tt} + zX_z^2\phi_{ttt}\,dz\end{aligned}$$

When $t = 0$ we have $k_zX_z = 0$, since $k$ is in the kernel of $D^2E$, and $X_z^2 = 0$ since that is the minimal surface equation. Hence the last integral vanishes at $t = 0$. Similarly the last term in the third integral vanishes. Collecting like terms we have

$$\begin{aligned}\left.\frac{\partial^4 E}{\partial t^4}\right|_{t=0} &= 8\,\mathrm{Re}\int zk_zk_{zt}\phi\,dz + 4\,\mathrm{Re}\int zk_{ttz}X_z\phi\,dz \\ &\quad + 12\,\mathrm{Re}\int zk_{zt}X_z\phi_t\,dz + 8\,\mathrm{Re}\int zk_z^2\phi_t\,dz\end{aligned}$$

That completes the proof of the lemma.

An important observation is that the *fourth variation is not intrinsic*, because it depends not only on the tangent vector $X_t = \psi X_\theta$, but also on $\phi_t$.

## 7 The cokernel equation

Recall the "weak inner product" of two tangent vectors to $X$ is defined by

$$\langle\langle k, h\rangle\rangle = \int_{S^1} k_r h\,ds$$

or, by abuse of notation,

$$\langle\langle k, h\rangle\rangle = \int_0^{2\pi} k_r h\,d\theta.$$

(That is an "abuse of notation" since technically $k$ and $h$ are defined on the closed unit disk $\bar{B}$, not the interval $[0, 2\pi)$.) Recall that the kernel of $D^2E(X)$ is the set of tangent vectors $k$ such that $D^2E(X)[h,k] = 0$ for all $h$. The kernel always contains at least the conformal directions; we say it is "one-dimensional" if it is really four-dimensional. Tromba's way of dealing with the conformal group is to choose a "transverse slice" $\mathcal{W}$ transverse (in the weak inner product sense) to the conformal orbits. Once that is done, if the kernel is one-dimensional, there is just one direction in both the kernel and the tangent space of $W$. We write $J_0$ for this subspace of the kernel, and loosely refer to $J_0$ as "the kernel of $D^2E(X)$" even though technically the kernel also includes the conformal directions. Then the *cokernel* $J_1$ is defined to be the set of tangent vectors lying in the tangent space of $\mathcal{W}$ and (weakly) orthogonal to $J_0$.

Assume that the kernel is one-dimensional, generated by $k = \psi X_\theta$. The "cokernel equation", which says that $h = \phi X_\theta$ is in the cokernel $J_1$, is

$$0 \quad = \quad \int_{S^1} h_r k \, ds = \int_{S^1} h k_r \, ds$$

Putting $h = \phi X_\theta$ and $k = \psi X_\theta$ the cokernel equation becomes

$$0 \quad = \quad \int_0^{2\pi} \phi X_\theta (\psi X_\theta)_r \, d\theta$$

The kernel equation tells us

$$X_\theta (\psi X_\theta)_r = X_\theta (\psi X_r)_\theta$$

so the cokernel equation can be written

$$0 \quad = \quad \int_0^{2\pi} \phi X_\theta (\psi X_r)_\theta \, d\theta$$

Integrating by parts we have

$$\begin{aligned} 0 \quad &= \quad -\int_0^{2\pi} (\phi X_\theta)_\theta \psi X_r \, d\theta \\ &= \quad -\int_0^{2\pi} \phi X_{\theta\theta} \psi X_r + \phi_\theta X_\theta \psi X_r \, d\theta \end{aligned}$$

Since $X_\theta X_r = 0$ the second term vanishes:

$$0 \quad = \quad -\int_0^{2\pi} \phi X_{\theta\theta} \psi X_r \, d\theta$$

# 8 The orthogonal path and the natural path

If $X$ is any minimal surface, we can consider a "path" through $X$ to be a family of harmonic surfaces $\tilde{X}(t)$ defined for $t$ belonging to some interval containing 0, such that $X(0) = X$. Thus technically $X(0)$ is a function from $[0, 2\pi)$ to $\mathbf{R}^3$; we write $X(t,\theta)$ for $X(t)(\theta)$. Unless we are explicitly also considering variations of the boundary curve, it is assumed that $\tilde{X}$ is bounded by the same curve as $X$, so $\tilde{X}(\theta) = X(\gamma(\theta))$ for some function $\gamma$. In that case, the partial derivative $\tilde{X}_t$ is a tangent vector to $\tilde{X}$. We can compute the variations of $E(X)$ by computing the derivatives of $E(X(t))$ and evaluating them at $t = 0$. Such variations are called "intrinsic" if they only depend on the tangent vector $X_t$. We have seen that the second variation is intrinsic, and the third variation is intrinsic on the kernel of the second variation.

The fourth derivative is not intrinsic, as we shall see below. In other words, it may depend on the particular path. When it comes to actual computation in a particular case (such as for example Enneper's surface), we want to choose a path along which it is comparatively easy to compute the variations of $E$. On the other hand, to draw conclusions from those computations, not just any path will do; so we need to make sure that the path we use for the computations is good enough to support our conclusions. Before proceeding further, we give an example in a finite-dimensional situation. Consider a function $f(x, y)$ defined on a neighborhood of the origin in $\mathbf{R}^2$, with $f(0,0) = 0$ and $\nabla f(0,0) = 0$, and suppose $f_{xx}(0,0) = c > 0$ but $f_{yy} = 0$ and $f_{xy} = 0$. Then the "kernel direction" is $(0,1)$. Suppose further that the third derivatives of $f$ all vanish at the origin. Now consider a path $\gamma$ through the origin of $\mathbf{R}^2$, defined on an interval containing 0, with $\gamma(0) = (0,0)$, and suppose that the fourth derivative of $f$ along that path is positive at the origin:

$$\frac{\partial^4}{\partial t^4} f(\gamma(t)) \bigg|_{t=0} > 0.$$

Can we conclude that $f$ has a relative minimum at $(0,0)$?

No, we cannot. Consider this example:

$$f(x, y) = x^2 - y^6$$

$$\gamma(t) = (t^2, t)$$

Then $\gamma_t(0,0) = (0,1)$ is in the kernel direction, and $f(\gamma(t)) = t^4 - t^6$ has fourth derivative positive at the origin, but $f$ does not have a relative minimum.

The reason that this fourth-derivative test failed to detect that $f$ does not have a relative minimum is that we computed the fourth derivative on "the wrong path." Had we used the path $\gamma(t) = (0,t)$, we would have gotten "the right answer"; the fourth derivative along *that* path is negative. The fourth derivative of $f$ is not intrinsic: it depends on more than just the tangent vector $\gamma_t(0)$. In particular, the problem here is that $\gamma$ has too large a component in the cokernel direction $(1,0)$, which allows $f(x, y)$ along $\gamma$ to pick up a quadratic term from $x$, masking the negative fourth derivative in the $y$ direction.

Something similar happens in the infinite-dimensional situation of the function $E$ defined on the space of $H^2$ harmonic surfaces with a given boundary. Suppose $X$ is a minimal surface, and suppose that $D^2E(X)$ has a one-dimensional kernel (aside from the conformal directions). Let $k = \psi X_\theta$ be a generator of the kernel. (Technically, we suppose $\mathcal{W}$ is a slice through $X$ transverse to the conformal orbits and that $k$ is tangent to $\mathcal{W}$ at $X$.) Then two paths of interest are defined as follows:

**Definition 1** *The* **natural path** *is defined by*

$$\tilde{X}(t,\theta) = X(\theta + t\psi).$$

*The (or for now "an")* **orthogonal path** *is the path such that for each sufficiently small $t$,*

$$DE(\tilde{X})[h] = 0$$

*for all tangent vectors $h = \psi \tilde{X}_\theta$, where $\psi X_\theta$ is in the cokernel of $D^2E(X)$.*

In the orthogonal path, $\tilde{X}$ is "almost minimal" in the sense that its first variation is zero in directions orthogonal to the kernel. Of course that statement is not quite an accurate translation of the definition, since "the kernel" exists in the tangent space of $\tilde{X}$ only when $t = 0$.

In the two-dimensional example, the condition for the orthogonal path would be that $f_x(\gamma(t)) = 0$ for all (sufficiently small) $t$. Since $f_x(x,y) = 2x$, that implies that the orthogonal path is $\gamma(t) = (0,t)$–the one on which we "got the right answer." Similarly, one can show in the infinite-dimensional context that one "gets the right answer" on an orthogonal path. But one also has to show that an orthogonal path exists (and by the way it is unique). We will discuss that issue in the next section.

The natural path is the path along which we wish to compute variations in specific examples. But it turns out that the natural path is in general not orthogonal. This apparent difficulty is reconciled by showing that the natural path is "close enough to orthogonal". Let me elucidate this concept. Consider a path $\tilde{X}$ through $X$, defined for $t$ in some interval $I$ about the origin. Since $\tilde{X}$ is bounded by the same curve as $X$, there is some function $\chi : I \times [0, 2\pi] \to \mathbf{R}$ such that

$$\tilde{X}(t,\theta) = X(\chi(t,\theta))$$

Consider the tangent vector $\tilde{X}_t = \chi_t \tilde{X}_\theta$. When $t = 0$ we can decompose $k = \tilde{X}_t(0)$ into its components $\psi X_\theta$ in the kernel and $\phi X_\theta$ in the cokernel. Then when $t = 0$ we have

$$\chi_t = \psi + \phi$$

For $t \neq 0$ we define $\phi(t,\theta) = \chi_t(t,\theta) - \psi(\theta)$, so for all $t$ we have

$$\chi_t = \psi + \phi$$

where $\psi$ does not depend on $t$, but $\phi$ does depend on $t$.

**Definition 2** *The path $\tilde{X} = X \circ \chi(t)$ is* **almost orthogonal** *if $\phi_t = 0$ when $t = 0$. That is, $\chi_{tt}$ vanishes when $t = 0$.*

**Lemma 4** *Let $\tilde{X}$ be any (sufficiently smooth) almost orthogonal path through $X$. Then the fourth variation of $E$ along the path $\tilde{X}$ is the same as along an orthogonal path. The natural path is almost orthogonal; hence the fourth variation is the same on the natural path as on any orthogonal path.*

*Proof.* Let $\tilde{X}(t,\theta) = X(\chi(t,\theta))$ be almost orthogonal. That means $\chi_{tt} = 0$ when $t = 0$. Define $\phi(t,\theta) = (\chi(t,\theta) - \theta)/t$, so that

$$\chi(t,\theta) = \theta + t\phi(t,\theta).$$

Then

$$\begin{aligned}
\chi_t &= \phi + t\phi_t \\
\chi_{tt} &= \phi_t + t\phi_{tt} + \phi_t \\
&= 2\phi_t + t\phi_{tt} \\
\chi_{tt}\Big|_{t=0} &= 2\phi_t\Big|_{t=0} \\
&= 0 \qquad \text{since } \tilde{X} \text{ is almost orthogonal}
\end{aligned}$$

Hence $\phi_t = 0$ when $t = 0$. But the formula for the fourth variation of $E(\tilde{X})$ shows that the fourth variation depends only on $\phi_t$ at $t = 0$. Hence we get the same value for any almost orthogonal path. In particular any orthogonal path is almost orthogonal, and the natural path is almost orthogonal since along the natural path $\phi_t$ is zero. That completes the proof.

One can show that the natural path is *not* in general orthogonal, although it is almost orthogonal. Here is one way to understand the situation. Consider a two-parameter family

$$\tilde{X}(t,s,\theta) = X(\theta + t\psi(\theta) + s\phi(t,\theta))$$

where $\psi X_\theta$ is in the kernel and $\phi(0)X_\theta$ is in the cokernel. Then $E(\tilde{X})$ can be expanded in a Taylor-MacLaurin series in $s$, whose coefficients are functions of $t$. For an orthogonal path, the coefficient of $s$ must be identically zero in $t$, as

$$DE(\tilde{X})[\phi(t)\tilde{X}_\theta]\Big|_{s=0}$$

must be zero for each $t$, by the definition of orthogonal, but that is the coefficient of the $s$ term. The coefficient of the $st$ term is zero because $k = \psi X_\theta$ is in the kernel. The condition that $\tilde{X}$ be almost orthogonal amounts to requiring that the $st^2$ term is zero. In other words, the coefficient of $s$ is $O(t^2)$. This is exactly what is required to prevent $E(\tilde{X})$ from picking up a quadratic term from the cokernel directions that would swamp the fourth derivative in the kernel directions.

# 9 The gradient of Dirichlet's energy

Let $X$ be a harmonic surface, or technically, its restriction to $S^1$, so that the actual surface is the harmonic extension $\hat{X}$. The space of "tangent vectors" $k$ (vectors of the form $\phi X_\theta$) admits an inner product

$$\langle\langle k,h\rangle\rangle := \int_0^{2\pi} k_r h \, d\theta.$$

It is written with two angle brackets to avoid confusion with the ordinary dot product. Note that by Green's theorem we also have

$$\langle\langle k, h\rangle\rangle = \int_B \nabla\hat{k}\cdot\nabla\hat{h}\,dxdy$$

Tromba proved the existence of a "gradient", that is, a vector field $W = W(X)$ such that

$$DE(X)[k] = \langle\langle W, k\rangle\rangle = \int_0^{2\pi} W_r k\,d\theta. \tag{5}$$

The vector field $W$ is a tangent vector satisfying the equation

$$W_r = X_r$$

where $W_r$ is the radial derivative of the harmonic extension $\hat{W}$ of $W$, and similarly $X_r = \hat{X}_r$. The existence of $W$ is proved by finding $W$ as the minimizer of the functional $\Phi$ defined by

$$\Phi(k) = \int_B |\nabla\hat{X} - \nabla\hat{k}|^2\,dxdy$$

in the Sobolev space $H_2^1(B, R^3)$. Tromba proves (see e.g. [11], pp. 406ff) that $W$ exists, is in $H^2$, and satisfies the equation $W_r = X_r$.

**Theorem 2 (Tromba)** $W(X) = 0$ *if and only if* $\hat{X}$ *is a minimal surface, and* $W$ *is the gradient of* $E$ *in the sense that*

$$DE(X)[k] = \langle\langle W, k\rangle\rangle = \int_0^{2\pi} W_r k\,d\theta.$$

*Proof.* First we prove the gradient equation. Let $W = W(X)$ be the vector field whose construction was described above, so $W_r = X_r$. Let $k = \phi X_\theta$ and $\tilde{X}(\theta) = X(\theta + t\phi)$. Then we have

$$\begin{aligned}
DE(X)[k] &= \frac{\partial}{\partial t}E(\tilde{X})\ \Big|_{t=0} \\
&= \frac{\partial}{\partial t}\frac{1}{2}\int_0^{2\pi}\tilde{X}_r\tilde{X}\,d\theta\ \Big|_{t=0} \\
&= \frac{1}{2}\int_0^{2\pi}\tilde{X}_{rt}\tilde{X} + \tilde{X}_r\tilde{X}_t\,d\theta\ \Big|_{t=0} \\
&= \frac{1}{2}\int_0^{2\pi}\tilde{X}_{rt}\tilde{X}\,d\theta\ \Big|_{t=0} + \frac{1}{2}\int_0^{2\pi}\tilde{X}_r\tilde{X}_t\,d\theta\ \Big|_{t=0} \\
&= \frac{1}{2}\int_0^{2\pi}\tilde{X}_t\tilde{X}_r\,d\theta\ \Big|_{t=0} + \frac{1}{2}\int_0^{2\pi}\tilde{X}_r\tilde{X}_t\,d\theta\ \Big|_{t=0} \\
&= \int_0^{2\pi}\tilde{X}_r\tilde{X}_t\,d\theta\ \Big|_{t=0} \\
&= \int_0^{2\pi} W_r k\,d\theta
\end{aligned}$$

as claimed in the theorem.

If $W = 0$, then by the gradient equation $X$ is a minimal surface. Conversely, if $X$ is a minimal surface then by (5) $\int_0^{2\pi} W_r k \, d\theta = 0$ for all tangent vectors $k$. Then for all $C^2$ functions $\phi$ on $S^1$ we have

$$\int_0 2\pi W_r \phi X_\theta \, d\theta = 0$$

whence (by the fundamental lemma of the calculus of variations) $W_r$ is identically zero. But then $\hat{W}_r = 0$ so $W$ is constant. But the only constant tangent vector is 0, so $W = 0$. That completes the first proof.

# 10 The flow of the gradient of $E$

**Theorem 3** *Let $X$ be an immersed minimal surface bounded by $\Gamma$. Suppose the kernel of $D^2E(X)$ has dimension $k$ (not counting the conformal directions). Then the flow lines of $W$ near $X$ consist of $2k$ analytic arcs $\beta(t)$. That is, there exist $k$ analytic maps $\beta$ such that for each $t$ in some interval containing 0, $\beta(t)$ is a harmonic surface bounded by $\Gamma$, $\beta(0) = X$, and for $t > 0$ we have*

$$\beta'(t) = W(X(\beta(t))).$$

*Moreover, the vectors $\beta'(0)$ are a basis for the kernel of $D^2E(X)$.*

*Proof.* Finish this.

# 11 Sufficient conditions for a relative minimum of Dirichlet's energy

In [**?**][1] Tromba used the gradient vector field $W$ discussed above to prove a version of the Morse lemma for Dirichlet's energy. The Morse lemma has the following immediate consequence:

**Theorem 4 (Tromba)** *Let $X$ be a minimal surface with positive definite second variation of Dirichlet's energy. Then $X$ is a relative minimum of Dirichlet's energy.*

Although this theorem is intuitively very appealing, it seems to require the Morse lemma to prove; in general if all we know is that the function $f$ has positive definite second variation at $X$, it does not follow that $X$ is a relative minimum of $f$. For that we need $D^2f(X)[k,k]$ to be bounded below by some positive constant. There is a problem, due to the fact that $D^2E(X)$ always has a (trivial) kernel: the three directions induced by the conformal group. In [11], §6.5, this problem is solved by taking a "slice" transverse to the orbits of the conformal group, but not without many technical difficulties.

[1] This work appears again in [11], §6.5, cf. especially Theorem 1, p. 425.

Next we ask, what happens if $D^2E(X)$ does have a non-trivial kernel? Then the instead of the generalized Morse lemma, we need the generalized Gromoll-Meyer splitting lemma, Theorem 2, p. 438 of [11]. In that theorem, we break the tangent space of $X$ into three pieces, orthogonal with respect to the inner product $\langle\langle h, k\rangle\rangle$: the conformal directions, the non-trivial part $J_0$ of the kernel of $D^2E(X)$, and the orthogonal complement $J_1$ of the other two pieces.

**Theorem 5 (Gromoll-Meyer-Tromba)** *Let $X$ be a minimal surface. Then with notation as above, there is a local diffeomorphism $\Phi$ defined on a neighborhood of 0 in $J_1 \times J_0$, and range including a neighborhood of $X$ in a slice of the space of harmonic surfaces transverse to the conformal orbits, and a function $h : J_0 \to J_1$, such that for $\ell$ in $J_1$ and $k$ in $J_0$ we have*

$$E(\Phi(\ell, k)) = \frac{1}{2}D^2\bar{E}(0)[\ell, \ell] + \bar{E}(h(k), k)$$

*where $\bar{E}(\ell, k) = E(\Phi(\ell, k))$ by definition.*

*Proof.* See [11], p. 438.

The diffeomorphism $\Phi$ in the theorem can be described explicitly. Let $k = \psi X_\theta$ and $h = \chi X_\theta$ be two tangent vectors to $X$, with $k$ in $J_0$ and $h$ in $J_1$. Then

$$\Phi(h, k)(\theta) = X(\theta + \psi + \chi)$$

Given a "variation" or "path" $\tilde{X}(t)$ with $\tilde{X}(0) = X$, we can calculate the third and fourth variations of Dirichlet's energy along that path, namely the third and fourth derivatives of $E(\tilde{X}(t))$ with respect to $t$. The fourth derivative is not intrinsic, as we have seen; it may depend on the particular path. One path of great interest is the "orthogonal" path defined by

$$DE(\tilde{X})[\chi\tilde{X}_\theta] = 0 \qquad \text{for all } t$$

This is the path defined by $\ell = h(k)$ in the Gromoll-Meyer-Tromba theorem, as we shall soon prove.

The actual definition of $\Phi$ is constructed so that for each $k$, $(h(k), k)$ is a critical point of the function $\bar{E}(\ell, k) := E(\Phi(\ell, k))$. In other words, $\Phi(\ell, k)$ is "almost minimal" in that the first variation of area is zero in all directions $\chi\tilde{X}_\theta$ with $\ell = \chi X$ in the cokernel $J_1$. (See the bottom of p. 438 *op. cit.*) That is, the path $\tilde{X}$ satisfies

$$DE(\tilde{X})[\phi\tilde{X}_\theta] = 0$$

for all $\phi$ such that $\phi X_\theta$ is in the cokernel $J_1$.

We now are in a position to prove the main result of this section:

**Theorem 6 (Conditions for a minimum)** *Suppose $X_0$ is an immersed minimal surface whose second variation has a one-dimensional kernel (aside from the conformal directions). Suppose $X(t, \theta)$ is a one-parameter family such that $k = X_t$ lies in that kernel when $t = 0$. Suppose that the third and fourth orthogonal variations of $E(X)$ are zero and positive, respectively. Finally, suppose that $\tilde{X}$ is an almost-orthogonal variation (for example the natural variation). Then $X_0$ is a relative minimum of $E$.*

*Remark.* The important point here is that, even though the fourth variation is not intrinsic (there are terms involving $\phi_t$ in the fourth variation), we just need to calculate the fourth variation along *one particular* path $X(t,\theta)$. As discussed above, we do not get to choose the path arbitrarily; but as long as we choose an almost orthogonal path, all will be well. In particular, we are allowed to compute along the natural path $\tilde{X}(t,\theta) = X(\theta + t\psi)$, where $k = \psi X_\theta$ is in the kernel.

*Proof.* Let $\Phi$ be as in the preceding theorem, and let $\Phi^{-1}$ be its inverse, defined on the slice $\mathcal{W}$ transverse to the conformal orbits. Let $\tilde{X}$ be the orthogonal path defined above; we may assume that $\tilde{X}$ lies in the slice $\mathcal{W}$ transverse to the conformal orbits. Then define

$$(\tilde{x}, \tilde{y}) = \Phi^{-1}\tilde{X}.$$

(So $\tilde{x}$ and $\tilde{y}$ are tangent vectors belong to the cokernel $J_1$ and the kernel $J_0$ respectively.) Then $\tilde{x} = h(\tilde{y}) = 0$ According to the theorem, we have

$$E(\tilde{X}) = \frac{1}{2}D^2\bar{E}(0)[\tilde{x},\tilde{x}] + \bar{E}(h(\tilde{y}),\tilde{y})$$

and along the orthogonal path we have $\tilde{x} = h(y) = 0$, so the

$$\begin{aligned} E(\tilde{X}) &= \frac{1}{2}D^2\bar{E}(0)[0,0] + \bar{E}(0,\tilde{y}) \\ &= \bar{E}(0,\tilde{y}) \end{aligned}$$

By hypothesis, we also have

$$E(\tilde{X}) = \bar{E}(0,0) + ct^4 + O(t^5)$$

for some $c > 0$. Hence

$$\bar{E}(0,\tilde{y}) = ct^4 + O(t^5)$$

Returning to the Gromoll-Meyer-Tromba formula, we have since $h(y) = 0$, for $\tilde{X}$ near $X$,

$$\begin{aligned} E(\tilde{X}) &= E(\tilde{x},\tilde{y}) \\ &= E(\Phi(\tilde{x},\tilde{y})) \\ &= D^2\bar{E}(0,0)[\tilde{x},\tilde{y}] + \bar{E}(0,y) \\ &= D^2\bar{E}(0,0)[\tilde{x},\tilde{y}] + \bar{E}(0,0) + ct^4 + O(t^5) \\ &> \bar{E}(0,0) \end{aligned}$$

since $c > 0$ and the second variation is positive definite on the cokernel. Hence $E(\tilde{X}) > E(X)$. Hence $X$ is a relative minimum of Dirichlet's energy. That completes the proof.

*Remark.* We did not need to know that the second variation is uniformly bounded away from zero on the cokernel. It is in fact so bounded, in terms of the second eigenvalue of the eigenvalue problem associated with $D^2A(X)$.

*Example.* The theorem can be applied to Enneper's surface $X$ defined in the unit disk. The formulas for the third and fourth variation above can be used to show that the hypotheses are satisfied; hence $X$ is a relative minimum. Of course, this was known already by Ruchert's uniqueness theorem [22], also presented in [19], p. 437, but it is nice to have another proof.

# 12 Bifurcations of a stable immersed minimal surface

Consider Enneper's surface (bounded by Enneper's wire $\Gamma$ with $R = 1$). When $R$ is increased to be slightly more than one, there occurs a "trifurcation"; in addition to Enneper's surface (which is unstable for $R > 1$) there are two "new" relative minima of area. Adding one more parameter, thus creating a two-dimensional family of nearby Jordan curves, at least sometimes leads to the "cusp catastrophe", as shown in [**?**]. What will happen if we distort $\Gamma$ in some other way? Could we get five new minimal surfaces? Twenty-seven? No. We shall see below that you can get two new relative minima, and no more.

In this section we suppose:

(i) $\Gamma$ is a Jordan curve, and $X$ is a minimal surface bounded by $\Gamma$, and

(ii) $D^2E(X)$ has a one-dimensional kernel (not counting the conformal directions), and

(iii) $\Gamma(\alpha)$ is a one-parameter family of Jordan curves depending on a real parameter $\alpha$, equal to $\Gamma$ when $\alpha = 0$.

The key to analyzing this situation is a generalization of the Gromoll-Meyer-Tromba theorem. It essentially says that the minimal surface $X$ is part of a family $X(\alpha)$ of harmonic surfaces bounded by $\Gamma(\alpha)$; we cannot claim that $X(\alpha)$ is minimal, but it is has zero first variation in all except possibly the "kernel direction" $\phi X_\theta$, that lies in the kernel of the second variation when $\alpha = 0$.

**Theorem 7 (Gromoll-Meyer-Tromba)** *With notation and assumptions as above, let $J_0$ be the kernel of $D^2E(X)$, and $J_1$ its weak orthogonal complement. Then there is an interval $I$ containing 0 (of $\alpha$-values), and a local diffeomorphism $\Phi$ defined on a neighborhood of 0 in $J_1 \times J_0 \times I$, and range including a neighborhood of $X$ in the slice of the space of harmonic surfaces $\mathcal{W}(\alpha)$ transverse to the conformal orbits, such that for $x$ in $J_1$ and $y$ in $J_0$ we have*

$$E(\Phi(x, y, \alpha)) = \frac{1}{2} D^2\bar{E}(0, 0, \alpha)[x, x] + \bar{E}(h(y), y, \alpha)$$

*where $\bar{E}(x, y, \alpha) = E(\Phi(x, y, \alpha))$ by definition.*

*Proof.* See [11], §6.5.

**Theorem 8** *Let $X$ be an immersed minimal surface bounded by a Jordan curve $\Gamma$. Suppose $D^2E(X)$ has a one-dimensional kernel (aside from the conformal directions). Suppose also that the orthogonal fourth variation of $E$ is positive at $X$. Let $\epsilon > 0$ and $\gamma > 0$ be given. Then any Jordan curve sufficiently close to $\Gamma$ (in $C^{6,\gamma}$ norm) bounds at most three minimal surfaces lying within $\epsilon$ of $X$ (in $C^6$ norm). Moreover, if it bounds three, two are relative minima of area and one is unstable; if it bounds two, one is a relative minimum and one is unstable; and if it bounds only one, that one is a relative minimum.*

*Proof.* Let $\Gamma(\alpha)$ be a path in the space of Jordan curves such that $\Gamma(0)$ is the given curve $\Gamma$ (so $\alpha$ is a real number, belonging to some interval $I$ containing the origin, and $\Gamma(0)$ is the boundary of $X$.) According to the Gromoll-Meyer-Tromba theorem, for small $\alpha$ we have for $y = 0$

$$vE(\Phi(x, y, \alpha)) = \frac{1}{2}D^2\bar{E}(0, 0, \alpha)[x, x] + \bar{E}(h(y), y, \alpha)$$

and by hypothesis $E(0, y, 0) = ct^4 + O(t^5)$ for some $c > 0$, when $y = tk = t\psi X_\theta$.

When $\alpha$ is small but not zero, we consider the function

$$f(t) := E(\Phi(0, t\psi\tilde{X}_\theta, \alpha)).$$

When $\alpha = 0$ it has the form $ct^4 + O(t^5)$. When $\alpha$ is small but not zero, it can pick up terms in lower powers of $t$. If it does not pick up such terms, then there is still just one minimum near $t = 0$. If it does pick up such terms, then considering possible forms of cubic equations near the origin, we see that there are either two minima and a relative maximum (corresponding to an unstable minimal surface), or one relative minimum and an inflection point (which also corresponds to an unstable minimal surface), or possibly still just one relative minimum. These are all minimal surfaces, because their first variation in the cokernel directions is still zero, by the Gromoll-Meyer-Tromba formula. That completes the proof of the theorem.

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