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The following invited lectures were presented at the Colloquium, but the text is not reproduced in the present volume.

A. DODD, The core model

J. FENSTAD, General recursive theory: a unified view

A. JOYAL, Sheaves in constructive mathematics

G. KREISEL, Gödel's later views

S. KRIPKE, Non-standard models and Gödel's theorem

M. MORLEY, Theories with few models

J. PARIS, Some problems in arithmetic

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CONTINUITY IN INTUITIONISTIC SET THEORIES

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I dedicate this paper to the memory of Karel de Leeuw. His real memorial lies in his influence on the lives of his students and friends. Let us not mourn but carry on his work.

The work presented here represents part of a continuing study of the connections between the two basic concepts <u>continuity</u> and <u>constructivity</u>. Both concepts have their roots deep in the soil of mathematical practice, and for as long a time as they have been considered, it has been felt that there is some deep relationship between the two. The histories of each of these concepts show striking parallels, in that (i) tremendous energies were devoted to the task of making an intuitive concept mathematically precise, and (ii) there was considerable controversy concerning the meaning and value of the resulting new mathematics. (Specifically, the ε-δ definition of continuity, and the Brouwer-Bishop-Heyting development of constructive mathematics.) Evidence that a connection between constructivity and continuity has long been perceived may be found, for example, in Hadamard's criteria for a "well-posed problem" in differential equations. One of these criteria is that the solution should depend continuously on the parameters of the problem. The rationale for this criterion was that if the problem corresponds to a physical situation, one is supposed to be able to compute the solution from (measured) approximations to the data. Further evidence lies in the central place accorded to continuity principles in the work of Brouwer. We may even go so far as to say that it was Brouwer's efforts to connect continuity and constructivity that led him in his development of intuitionism (specifically, the theory of free choice sequences).

What we believe we have now accomplished is this: we have done for the <u>connection between continuity and constructivity</u> what ε and δ did for continuity. More precisely, one can formulate informally the "Principle of Continuity":

> <u>If a problem is constructively solved, then the solution depends continuously on its parameters.</u>

Our claim is to have made this principle precise, in its most general form, by specifying

(i) exactly which "problems" it applies to.
(ii) what "constructively solved" means.
(iii) exactly what "depends continuously on parameters" means.

We consider first point (ii). We find ourselves in a unique historical position, in that for the first time formal systems are available in which the bulk of constructive mathematics can be readily formalized. Thus one way of making the Principle of Continuity precise is as a derived rule of inference: if the problem can be proved in a constructive formal system T to have a solution, then the solution depends continuously on the parameters of the problem. And, we may add, provably so in the theory T. (For those unfamiliar with the present "historical position": in 1967, Bishop published his book, demonstrating that the scope of constructive mathematics is vastly wider than was previously suspected, and also demonstrating vividly the clarity and power of the constructive approach. This work stimulated the development of formal systems by Feferman, Friedman, Myhill, and Martin-Löf, which are intended to be suitable for formalizing Bishop's work.)

Next we discuss (i). It seems that the "correct" answer here is that the principle applies to problems of the form, given a in X, find b in Y such that $\langle a,b\rangle$ is in P, where X and Y are complete separable metric spaces, and P is any subset of their Cartesian product such that for each a in X, $\{b \in Y : \langle a,b\rangle \in P\}$ is closed in Y. (There are some variants on this form and much more discussion in [B1]. It may be that the condition on X may permit some generalization, so we should thus qualify our claim to the "most general form".)

Finally, the answer to (iii) is a bit subtle. One cannot require that b be found by a continuous function of a, defined on X, as the example $\forall a \in R \exists b \in N\ a \leq b$ shows, where N is the integers. This example might seem to "sink the whole ship" of the Principle of Continuity, until one sees that what should really be formulated is a Principle of Local Continuity: We should require that for each a in X, we can find a stable solution b in Y, where b is called stable if $\forall \varepsilon > 0 \exists \delta > 0$ such that to any c within δ of a, there corresponds a solution d within ε of b. As a matter of fact, this definition of stability is a common one in mathematical practice (for example, see [T]). To require that b should be given by a continuous function defined on some neighborhood of a is too strong in the case of problems without a unique solution. For example, every complex number has a square root, but there is no continuous square-root function defined in a neighborhood of zero. (Thanks to M. Hyland for showing me this example.)

The proper formulation of the Principle of Local Continuity opens up two distinct lines of research:

(1) A mathematical program, in which one wants to systematize and clarify various stability and continuity results in mathematics, and discover new ones, by the light of the Principle of Continuity, and by making use of the body of already-developed constructive mathematics.

(2) A metamathematical program, in which one wants to clarify the nature of constructive formal systems by considering their properties in the light of the Principle of Continuity.

At the present time, the mathematical program has been carried forward mostly in unpublished work. We may mention, as illustrative examples, the following known theorems which come within the scope of the Principle of Continuity together with constructively proved existence theorems:

(a) The continuous dependence on initial or boundary conditions of the solutions of any differential equation which can be solved by a method of successive approximation (contraction mappings); for example the well-known equation $y' = f(x,y)$, where f is Lipschitz in y, and the initial value of y is the parameter.

(b) The continuous dependence on the domain D in the plane of the eigenvalues of the vibrating-membrane equation $\Delta\phi + \lambda\phi = 0$ in D, $\phi=0$ on the boundary of D.

(c) The continuous dependence on the rectifiable Jordan curve C in R^3 of the infimum of areas of surfaces bounded by C.

In [B2] a new theorem is proved, whose (ordinary mathematical) proof was first discovered by means of the Principle of Continuity. See also [B5] for another example of mathematical work inspired by the Principle of Continuity.

The metamathematical program, on the other hand, is at present nearly complete. Our aim has been to show that various formal systems have various pleasing metamathematical properties related to continuity. These properties fall roughly into two categories: derived rules of inference, and consistency/independence results. It has turned out that the Principle of Local Continuity has surprising and sweeping power to systematize and organize the various continuity properties which have been considered in the past. We may draw evidence for two conclusions from the success of this program:

(1) We have in fact found the right connection between constructivity and continuity.

(2) The formal systems in question are in fact good ones, in some sense, for formalizing constructive mathematics.

(Of course some moderation is called for, especially in relation to (2), since there may be objections to a given system having nothing to do with continuity.)

The metamathematical program outlined above was begun in [B1]. In that paper two things are accomplished: (1) General conditions on a theory T are given, such that if they hold then T is closed under the various derived rules related to continuity, which we shall describe in more detail below. (2) These conditions are verified, and a number of related consistency/independence results are obtained, for the particular formal systems introduced by Feferman [Fe] for constructive mathematics.

In this paper, our purpose is to treat the intuitionistic set theories developed by Friedman and Myhill after the same fashion as we previously treated Feferman's theories. After the work mentioned above, we do not need to consider the continuity rules directly, but only to establish that the necessary metamathematical closure properties (explicit definability etc.) are satisfied. In practice, what this entails is the development of suitable realizability and forcing interpretations for these theories; these interpretations also enable us to establish the related consistency and independence results. Before turning to a discussion of these various theories, we first wish to summarize the metamathematical <u>conclusions</u> of the work, by stating exactly some of the derived rules which are under discussion. A more complete list and exhaustive discussion can be found in [B1]. In this list, X and Y are complete separable metric spaces, and C(X,Y) is the set of continuous functions from X to Y which are uniformly continuous on each compact subset.

(1) (Principle of Continuity). Provably well-defined functions from X to Y can be proved to be in C(X,Y).

(2) (Continuous Choice). If $\forall a \in X\, \exists! b \in Y\, (\langle a,b\rangle \in P)$ is provable, then so is $\exists f \in C(X,Y)\, \forall a \in X\, (\langle a,f(a)\rangle \in P)$.

(3) (Heine-Borel's rule). If a sequence of neighborhoods I_n can be proved to cover a compact space, then for some k, the union of the first k neighborhoods can be proved to cover the space.

(4) (Principle of Local Continuity). Suppose $\forall a \in X\, \exists b \in Y (\langle a,b\rangle \in P)$ is provable, and the hypothesis on P mentioned above is also provable. Then $\forall a \in X\, \exists b \in Y (\langle a,b\rangle \in P \;\&\; b \text{ is stable})$ is provable.

There is also a Principle of Local Uniform Continuity, of which we shall say more in Section 7 below. In [B1] is is shown how the above rules all flow from the Principle of Local Continuity, with Uniform Continuity being used for Heine-Borel's rule.

We now turn to a discussion of the various formal systems to which these results apply, namely, the systems of Feferman and those of Friedman. The systems of

Feferman and Friedman are quite different. Feferman's systems are based on the idea that every object is a construction, and constructions may apply to other constructions, so we have a sort of λ-calculus of constructions; in addition, we have "classifications" (similar to sets) and an ε-relation. We do not, however, have extensionality, as there is no reason to assert it for Feferman's underlying conception. Friedman's systems, on the other hand, are modifications of classical set theory, which do contain extensionality, but are made "constructive" in some sense by weakening the axiom of choice and using intuitionistic logic (for instance, they are consistent with Church's thesis). There has been considerable discussion (in fact "controversy" is not too strong a word) over the relative merits of the two types of systems, and over the question whether they are in accord with constructive mathematics from a foundational point of view. The present contribution to this discussion is that all the systems (except perhaps the weakest) share the same closure properties under rules related to continuity, the corresponding principles of continuity are consistent with very strong intuitionistic set theories.

In this paper, besides developing forcing and realizability for these set theories, we spend considerable effort analyzing the role of the axiom of extensionality. We prove that this axiom can be eliminated from the proofs of theorems mentioning only objects of low type, such as reals or natural numbers. This seems to be necessary from a technical standpoint (or at least the most convenient way to proceed) in order to obtain the explicit definability results we need. However, it is also interesting in its own right, principally because nearly every theorem of mathematical practice can be expressed at low types (since complete separable metric spaces can be regarded as subsets of N^N). Thus extensionality is essentially irrelevant to mathematical practice. (This is not to say that it is irrelevant from the philosophical, foundational viewpoint.) Another interesting thing about this theorem is that it has applications; see [B3] and [B4].

It is a pleasure to have this opportunity to thank those who have contributed to this work, by their interest, by their criticisms, by encouraging me to prove these theorems, and by inviting me to speak at the Colloquium in Mons: H. Barendregt, D. van Dalen, S. Feferman, H. Friedman, and D. Scott. I also would like to mention that the dedication to Karel de Leeuw is especially appropriate, since this paper was written in his house, while his companionship brightened my days.

§1. Description of Some Intuitionistic Set Theories

In this section we describe the principal intuitionistic set theories, which have been invented and studied by Friedman and Myhill. First we describe Friedman's systems. (Precise statements of the axioms will be given below.)

Let ZF^- be Zermelo-Fraenkel set theory, with intuitionistic logic, and with the foundation axiom expressed as (transfinite) induction on $\in$, instead of the usual way. (The usual foundation axiom implies the law of the excluded middle, see [M1].) We cannot add the axiom of choice AC without getting the law of the excluded middle, but we can add (some forms of) dependent choice. The strongest set theory we consider is thus $ZF^- + RDC$ (relativized dependent choice). (Introduced in [Fr1].)

Friedman and Myhill have directed their attention to finding subsystems of $ZF^- + RDC$ which are formally weak and practically strong: that is, which are strong enough to formalize known constructive mathematics (e.g. Bishop's book [B1]) and yet are proof-theoretically weak. There are two principal ideas here: one is to replace the power set axiom by the axiom of exponentiation, which says that A^B exists if A and B are sets. (This was introduced by Myhill in [M1].) The other is to restrict induction to sets instead of formulae, i.e. to consider $0 \in X \;\&\; \forall n(n \in X \rightarrow n+1 \in X) \rightarrow \forall n(n \in X)$ instead of $A(0) \;\&\; \forall n(A(n) \rightarrow A(n+1)) \rightarrow \forall n A(n)$. (Note that classical second-order arithmetic with restricted induction and arithmetic comprehension is a conservative extension of arithmetic.) The use of restricted induction is the germinal idea of Friedman's work. If we use exponentiation instead of power set, and restrict induction, and restrict separation to Δ_0 formulae (no unbounded quantifiers), and add a restricted form of dependent choices, we get Friedman's theory T_1, which he showed has the same strength as arithmetic.

Friedman also studied a variant of T_1 called $\underset{\sim}{B}$, which differs from T_1 in that $\underset{\sim}{B}$ has no foundation axiom, and collection is replaced by Δ_0-abstraction, which says $\{\{u \in x: A(y,u)\} : y \in x\}$ exists, where A is a Δ_0 formula. The point of this is that $\underset{\sim}{B}$ has a model in sets of rank $< \omega + \omega$, and so is easier to justify by some constructive philosophy (see [Fr2, PartI]). $\underset{\sim}{B}$ also has the same strength as arithmetic.

In between $\underset{\sim}{B}$ and $ZF^- + RDC$, Friedman considers several intermediate theories, which all have the full induction schema, and have additional axioms as follows:

T_2 : T_1 + induction + RDC	Z : Zermelo set theory
T_3 : T_2 + transfinite induction	
T_4 : T_3 + full separation.	

Thus $ZF^- + RDC$ is just T_4 + power set.

Myhill's first published intuitionistic set theory CST [M1] is closely related to T_2, as discussed in [Fr 2]. We do not consider CST explicitly.

One feature of all these set theories worth remarking is that they include extensionality. This is one feature which distinguishes them from other formal systems which have been shown adequate for formalizing Bishop's book, such as Feferman's systems. We shall return to this point in §3. We find it necessary (as well as interesting) to consider set theories without extensionality (even if we want results only for extensional theories). We adopt the notation T-ext for the set theory T minus the axiom of extensionality; the proper formulation of these theories requires a little care, and we give a more complete description below. One difference between the extensional and non-extensional theories is that the syntax of the extensional theories is much simpler--we need only the one binary relation of membership. We do not include equality in the extensional theories. On the other hand, we must include equality in the intensional case, as well as some constants and function symbols to be described below.

We use $\langle x,y\rangle$ for the ordered pair, defined in the usual way from unordered pairs. The integers can be developed in set theory in the usual (von Neumann) way. Each formula of arithmetic has a natural translation into set theory.

We now list the axioms we will be considering; we give them first in the form suitable when extensionality is present, i.e. in the form used by Friedman. Afterwards we shall indicate the modifications which are necessary when extensionality is dropped.

A. (extensionality) $x = y \leftrightarrow \forall a(x \in a \leftrightarrow y \in a)$

B. (pairing) $\exists x \forall y(y \in x \leftrightarrow y = a \vee y = b)$

C. (infinity) $\exists x(0 \in x \;\&\; \forall y(y \in x \rightarrow y \cup \{y\} \in x) \;\&\; \forall z(0 \in z \;\&\; \forall y(y \in z \rightarrow y \cup \{y\} \in z) \rightarrow x \subseteq z))$

D. (union) $\exists x \forall y(y \in x \leftrightarrow \exists z(y \in z \;\&\; z \in a))$

E. Δ_0-separation) $\exists x \forall y(y \in x \leftrightarrow (y \in a \;\&\; \emptyset))$ where $\emptyset$ is Δ_0 and x is not free in $\emptyset$

F. (strong collection). $\forall x \in a \exists y\ \phi(x,y) \to \exists z(\forall x \in a \exists y \in z\ \phi(x,y)$ $\& \forall y \in z \exists x \in a\ \phi(x,y))$ (Ordinary collection doesn't have the second clause on z.)

G. (foundation). $(\forall a,b\ ((a \in b\ \&\ b \in x) \to a \in x)$ $\&\ \forall y(y \in x\ \&\ y \subseteq z) \to y \in z)) \to x \subseteq z$, (in other words, transfinite induction on $\in$ with respect to sets only, not formulae.)

H. (exponentiation). $\exists x \forall y(y \in x \leftrightarrow \mathrm{Fcn}(y)\ \&\ \mathrm{Dom}(y)=a\ \&\ \mathrm{Rng}(y) \subseteq b)$

I. (bounded dependent choice). $\forall x \in a \exists y \in a\ Q(x,y) \to$ $\forall x \in a \exists z(\mathrm{Fcn}(z)\ \&\ \mathrm{Dom}(z) = \omega\ \&\ z(0)=x\ \&\ \forall n \in \omega\ Q(z(n),z(n+1))$ $\&\ \mathrm{Rng}(z) \subseteq a)$, for Q a Δ_0 formula.
This much constitutes T_1; note that restricted induction follows from foundation together with our basic axioms.

J. (induction). $A(0)\ \&\ \forall n(A(n) \to A(n')) \to \forall n A(n)$, for all formulae A.

K. (relativized dependent choice RDC). Like axiom I. except that the set a is replaced by an arbitrary formula A, and Q is not required to be Δ_0. This much constitutes T_2.

L. (transfinite induction). $\forall x(\forall y(y \in x \to P(y)) \to P(x)) \to \forall x P(x)$, where y does not appear in $P(x)$. This much constitutes T_3.

M. (separation). $\exists x \forall y(y \in x \leftrightarrow (y \in a\ \&\ Q))$, where x is not free in Q. This much constitutes T_4. To obtain ZF^- + RDC, we add:

N. (power set). $\exists x \forall y(y \subseteq a \to y \in x)$.
The theory $\underset{\sim}{B}$ consists of A-E, G, H and the axiom of "abstraction":

O. (abstraction). $\{\{u \in x: A(u,y)\}: y \in x\}$ exists.

Note that abstraction follows from collection. (If abstraction is formulated as in [Fr2], we need extensionality to deduce it from collection; we shall discuss the non-extensional theories further below.) Thus $\underset{\sim}{B}$ differs from T_1 in that collection is dropped, and a weaker consequence is added back in. It is worth noting that abstraction is restricted to Δ_0-formulae, while collection is not.

Intuitionistic Zermelo set theory Z consists of $\underset{\sim}{B}$ with full separation (so abstraction is unnecessary); dependent choice, induction, and power set. That is, Z differs from ZF^- + RDC in that it does not have foundation or transfinite induction and does not have collection. To list its axioms: A,B,C,D,J,K,M,N.

Myhill has given still another theory in [M2]; this theory has variables for both functions and sets. We do not deal with this theory here, but we expect that it can be handled (as well as $\underset{\sim}{B}$ can be) by using the interpretation in $\underset{\sim}{B}$ given by Myhill.

Now we discuss carefully the formulation of our non-extensional set theories. If T is one of the set theories including the axiom of extensionality, then T-ext is not just T with axiom A deleted. We must also (i) include a system of terms such as $\{x \in a: P(x)\}$ and (ii) modify the exponentiation axiom and abstraction axiom, choosing a form which is equivalent to the above when extensionality is present, but which is non-extensionally correct. Let $\mathrm{Fcn}(f)$ be $\forall x,y,z(\langle x,z\rangle \in f\ \&\ \langle x,y\rangle \in f \to \forall u(u \in y \leftrightarrow u \in z))\ \&\ \forall a \in f \exists b,c(a=\langle b,c\rangle)$. Let $\mathrm{Dom}(f)=a\ \&\ \mathrm{Rng}(f) \subseteq b$ abbreviate $\forall x \in a\ \exists y \in b(\langle x,y\rangle \in f)\ \&\ \forall \langle x,y\rangle \in f(x \in a\ \&\ y \in b)$. Then the exponentiation axiom says $\exists X\ \forall g(\mathrm{Fcn}(g)\ \&\ \mathrm{Dom}(g) = A\ \&\ \mathrm{Rng}(g) \subseteq B \to \exists f \in X\ \forall x \in A(f(x) = g(x))$. Similarly, in the abstraction axiom, we assert $\exists X(\forall y \in a\ \exists w \in X(x \in w \leftrightarrow (x \in a\ \&\ \phi(x,y)))\ \&$ $\forall w \in X\ \exists y \in A(x \in w \leftrightarrow (x \in a\ \&\ \phi(x,y))))$.

We now specify the exact system of terms to be included in our non-extensional set theories; these terms are built up from the following constants and function symbols. We also give the defining axioms for these symbols.

(i) a constant symbol $\emptyset$, and the axiom $\forall x(x \notin \emptyset)$.

(ii) a function symbol $\{\ \}$, and the axiom $z \in \{x,y\} \leftrightarrow z = x \vee z = y$.

(iii) a function symbol for union, and the axiom

$$y \in \bigcup_{z \in a} z \leftrightarrow \exists z \in a(y \in z)$$

Then $a \cup b$ abbreviates $\bigcup_{z \in \{a,b\}} z$.

(iv) a constant symbol ω and axiom

$$\emptyset \in \omega\ \&\ \forall z(z \in \omega \to z \cup \{z\} \in \omega)\ \&$$
$$\forall X(\emptyset \in X\ \&\ \forall z(z \in X \to z \cup \{z\} \in X) \to \omega \subseteq X)$$

(v) for each formula P for which separation is allowed, a function symbol $\{x \in a: P(x,a)\}$ and the obvious axiom.

(vi) Symbols for dependent choice: if P is a formula for which dependent choice is allowed, we have a function symbol i_p with the axiom $\forall x \in \omega \exists! y \in \omega P(x,y)\ \&\ x_0 \in \omega \to i_p(x_0) \in \omega^\omega\ \&\ i_p(x_0)(0) = x_0\ \&$ $\forall n \in \omega\ P(i_p(x_0)(n),\ i_p(x_0)(n+1))$.

Note that we include choice symbols only for functions from ω to ω. Thus at least we have terms for all the primitive recursive functions, so that our non-extensional theories contain arithmetic in a natural sense. Note that there are no terms corresponding to the collection axiom. Generally speaking, it seems that we get several theories of different strength by including or not including constants and functions symbols corresponding to the various axioms. We certainly need to include separation terms in order to achieve the technical results we want; the rest seem to be optional.

The above description requires a little elaboration, since the formula P in a term $\{x \in a: P(x)\}$ may itself contain other terms. One way to make our definitions completely precise is as follows: Start by adding a list f_n of function symbols to the language (for the separation terms) and similar lists for the other types of terms required. Then Gödel number all the formulae of the language, and then write $\{x \in a: P(x,y)\}$ for $f_n(a,y)$, where P has Gödel number n. Of course, we now have <u>more</u> terms than we want, since we only want such terms for certain formulae P. One can either delete the extra terms from the language, or leave them in, but add no axioms about them. To specify which formulae P are allowed, for example in the case of Δ_o-separation, we add a clause to the definition of a Δ_o-formula specifying that if terms $\{x \in b: Q(x)\}$ occur in the component formulae, then Q is already Δ_o, and similarly for abstraction and choice terms occurring in the component formulae. Note that generally when we add more symbols to the language, there are more Δ_o-formulae.

§2. Complete Metric Spaces and some Auxiliary Theories

Since the derived rules which we wish to establish mention metric spaces, we have to discuss the formalization in intuitionistic set theories of the mathematics of complete separable spaces. This is quite straightforward and offers no difficulties. (X,ρ) is a complete separable metric space if it is a metric space, and is has a dense subset, which is the range of a function whose domain is ω, and every Cauchy sequence converges. Using the axioms of $\underset{\sim}{B}$ only, we see that from every Cauchy sequence we can extract a subsequence x_n satisfying $\rho(x_n,x_m) < 1/n + 1/m$. Letting σ be the metric on the integers induced by "pulling back" ρ from the countable dense subset of S, we see that (X,ρ) is isometric to the space of all functions $y \in N^N$ satisfying $\sigma(y_n,y_m) < 1/n + 1/m$ (using the convention $y_n = y(n-1)$ to avoid the problem of subscripts beginning at 1 and functions beginning at 0). Here we follow Bishop in not passing to equivalence classes of such functions, but allowing instead a broader equality relation in the space X; equality of elements of X will not necessarily be set-theoretic equality. It is worth noting that having the axiom of extensionality does not force us to use equivalence classes. Thus any complete separable space can be brought to "standard form" as a set of sequences of integers. The metric on such a space will be $\rho(x,y) = \lim_{n\to\infty} \sigma(x_n,y_n)$.

The "standard form" considered above is not the most useful form for a compact space. For instance, 2^N is most naturally thought of as the space of all y in N^N with $y_n = 0$ or 1. A compact space is one such that for each n, we can find a finite $1/n$-approximation to the space, i.e. a finite set $y_1,\ldots,y_k$ such that each point of the space is within $1/n$ of some y_k. Using bounded dependent choice, we can select a countable base consisting of such points y_j for the various values of n, and associate to each point of the space a sequence of the y_i's such that $\rho(y,y_i) < 1/i$. It follows that every compact space can be brought to standard form as follows: for some non-decreasing sequence of integers M_i, X consists of all y in N^N with $y_n \le M_n$, and the metric has the same form as in the standard form for complete spaces above.

Let T be any one of the theories considered in this paper, and let X be some (provably) complete metric space, in standard form. (To be precise, this means there is a formula Q such that T proves $\exists! X\, Q(X)$ and $Q(X) \to X$ is a complete separable space in standard form.) We shall use X both for the space defined informally by the formula Q, and also to abbreviate formal expressions; thus $y \in X$ means $\forall X(Q(X)) \to y \in X)$. Let b be an (arbitrary but fixed) element of X; we shall have occasion to consider the auxiliary theory Tb, which is formed from T by adding a constant symbol <u>b</u>, the axiom $\underline{b} \in X$, and axioms $\underline{b}(\bar{n}) = \bar{m}$ where $m = b(n)$, and $\bar{n}$ is the numeral for n.

In case X is a compact space in standard form, we shall have occasion to consider another auxiliary theory, Ta. This theory is formed by adding a constant symbol <u>a</u> to T, and the axiom $\underline{a} \in X$, but no other axioms.

It will sometimes be convenient to assume that the metric on the countable base of a space in one of the two standard forms is actually a recursive function. In case X is provably a complete separable space (or a compact space) this can be done without loss of generality, since for the theories we shall consider, a function provably in N^N is (provably) recursive. (See §5.)

If P is a subset of a metric space X in standard form, we say "P is extensional" if $\rho(x,y) = 0$ & $P(x) \to P(y)$. Note that this concept has nothing to do with whether the axiom of extensionality is assumed or not; for instance, as long as we take the reals to be defined by Cauchy sequences instead of equivalence classes, there will be non-extensional sets, in this sense.

§3. The role of extensionality

The intuitionistic set theories propounded by Friedman and Myhill contain the axiom of extensionality, while the theories of Feferman do not. The actual practice of constructive mathematics can be done straightforwardly without any underlying notion of extensionality. Of course, in the practice of mathematics we define various notions of extensional equality; in fact, Bishop takes the view that each set should come equipped with an equivalence relation to be used as an equality relation. These equivalence relations can be used quite straightforwardly without assuming that equivalent objects are equal; for instance, many different Cauchy sequences of rationals determine the same real number. In [Fr2], Friedman goes into some detail as to exactly how to formalize Bishop's book in intuitionistic set theory. Extensionality is hardly made use of; and where it is, it is easily eliminated. Why then include extensionality at all? The answer to this is that Friedman wished to make constructive mathematics formalized in his system look as much like classical mathematics as possible, in order to make it easier for the classical mathematicians to appreciate constructive mathematics.

Be that as it may, in this paper we are trying to obtain derived rules related to continuity for intuitionistic set theories, both with and without extensionality. These results rest on the following theorem.

Theorem 3.1

Let T be any of the theories (with extensionality) discussed in this paper (so $T = \underset{\sim}{B}$, T_1, T_2, T_3, T_4 or $ZF^- + RDC$); or let T be one of the auxiliary theories T^*a or T^*b, where T^* is one of the theories considered in this paper. Then

(i) T can be interpreted in T-ext (without extensionality); that is, we can assign to each formula A an interpretation A^* such that $T \vdash A$ implies $T\text{-ext} \vdash A^*$. Furthermore, we have $T \vdash (A \leftrightarrow A^*)$, for A a Δ_o-formula.

(ii) T is conservative over T-ext for arithmetical sentences, in fact for sentences with quantifiers over a fixed (definable) subset of N^N allowed.

(iii) Both (i) and (ii) are provable in arithmetic.

Proof: We interpret T in T-ext , assigning to each formula A a formula A^* in which $\in$ is replaced by a formula ε, and sets are relativized to a formula $M(a)$. We shall show that $T \vdash A$ implies $T\text{-ext} \vdash A^*$. In other words, we shall explain how to give a definable model of T in T-ext. In order to make the model intelligible, we first give a false attempt. The most natural thing to do is to define $x \sim y$ if

$\forall a \in x \exists b \in y(a \sim b) \;\&\; \forall a \in y \exists b \in x(a \sim b)$; and then to set $x \in y$ if $\exists z \in y(x \sim z)$. The first problem with this is that $\sim$ is inductively defined, instead of being given by a formula. There are ways to overcome this, and we shall explain them. The simplest way to think of what we are doing is to think we are defining a model of T (given by a class, not a set) assuming only that the axioms of T-ext are true in the universe. This can be recase in official language as an interpretation, as above.

First of all, let us discuss the case $T = ZF^- + RDC$, where we have both power set and collection. Then we can make the above inductive definition of $\sim$ explicit in the most straightforward manner, so that if X is any transitive set, then $\sim_X$ ($\sim$ restricted to X) is a set. (It is the intersection of all binary relations R in the power set of X^2 which satisfy the appropriate inductive condition.) We need collection in order to prove that every set has a transitive closure $TC(x)$. Then we can define the model (interpretation), using for $x \sim y$ the formula

$$\forall R \in \wp(TC(\{x,y\}))(I(R,TC(\{x,y\}) \rightarrow \langle x,y\rangle \in R),$$

where $I(R,z)$ says that R satisfies the inductive conditions for $\sim$ on the transitive set z . It is straightforward to verify that the axioms of $ZF^- + RDC$ are valid on this interpretation, using the axioms of $ZF^- + RDC$.

Several of the theories we have to consider, however, are not strong enough to prove the existence of $TC(x)$ for each x . Our model for such theories will therefore have to be somewhat more complicated; we take the "sets" to be pairs $\langle x,y\rangle$, where y is the transitive closure of x. To be precise, we write $Trans(y)$ for $\forall a \in y \forall b \in a(b \in y)$ and we write $y = TC(x)$ for

$Trans(y) \;\&\; x \in y \;\&\; \forall b \in y(b{=}x \vee \exists n \in \omega \exists a_1, a_2, \dots, a_n\, (b \in a_1, \dots, a_n \in x))$.

Note that $y = TC(x)$ is a Δ_o-formula (in predicates definable in $\underset{\sim}{B}$, although it is not strictly Δ_o.)

To deal with the case T = Zermelo set theory, which lacks collection but has power set, we can proceed with these sets as we did above for $ZF^- + RDC$, using power set to make an inductive definition explicit. However, to deal with weaker set theories, which lack power set, more work is required. Let $Q(R,r)$ express that y is transitive and R is $\sim_y$; to be precise, $Q(R,y)$ is

$$Trans(y) \;\&\; \forall a,b \in y(\langle a,b\rangle \in R \leftrightarrow (\forall p \in a \exists q \in b \langle p,q\rangle \in R \;\&\; \forall q \in b \exists p \in a \langle p,q\rangle \in R))$$

Note that Q is a Δ_o-formula. Note that with the aid of the foundation axiom, we can prove that for each transitive y , if $Q(R,y)$ and $Q(S,y)$, then R and S are extensionally equal relations, i.e. $\langle a,b\rangle \in R \leftrightarrow \langle a,b\rangle \in S$. (We don't need transfinite induction to prove this.)

Suppose W is a fixed transitive set; then for $x \in W$, $TC(x)$ exists. What axioms does it take to prove $\forall x \in W \exists R\, Q(R,TC(x))$?
It seems to require collection and transfinite induction, as well as union. If we know this to be provable, we could go ahead and define the model; setting $x \sim y$ iff $R(Q(R,TC(\{x,y\}))\ \&\ \langle x,y\rangle \in R)$, or more precisely, $\langle x,u\rangle \sim \langle y,v\rangle$ iff the same condition holds, where the TC's occuring are to be extracted from u,v .
This model allows us to handle T_4 , which has collection, transfinite induction, and full separation. In order to handle the weaker theories, we must refine the model more.

The first thing that occurs to one is to restrict attention to those "sets" $(x,TC(x))$ such that there is an R such that $Q(R,TC(x))$. This is not enough. Let us call pairs $\langle x,y\rangle$ such that $y = TC(x)$, M_1-sets. Consider an M_1-set $\langle x,y\rangle$ such that for all M_1-sets $\langle a,b\rangle$, there is an R such that $Q(R,TC(\{x,a\}))$. Such sets $\langle x,y\rangle$ we call M_2-sets. These are the "sets" of our next model. We define $\sim$ as above, $x \sim y$ iff (more precisely) $\langle x,TC(x)\rangle \sim \langle y,TC(y)\rangle$ iff $\exists R(Q(R,TC(\{x,y\})\ \&\ \langle x,y\rangle \in R)$ iff $\forall R(Q(R,TC(\{x,y\}) \to \langle x,y\rangle \in R)$. This model will work for T_1,T_2,T_3 , and incidentally for T_4 as well. For simplicity we write x instead of $\langle x,TC(x)\rangle$ for M_1-sets. Suppose x and y have the same ε-members, that is, every element of x is equivalent to a member of y and vice-versa. We have to show that x and y are equivalent (hence are ε-members of the same set). Since x and y are M_2-sets, there is an R such that $Q(R,TC(\{x,y\}))$. If $z \in TC(\{x,y\})$, and $Q(R^*,TC(z))$, then R^* is a subset of R; hence every member of x is equivalent to a member of y and vice-versa, using R for the equivalence; hence $\langle x,y\rangle \in R$; this verifies extensionality.

We now check Δ_o-separation. If a is a fixed M_2-set, and A is a Δ_o-formula, we have to find an M_2-set x such that for all M_2-sets z , $z \,\varepsilon\, x$ iff $z \,\varepsilon\, a\ \&\ A^*(z)$. (Here A^* is the interpretation of A .) Since ε is not given by a Δ_o-formula, and neither is M_2, the range of the quantified variables, it is not obvious how to produce x , using only Δ_o-separation. We proceed by induction on the complexity of A . We first have to handle the case of atomic A ; here there are two possibilities, either A is $z \in b$ or A is $b \in z$. First suppose A is $z \in b$. Take x to be $\{ z: z \in a\ \&\ \exists y \in b(y \sim z)\}$. This set can be formed using Δ_o-separation, since we have $\{\langle z,y\rangle: z \in a\ \&\ y \in b\ \&\ z \sim y\}$ available to use as a parameter, since a and b are M_2-sets.
Next suppose A is $b \in z$. This time take x to be $\{z \in a: \exists b' \in z(b' \sim b)\}$. This set can be formed, using as a parameter some relation R such that $Q(R,TC(a))$. It is easy to check that these two sets x are actually M_2-sets. This takes care of the case A atomic. Similar arguments take care of the induction steps in which A is of the form $B\ \&\ C$, $B \vee C$, or $B \to C$.

Next, observe that an easy induction on the complexity of A shows that $x \sim y\ \&\ A^*(x) \to A^*(y)$ is provable, for each fixed A . Now consider the case in which A is $\forall w \in y\, B(z,w,y)$. Since every ε-member of y is equivalent to some $\in$-member, the remark just made shows that it suffices to quantify over ε-members; that is the key to the verification in this case, which we now give in more detail. By the induction hypothesis, we have that $\forall w \in y \exists P(P$ is an M_2-set & $(\langle w,z\rangle \in P \to z \in a\ \&\ B^*(z,w,y)))$. By strong collection, there is a set T such that $\forall w \in y \exists P \in T(\ldots)\ \&\forall P \in T\, \exists w \in y(\ldots)$.As a matter of fact, T is not only a set but an M_2-set, if y is an M_2-set. To check this, first note that the transitive closure of T can be obtained by taking the union of the transitive closures of the elements of T and $\{T\}$; thus T is an M_1-set; if T' is some M_1-set, we have to get $\{\langle a,b\rangle : a \sim b\ \&\ a \in TC(T)\ \&\ b \in TC(T')\}$ to exist. This relation R is just the union of the corresponding relations for the members of T (which can be formed using collection), union the set $\{\langle T,b\rangle : b \in TC(T')\ \&\ T \sim b\}$, union $\{\langle a,T'\rangle : a \in TC(T)\ \&\ a \sim T'\}$, both of which are easily defined. Hence T is an M_2-set. Now form the set $S = \{z \in a : \forall w \in y \exists P \in T\ \ \langle w,z\rangle \in P\}$, which can be formed using Δ_o-separation. We have

$$z \,\varepsilon\, S \to z \,\varepsilon\, a\ \&\ \ w \,\varepsilon\, y\ B^*(z,w,y) \quad .$$

This completes the verification of the case in which A is formed by bounded universal quantification. The case of bounded existential quantification is much easier, and we leave it to the reader. This completes the verification of Δ_o-separation.

We turn our attention to the axiom of infinity. Here the only difficult part of the proof is to prove that $\langle\omega,\omega\rangle$ is an M_2-set; we first have to prove that if n is an integer, then $\langle n,n\rangle$ is an M_2-set. (Each integer is its own transitive closure). This is, we have to prove that

$$\forall n \in \omega \forall \text{transitive } x \exists R(\langle p,q\rangle \in R \to p \in n\ \&\ q \in x\ \&\ \langle p,p\rangle \sim \langle a,TC(q)\rangle)$$

The obvious way to prove this is by induction on n; however, in some of the weak theories we do not have full induction available, and it seems to require at least Δ_1-induction. Fortunately, we can prove it without induction. Fix an integer n and a transitive set x ; then R can be taken to be R_n , where

$$R_o = \emptyset \text{ and } R_{k+1} = \{\langle p,q\rangle: p \in n\ \&\ q \in x\ \&\ \forall a \in p \exists b \in q\ \ \langle a,b\rangle \in R_k\ \&$$
$$\forall b \in q \exists a \in p\ \ \langle a,b\rangle \in R_k\} \quad .$$

Now the desired property of R can be proved by a bounded induction.

We next undertake to verify the sound interpretation of the axiom of exponentiation. In our non-extensional set theories, this axiom takes the following form:

$$(*)\ \forall A,B\,\exists X\,\forall q(\mathrm{Fcn}(q)\ \&\ \mathrm{Dom}(q) = A\ \&\ \mathrm{Rng}(q) \subseteq B \rightarrow$$
$$\exists f \in X(\mathrm{Fcn}(f)\ \&\ \mathrm{Dom}(f) = A\ \&\ \mathrm{Rng}(f) \subseteq B\ \&\ \forall x \in A(f(x) = q(x)))).$$

Note it is not necessary to put in that X is a set of functions from A to B, because Δ_o-separation can alway be applied to get $\{f \in X : \mathrm{Fcn}(f)\ \&\ \mathrm{Dom}(f) = A\ \&\ \mathrm{Rng}(f) \subseteq B\}$.

Now we have to use this axiom to verify that the ordinary exponentiation axiom is satisfied in the model. The first point to make is that if A and B are sets of the model, then so is B^A (and indeed any X as in $(*)$). This is because any descending $\in$-chain has one of the forms

$$x_1 \in x_2 \ldots \in a \in \{a\} \in \langle a,b\rangle \in f \in X$$
$$x_1 \in x_2 \ldots \in a \in \{a,b\} \in \langle a,b\rangle \in f \in X$$
$$x_1 \in x_2 \ldots \in b \in \{a,b\} \in \langle a,b\rangle \in f \in X\ .$$

Hence the transitive closure $TC(X)$ can be defined as the union over all members of such sequences, using the fact that $TC(A)$ and $TC(B)$ are sets.

Now to verify $(*)$. Fix A and B. Let X be given by axiom $(*)$. (This X is not however the X^* we choose to verify $(*)$ - we give this X^* later). Suppose q satisfies the hypothesis of $(*)$ in the model; that is

(i) $\forall a' \in A\,\exists b' \in B\ \langle a',b'\rangle \in q$

(ii) $\langle a',b'\rangle \in q \rightarrow a' \in A\ \&\ b' \in B$

(iii) $\langle a',b'\rangle \in q\ \&\ \langle a',c'\rangle \in q \rightarrow b' \sim c'$

Let $f = \{\langle a,b\rangle : a \in A\ \&\ b \in B\ \&\ \exists a'\exists b'\ \ \langle a',b'\rangle \in q\ \&\ a \sim a'\ \&\ b \sim b'\}$.

Now f can be defined in $\underset{\sim}{B}$-ext , because : we can fix a transitive set W containing both A,B,and q, and then as we have shown above, $\in$ and $\sim$ restricted to W are sets and the quantifiers $\exists a'\exists b'$ can be relativized to W .

We claim f is satisfied to be a function from A to B. Suppose $\langle a',b'\rangle \in f$; then $a' \sim a\ \&\ b' \sim b$ for some $\langle a,b\rangle \in f$, so $a \in A\ \&\ b \in B$, so $a' \in A\ \&\ b' \in B$. Next suppose $\langle a,b\rangle \in f\ \&\ \langle a,c\rangle \in f$. Then $a \sim a''$, $b \sim b'$, $a \sim a'$, $c \sim c'$ with $\langle a',b'\rangle \in q\ \&\ \langle a'',c'\rangle \in q$. Hence, by (iii) $b' \sim c'$; hence $b \sim c$. Hence f is satisfied to be a function from A to B.

Moreover, f satisfies $\forall x \in A\,(f(x) = q(x))$, i.e. $\forall a \in A\,\forall b,c \in B(\langle a,b\rangle \in f\ \&\ \langle a,c\rangle \in q \rightarrow b \sim c)$, since if $\langle a,b\rangle \in f$, then $a \sim a'\ \&\ b \sim b'$ where $\langle a',b'\rangle \in f$, so $a \sim a''\ \&\ b' \sim b''$ where $\langle a'',b''\rangle \in q$, and by (iii), $a \sim a''$ implies $c \sim b''$; since we have $b \sim b' \sim b'' \sim c$, we get $b \sim c$.

After these preliminaries we can give the set X^* which is to work for X in $(*)$ in the model. Note that if f is satisfied to be a function from A to B , f may not actually be a function; but it induces a function from A to the set $\{[b] : b \in B\}$, where $[b]$ is the equivalence class of b in the equivalence relation $\sim$ on B .

Note that $\{[b] : b \in B\}$ can be formed in $\underset{\sim}{B}$-ext by the abstraction axiom; hence by exponentiation we can form the set S of all functions from A to $\{[b] : b \in B\}$ Now if $H \in S$, let $H^\#$ be defined by

$$\langle a,b\rangle \in H^\# \leftrightarrow [b] \in H(a)$$

Take $X^* = \{H^\# : H \in S\}$. X^* can be produced using the abstraction axiom of $\underset{\sim}{B}$-ext. To see this explicitly, we write

$$X^* = \{\{\langle a,b\rangle : \exists z \in \{[b] : b \in B\}\ (\langle a,z\rangle \in H\ \&\ b \in z)\} : H \in S\}\ .$$

Note also that $TC(X^*)$ exists, so X^* is a "set" of the model.

Now suppose the model satisfies

$$\mathrm{Fcn}(q)\ \&\ \mathrm{Dom}(q) = A\ \&\ \mathrm{Rng}(q) \subseteq B\ .$$

As above, we can produce f such that f is satisfied to be a function from A to B and $\forall x(f(x) = q(x))$ holds in the model. It remains to show $f \in X^*$. Let $f^* = \{\langle a,b\rangle : a \in A\ \&\ b \in B\ \&\ \langle a,b\rangle \in f\}$. We claim $f^* \in X^*$, and $f \sim f^*$. To see $f \sim f^*$, note that $f^* \subseteq f$, and if $\langle a',b'\rangle \in f$, then $a' \in A\ \&\ b' \in B$, so $a' \sim a \in A\ \&\ b' \sim b \in B\ \&\ \langle a'',b''\rangle \in q$ for some a'' and b'' with $a' \sim a''\ \&\ b' \sim b''$. Hence $\langle a,b\rangle \in f$ also; thus every member $\langle a',b'\rangle$ of f is equivalent to a member $\langle a,b\rangle$ of f^* . Hence $f \sim f^*$.

Finally we prove $f^* \in X^*$. Define H by $H(a) = [f(a)]$, the equivalence class of $f(a)$ in B . Technically, f is not a function (though it is satisfied to be a function) so we really must define $H(a) = \{b \in B : \exists b' \in B\ \ \langle a,b'\rangle \in f\ \&\ b' \sim b\}$. Actually, looking at the definition of f it is enough to take $H(a) = \{b \in B : \langle a,b\rangle \in f\}$. In any case H is a function from A to $\{[b] : b \in B\}$, and $f^* \sim H$, in fact f^* and H have exactly the same members. But $H^\# \in X^*$ by definition of X^* ; hence $f^* \in X^*$.

Since we have proved $f^{\#} \varepsilon X^{\#}$ & $f^{\#} \sim f$, it follows that $f \varepsilon X^{\#}$. Hence the conclusion of the exponentiation axiom is verified.

This leaves pairing, union, collection, foundation, induction, dependent choice and transfinite induction still to check. Because of limitations of space, we omit the details of these verifications. Thus we take it as proved that our interpretation is sound for T_1, T_2, T_3 and T_4 . We have already done ZF^- + RDC and Z , which leaves only $\underset{\sim}{B}$ to consider.

The above model uses collection quite heavily; however, in the case of $\underset{\sim}{B}$, we should be able to describe the model consisting of sets of rank less than $\omega + \omega$ quite explicitly (incidentally giving another interpretation that works for Zermelo). Define a set x to be of rank less than $\omega + n$ if every descending $\in$-chain from x terminates in an integer in $\leq n$ steps. (This is not exactly the usual definition, but it is convenient). Write $S(x)$ if for some n, x is of rank less than $\omega + n$.

Again we shall consider the "sets" of the model to be pairs $(x, TC(x))$. Note that $S(x)$ is Δ_o in x and $TC(x)$ (we need $TC(x)$ to be able to quantify over descending $\in$-chains). Let W be a fixed transitive set of rank $\leq \omega + n$. We prove that $\sim_W$ is a set, that is, $\exists R\, Q(R,W)$, where Q is as above.

Namely, R is R_n , where $R_o = \{\langle n,n\rangle : n \in \omega\}$ and
$R_{j+1} = \{\langle x,y\rangle \in W^2 : \forall a \in x \exists b \in y \ \langle a,b\rangle \in R_j \ \& \forall b \in y \exists a \in x \ \langle a,b\rangle \in R_j\}$

As we discussed near the beginning of this proof, this is all we need to make the interpretation work. We interpret $x \in y$ as $\exists z \in y (z \sim x)$, where $z \sim x$ is

$$\exists R (Q(R, TC(\{x,y\})) \ \& \ \langle z,x\rangle \in R) \ .$$

We interpret sets, as mentioned, as pairs $\langle x,y\rangle$ with $S(x)$. We leave the reader to verify that the interpretation is sound for $\underset{\sim}{B}$.

Next we turn to the case of the auxiliary theories Ta and Tb, where T is ZF^- + RDC , Z , T_1 , T_2 , T_3 or T_4 . We use the same interpretation for Ta and Tb as for T ; to finish the proof, we have to verify that the extra of Ta and Tb are soundly interpreted.

We have to tell how the constant a (or b) is to be interpreted; of course it is as $\langle a, TC(a)\rangle$ (in the case of the interpretation that works for the T_i) and just a in the case of the interpretation that works for ZF^- + RDC and Z . This depends on the fact that $TC(a)$ is definable from a . One easily verifies that every member of N^N (remember a is a member of a subset of N^N) has a transitive

closure definable from a ; note that each integer is its own transitive closure. Next we have to verify that each member of N^N is actually (with its transitive closure) an M_2-set; that is, if x is any transitive set, we can form $\{\langle p,q\rangle : p \in TC(a) \ \& \ q \in x \ \& \ p \sim q\}$. (More precisely, p and q should be paired with their transitive closure). This is easy, once we know that each integer is an M_2-set, which we have already discussed. Finally, we have to verify that the axiom $a \in X$ and the axioms $b(\overline{n}) = \overline{m}$ where $m = f(n)$, for some fixed f in X , are soundly interpreted. Recall from §2 that membership in X is given by a purely universal condition on the values of a. Below we give a proof that the interpretation preserves arithmetic sentences; the same proof applies to show that it preserves the axioms in question .

We have now given the interpretation A^* for each of the set theories discussed in this paper, and proven the soundness of the interpretation. Next we prove that $T \vdash A \leftrightarrow A^*$, for Δ_o-formulae A . This is established by induction on the complexity of A ; to be quite precise, $A \leftrightarrow A^*$ is only for closed formulae ; for formulae with free variables, say x , we should say

$$S(\langle x,y\rangle) \rightarrow (A^*(\langle x,y\rangle) \leftrightarrow A(x)).$$

(Here we are considering the interpretation that works for T_1, T_2, T_3, T_4). The basis is $S(\langle x,y\rangle) \ \& \ S(\langle a,b\rangle) \rightarrow (\langle x,y\rangle \ \varepsilon \ \langle a,b\rangle \rightarrow x \in a)$, which can be established using the foundation axiom; extensionality is used here. The induction step proceeds smoothly, using the fact that members of M_2-sets determine M_2-sets; we leave the details to the reader. Note that we cannot seem to get $A^* \leftrightarrow A$ for all formulae A but only for Δ_o-formulae. (For ZF^- + RDC we can get it for all formulae, because the interpretation does not require that transitive closures be tacked on.) The same argument works for the interpretation of $\underset{\sim}{B}$ using sets of rank less than $\omega + \omega$; here the induction step over a bounded quantifier uses the fact that the members of sets of rank less than $\omega + \omega$ are themselves sets of rank less than $\omega + \omega$. A similar induction works for the interpretation used for Zermelo set theory. This completes the proof of (i) of the theorem.

We next consider the question of which sentences are preserved by the interpretations; it is for these sentences that we get a conservative extension result. First arithmetic sentences are preserved. This is shown by induction on the complexity of an arithmetic formula; actually, as above we have to prove

$$\langle m,x\rangle \in \omega \ \& \ n \in \omega \ \& \ \langle m,x\rangle \sim \langle n, TC(n)\rangle \rightarrow (A(\langle m,x\rangle) \leftrightarrow A(n)) \quad \text{(this time without extensionality)}$$

Note that every integer has a transitive closure, namely itself; in fact, every integer is (part of) an M_2-set.
Here A is a formula of set theory translating a formula of arithmetic, which we also call A ; the induction is on the complexity of the arithmetic formula. The details are easy but cumbersome; we leave them to the reader. Next note that every f in N^N has a transitive closure; this allows us to extend the above induction to formulae involving quantifiers over such objects. Actually, we must verify that each such f is (part of) an M_2-set; to do this, we must be able to form for each transitive set x , the set $\{\langle a,b\rangle : a \in TC(f)\ \&\ b \in x\ \&\ a \sim b\}$. This boils down to the fact that we can form the corresponding set with an integer m in place of f , in other words that each integer is an M_2-set, a fact alluded to above.

This completes the proof of part (ii) of the theorem. Part (iii) of the theorem, which says that the first two parts are provable in arithmetic, is proved by examining the above proof, and noticing that only arithmetic is needed. In other words, we proved by induction on (Gödel numbers of) proofs in T that the interpretation of the last formula of the proof is provable in T-ext . This completes the proof of theorem 3.1.

§4. Realizability for Set Theories

In this section, we give a variant of q-realizability adapted to set theories. This type of realizability has been used before for arithmetic and the theory of species to obtain explicit definability theorems [Tr]. Here we extend this program to set theories. The extension to set theories without extensionality is relatively straightforward, but there seems to be no simple way to handle set theories with extensionality. (Myhill gave [M1] a complicated realizability for his extensional set theory; but it cannot be made to work for our purposes.) For this reason, even if we want to obtain derived rules only for extensional theories, we have to consider the non-extensional ones and use the results of the previous section.

The plan of the present is to give the realizability interpretation we need and prove its soundness both for the basic set theories T-ext and for Ta-ext and Tb-ext . Our definition of realizability will proceed by associating to each formula A another formula e r A ("e realizes A"). We will then prove soundness theorems of the form, if $T \vdash A$, then for some e , $T \vdash \bar{e}\ r\ a$. Here e is an integer; all our realizing objects are integers, not arbitrary sets. (We use e , n , m etc. to indicate variables whose range is restricted to ω .)

We begin by assigning to each set variable x another variable x^* , in the manner discussed in [B1]. The free variables of e r A are e,x and x^* , where x are the free variables of A .
(Our convention is that a single letter can denote a finite list of variables.) We now give the clauses defining e r A , for the notion of realizability that works for theories without extensionality:

$e\ r\ x = y$	is	$x^* = y^*$
$e\ r\ x \in y$	is	$\langle e,x,x^*\rangle \in y^*$
$e\ r(A\ \&\ B)$	is	$(e)_0\ r\ A\ \&\ (e_1)\ r\ B$
$e\ r(A \vee B)$	is	$((e)_0 = 0 \rightarrow (e)_1\ r\ A\ \&\ A)$ $\&\ ((e)_0 \neq 0 \rightarrow (e)_1\ r\ B\ \&\ B)$
$e\ r(A \rightarrow B)$	is	$a\ r\ A\ \&\ A \rightarrow \{e\}(a)\ r\ B$; or more precisely $\forall a(a\ r\ A\ \&\ A \rightarrow \exists k(T(e,a,k)\ \&\ U(k)\ r\ B)$
$e\ r \forall x\ A$	is	$\forall x,x^*\ e\ r\ A$
$e\ r \exists x\ A$	is	$\exists x,x^*\ (A\ \&\ e\ r\ A)$

In order to complete the definition, we have to define e r A for atomic A involving terms t of the non-extensional set theories. This can be done by the same clauses as above, as soon as we associate to each term t another term t^* which intuitively defines $\{\langle e,x,x^*\rangle : e\ r\ x \in t\}$. These terms t^* will be given in the course of the soundness proof below; they could be listed here, but would be unintelligible. For instance we define $\omega^* = \{\langle n,\langle n,n\rangle\rangle : n \in \omega\}$. As another example, if $t = \{y \in a : B(y)\}$, then $t^* = \{\langle e,y,y^*\rangle : \langle (e)_0,y,y^*\rangle \in a^*\ \&\ (e)_1\ r\ B(y)\}$. We give this example in order to clarify the following point: There is no vicious circle in the fact that $(e)_1\ r\ B(y)$ appears in the definition of t^*, which must precede the definition of $e\ r\ x \in t$; for, as discussed above, the definitions of Δ_0-formulae and terms proceed by simultaneous induction, so that B(y) contains only less-complicated terms than t . To make this completely precise, we could assign a measure of complexity to both terms and formulae, say C(t), giving atomic formulae without compound terms complexity zero, and atomic formulae $t=s$ or $t \in s$ the complexity $\max(C(t),C(s))$; let propositional connectives and quantifiers increase the complexity by 1 , and let separation terms $\{x \in a : B(x)\}$ have complexity $1+\max(C(a),C(B))$; similarly for union, pair, and choice terms. Then our definition of e r A proceeds by induction on the complexity of A .

Remarks:

(1) If we were doing 1945-realizability (see [Tr]), we would not need the extra variables with stars, but could avoid them by defining $e \; r \; x \in a$ to be $\langle e,x\rangle \in y$. Trying to do something similar for q-realizability is more trouble than it is worth.

(2) One cannot define $e \; r \; x \in y$ to be $\langle e,x\rangle \in y^*$, though this may seem tempting. In this case, all the axioms except dependent choice will be realized (including extensionality), but one will not be able to get anything realized to be a function. Consider $Fcn(f)$ which says

$$\forall x,y,w \; (\langle x,y\rangle \in f \;\&\; \langle x,w\rangle \in f \to y = w).$$

In order to get $y = w$ realized, there will have to be some relation between y^* and w^*, which we cannot get from having the antecedent realized, with this definition of realizability. This is somewhat interesting because it points up the absolute necessity of the axiom of choice in proving the existence of functions.

(3) The motivation behind the definition of $e \; r \; x \in y$ is that y^* is thought of as the set of $\langle e,x\rangle$ such that e proves, or verifies, or realizes, that $x \in y$. Remember that Kleene's original motivation for realizability was that realizing numbers were thought of as like proofs. It is no wonder that extensionality gives trouble, because one can have x and y extensionally equal, without any relationship at all between x^* and y^*; yet if $\forall a (x \in a \leftrightarrow y \in a)$ is to be realized, there has to be some relationship between x^* and y^*.

Theorem 4.1. (soundness of q-realizability).

Let T be any of the set theories considered in this paper, without extensionality. Then for the notion of realizability just given, if $T \vdash A$, then for some number e, we have $T \vdash \bar{e} \; r \; A$.

Proof: As usual for realizability soundness theorems, we proceed by induction on the length of the proof of A, proving that the universal closures of all statements in the proof are realized. Thus we have to verify that the universal closures of all statements in the proof are realized. Thus we have to verify that the universal closures of all the axioms are realized, and the rules of inference preserve realizability. The logical axioms and rules of inference are handled in the usual way (see [Tr]). We have to check the non-logical axioms.

B. (Pairing). $\langle p,q\rangle \; r \; \exists x \forall y (y \in x \to (y = a \vee y = b))$

Take $x^* = \{\langle e,y,y^*\rangle : e \in \omega \;\&\; y \in \{a,b\} \;\&\; y^* \in \{a^*,b^*\} \;\&\; e \; r (y = a \vee y = b)\}$.

This set can be formed in $\underset{\sim}{B}$ without extensionality. Take $x = \{a,b\}$.

C. (Infinity).

$$\emptyset \in \omega \;\&\; \forall y \in \omega \, (y \cup \{y\} \in \omega) \;\&\; \forall z (\emptyset \in z \;\&\; \forall y \in z (y \cup \{y\} \in z) \to \omega \subseteq z)$$

Take $\omega^* = \{\langle n, \langle n,n\rangle\rangle : n \in \omega\}$

Then $\emptyset \in \omega \;\&\; \forall y \in \omega \, (y \cup \{y\} \in \omega)$ is realized and true.

We have to show that

$\forall z (\emptyset \in z \;\&\; \forall y \in z (y \cup \{y\} \in z) \to \omega \subseteq z)$ is realized and true. Suppose z and z^* are given, so that $\emptyset \in z \;\&\; \forall y \in z (y \cup \{y\} \in z)$ is true and realized, say by $\langle a,b\rangle$.

Then, first of all, $\omega \subseteq z$ is true; in order to get $\omega \subseteq z$ realized, we introduce a recursive function $\{p\}$ by the recursion theorem to satisfy the equation

$$\{p\}(0) = a$$
$$\{p\}(y+1) = \{b\}(\{p\}(y))$$

Then we prove by induction that $\{p\}(y) \; r \; y \in z$; that is, $\langle \{p\}(y), y, y^*\rangle \in z^*$. (What we are proving by induction has a free variable y^*.) Note that only the restricted induction axiom is needed.

D. (Union). $y \in \bigcup_{z \in a} z \leftrightarrow \exists z (y \in z \;\&\; z \in a)$. If t is the term $\bigcup_{z \in a} z$, we take t^* to be $\{\langle e,y,y^*\rangle : e \; r \; \exists z (y \in z \;\&\; z \in a)\}$; this can be formed in $\underset{\sim}{B}$.

E. (Separation). Let t be the function symbol such that the following is an axiom: $\forall y (y \in t(a) \leftrightarrow (y \in a \;\&\; B(y)))$. To get this realized, we define a function t^* by

$$t^*(a) = \{\langle e,y,y^*\rangle : \langle (e)_0, y, y^*\rangle \in a^* \;\&\; (e)_1 \; r \; B(y)\}$$

t^* can be proved to exist in $\underset{\sim}{B}$-ext, since $u \; r \; B$ is a Δ_0-formula if B is.

Then $t^*(a) = \{\langle e,y,y^*\rangle : e \; r \; (y \in a \; \& \; B(y))\}$; this finishes the verification. Note that Δ_o-separation suffices to interpret Δ_o-separation, and full separation for full separation.

F. (strong collection).

$$\forall a(\forall x \in a \exists y \, A \to \exists z(\forall x \in a \exists y \in z \; A \;\; \& \forall y \in z \; \exists x \in a \; A)).$$

Suppose a and a^* are given, and suppose $p \; r \; \forall x \in a \exists y \; A$, and $\forall x \in a \exists y \; A$. Let $Q_{x,x^*} = \{c: c \; r \; x \in a\} = \{c: \langle c,x,x^*\rangle \in a^*\}$. Then $\forall x \in a \forall x^* \in \text{Rng} \;\; \text{Rng}(a^*) \forall c \in Q_{x,x^*} \exists y(A \; \& \; \{p\}(c) \; r \; A)$; applying collection, we get the existence of some z_o such that

$$\forall x \in a \forall x^* \in \text{Rng} \;\; \text{Rng}(a^*) \; \forall c \in Q_{x,x^*} \exists y \in z_o \ldots$$

Also, applying collection to $\forall x \in a \exists y \; A$, we get some z_1 such that $\forall x \in a \exists y \in z_1 \;\; A \; \& \forall y \in z_1 \; \exists x \in a \; A$. Take $z = z_o \cup z_1$. Then $\forall x \in a \exists y \in z \;\; A \;\; \& \forall y \in z \;\; \exists x \in a \; A$, i.e. the conclusion of axiom F is true. Note that this works because we have strong collection, not just plain collection; the extra conclusion indicated by ... in the choice of z_o is needed. We need to show that this conclusion is not only true but realized.

Define

$$z^* = \{\langle\langle e,c,x,x^*\rangle,y\rangle : c \; r \; x \in a \; \& \; x \in a \; \& \; y \in z \; \& \; x^* \in \text{Rng} \;\; \text{Rng}(a^*) \; \& \; e \; r \; A(x,y)\}$$

First we show that $\forall x \in a \exists y \in z \; A$ is realized (by a number depending recursively on p). Suppose $c \; r \; x \in a$ and $x \in a$. Then for some y in z_o, $A(x,y)$ and $\{p\}(c) \; r \; A(x,y)$. Hence $\langle\langle\{p\}(c),c,x,x^*\rangle,y\rangle \in z^*$, so $\forall x \in a \exists y \in z \; A$ is realized.

Similarly, we have to show $\forall y \in z \exists x \in a \; A$ is realized. Suppose $b \; r \; y \in z$; then b has the form $\langle\langle e,c,x,x^*\rangle,y\rangle$ where $e \; rA(x,y)$ and $c \; r \; x \in a$ and $x \in a$. Hence $\forall y \in z \exists x \in a \; A$ is realized, by a simple combination of unpairing functions.

G. (foundation). $\forall a,b(a \in b \; \& \; b \in x \to a \in x) \; \& \forall y \in x(y \subseteq z \to y \in z) \to x \subseteq z$.
Suppose z,z^* are given, and $p \; r \forall a,b(a \in b \; \& \; b \in x \to a \in x)$, and $q \; r \forall y \in x \; (y \subseteq z \to y \in z)$, and the formulae realized by p and q are true. Introduce a recursive function $\{f\}$ by the recursion theorem so that

$$\{f\}(e) = \{\{q\}(e)\}(\Lambda u\{f\}(\{p\}(\langle u,e\rangle))) \qquad .$$

(Here $\Lambda u \; h$ is an index of $\lambda \; u \; h$, so $\{\Lambda \; u \; h(u,v)\} \; (u) = h(u,v)$; this is an old and useful notation of Kleene.) We claim $f \; r \; x \subseteq z$, the conclusion of the foundation axiom. (Which will finish the proof, since the conclusion of the axiom is true, because we have assumed the hypothesis.) We must show $f \; r \forall y(y \in x \to y \in z)$; that is, whenever $y \in x$, and $e \; r \; y \in x$, we have $\{f\}(e) \; r \; y \in z$. We prove this by transfinite induction on $\langle y,y^*\rangle$ (then later show how to get by with only the foundation axiom). Our induction hypothesis is that for all $a \in y$ and $a^* \in \text{Rng} \;\; \text{Rng}\;(y^*)$, $(e \; r \; a \in x) \to \{f\}(e) \; r \; a \in z$. Suppose $e \; r \; y \in x$; we must show $\{f\}(e) \; r \; y \in z$. Note that if $u \; r \; a \in y$, then $\{p\}(\langle u,e\rangle) \; r \; a \in x$ by our hypothesis on p. Applying our induction hypothesis (substituting $\{p\}(\langle u,e\rangle)$ for e), we see that if $a \in y$ and $u \; r \; a \in y$, we have $\{f\}(\{p\}(\langle u,e\rangle)) \; r \; a \in z$. That is, $\Lambda \; u\{f\}(\{p\}(\langle u,e\rangle)) \; r \; y \in z$. Now, applying the hypothesis on q, and the definition of f, we reach the desired conclusion, that $\{f\}(e) \; r \; y \in z$. This completes our proof by transfinite induction; now we have to show how to get by with only foundation. The foundation axiom amounts to proof by transfinite induction, where what is proved is membership in some set. Here the set in question is

$$\{\langle y,y^*\rangle \in x \times \text{Rng Rng}\;(x^*) \; ; \; \forall e \in \omega \; (\langle e,y,y^*\rangle \in x^* \to \langle\{f\}(e),y,y^*\rangle \in z^*)\}$$

(Recasting the induction on the pair $\langle y,y\rangle$ as an induction on a single variable is left to the reader.) This completes the verification of foundation.

H. (Exponentiation). $\exists x \forall y(y \in x \leftrightarrow \text{Fcn}(y) \; \& \; \text{Dom}(y) = a \; \& \; \text{Rng}(y) \subseteq b)$.
(Even the strong version of the exponentiation axiom is realized.) Suppose a,b,a^*, and b^* are given; we have to produce x and x^*. Let x be b^a; the problem is to produce x^*. Suppose for the moment that we had a set Q such that if $\text{Fcn}(y) \; \& \; \text{Dom}(y) = a \; \& \; \text{Rng}(y) \subseteq b$ is realized, then $y^* \in Q$. (An "a priori bound" on the complexity of y^*.) Then we would like to take x^* to be

$$\{\langle e,y,a^*\rangle \in \omega \times x \times Q: e \; r \; (\text{Fcn}(y) \; \& \; \text{Dom}(y) = a \; \& \; \text{Rng}(y) \subseteq b).\}$$

Since $\text{Fcn}(y)$ involves an unbounded quantifier, it is not immediate that this set can be formed using the axioms of $\underset{\sim}{B}$. However, we can instead take

$$x^* = \{\langle e,y,y^*\rangle \;\; \omega \times x \times Q: e \; r \; (\forall s \in a \forall t,t' \in b(\langle s,t\rangle \in y \; \& \; \langle s,t'\rangle \in y \to t = t')$$
$$\& \;\; \text{Dom}(y) = a \; \& \; \text{Rng}(y) \subseteq b)\} \text{ (here } = \text{ is extensional equality).}$$

From a member of this x^*, we can recursively pass to a realizer of $\text{Fcn}(y) \; \& \; \text{Dom}(y) = a \; \& \; \text{Rng}(y) \subseteq b$ and vice versa, so that the exponentiation axiom will be realized. The difficult part, namely producing the set Q, is still ahead of us. At first, this seems to be a serious problem. A typical element of y^* will be $\langle e,\langle s,t\rangle,\langle s^*,t^*\rangle\rangle$ with $t=y(s)$: but t^* is not uniquely determined (even extensionally) so we cannot seem to use the exponentiation axiom to get a set in which y^* must lie, and power set is not available. However, the following argument gets us around the difficulty. First, set $x^* \sim y^*$ iff $\forall z \in b(z \in x \leftrightarrow z \in y)$ is realized. Thus $\{\langle x^*,y^*\rangle \in \mathrm{RngRng}(b^*)^2 : x^* \sim y^*\}$ can be formed in $\underset{\sim}{B}$. Now, we can form in $\underset{\sim}{B}$ the equivalence classes $[x^*]$ under the relation $\sim$ and the set $B_o = \{[x^*] : x^* \in \mathrm{RngRng}(b^*)\}$ (using the abstraction axiom). Next, observe that if $\langle x,z\rangle \in y$ and $\langle x,w\rangle \in y$ are realized, and $\mathrm{Fcn}(y)$ is realized then $x^* \sim w^*$. Hence, although t^* is not uniquely determined, the equivalence class $[t^*]$ is uniquely determined (extensionally). Suppose

$$p\ r\ s \in a \rightarrow \exists t \langle s,t\rangle \in y; \quad \text{that is,}$$

$$\langle e,s,s^*\rangle \in a^* \rightarrow \exists t,t^*(\langle \{p\}(e),\langle s,t\rangle,\langle s^*,t^*\rangle\rangle \in y^*) .$$

Then if y^* is such that $\mathrm{Fcn}(y)$ & $\mathrm{Dom}(y) = a$ & $\mathrm{Rng}(y) \subseteq b$ is realized, then y^* has the form

$$\{\langle d,\langle s,y(s)\rangle,\langle s^*,t^*\rangle\rangle : t^* \in f(s,s^*) \ \&\ d \in p\}$$

for some function f from $a \times \mathrm{RngRng}(a^*)$ to B_o and some set $P \subseteq \omega$, where P has the form $\{\{q\}(\{p\}(e)) : \langle e,s,s^*\rangle \in a^*\}$, for some q in ω. We do not need the power set axiom to form the set of all such P; quantification over integers q is enough. Using exponentiation, we can form the set of all such functions f; thus the set Ω of all such y^* can be formed; this completes the verification of the exponentiation axiom.

I. (Bounded dependent choice). $\forall x \in a \exists y \in a\ P \rightarrow \forall x \in a \exists f(\mathrm{Fcn}(f)$ & $\mathrm{Rng}(f) \subseteq a$

$$\&\ \mathrm{Dom}(f) = \omega \ \&\ f(0) = x \ \&\ \forall n \in \omega\ P(f(n),f(n+1)))\text{, with } P\ \Delta_o .$$

Suppose $p\ r\ \forall x \in a \exists y \in a\ P$ and $s\ r\ x \in a$, and these formulae are true, as well as realized. Define a recursive function h by

$$h(0) = \langle s,0\rangle$$

$$h(n+1) = \{p\}((h(n))_o)$$

and introduce e by

$$\{e\}(0) = \langle s,(\{p\}(s))_1\rangle$$

$$\{e\}(n+1) = h(n)$$

Now, we will prove the existence of a (set-theoretic) function f such that $\{e\}(n)\ r\ (f(n) \in a \ \&\ P(f(n),f(n+1)))$. First note that the principle of countable

independent choice $\forall n \in \omega \exists y \in a \ldots$ can be derived from dependent choice. Then prove by induction on n that

$$(*)\ \forall n \in \omega \exists y_o,\ldots,y_n \in a\ \exists y_o^*,\ldots,y_n^* \in \mathrm{RngRng}(a^*)\ \forall j \le n$$

$$(\{e\}(j)\ r\ (y_j \in a \ \&\ P(y_j,y_{j+1}))) \ \&\ y_o = x \ \&\ P(y_j,y_{j+1}) \ \&\ y_o^* = x^*)$$

This can be done in $\underset{\sim}{B}$ since the formula being proved by induction is a Δ_o-formula. Apply countable choice to get two functions $\bar{f}$ and $\bar{h}$ such that $\bar{f}(n) = \langle y_o,\ldots,y_{n-1}\rangle$ and $\bar{h}(n) = \langle y_o^*,\ldots,y_{n-1}^*\rangle$ as in (*). Then the functions f and h themselves can be defined. Let f^* be defined by

$$f^* = \{\langle u,\langle n,y\rangle,\langle n,y^*\rangle\rangle \in \omega \times (\omega \times a) \times (\omega \times \mathrm{RngRng}(a^*)) :$$

$$y = f(n) \ \&\ y^* = h(n)\}$$

Then we will show that, with f and f^* substituted in, the conclusion of the dependent choices axiom is true and realized; that is,

$$(\mathrm{Fcn}(f) \ \&\ \mathrm{Dom}(f) = \omega \ \&\ f(0) = x \ \&\ \forall n \in \omega\ P(f(n),f(n+1) \ \&\ \mathrm{Rng}(f) \subseteq a)$$

is true and realized.

First we show $\forall n \in \omega\ P(f(n),f(n+1))$ is true and realized. The truth follows from the last clause in the formula (*) used to define f. For the realizability, we have to use the definition of ω^*, which tells us that a realizer of $n \in \omega$ contains n, specifically, if $s\ r\ n \in \omega$ then $(s)_o = n$. Now, $P(f(n),f(n+1))$ is actually an abbreviation for

$$(\langle n,y\rangle \in f \ \&\ \langle n+1,w\rangle \in f \rightarrow P(y,w)). \text{ Take}$$

$$q = \Lambda s\ \Lambda p\ (\{e\}(\ (s)_o\ ))_1 \text{ ; then}$$

$$q\ r\ \forall n \in \omega\ P(f(n),f(n+1)) .$$

Next we show $\mathrm{Fcn}(f)$ is true and realized; $\mathrm{Fcn}(f)$ is $\langle n,y\rangle \in f$ & $\langle n,w\rangle \in f \rightarrow y = w$. Suppose p realizes the left side of this; then p is a pair $\langle k,j\rangle$ with $\langle k,\langle n,y\rangle,\langle n,y^*\rangle\rangle \in f^*$ and $\langle j,\langle n,w\rangle,\langle n,w^*\rangle\rangle \in f^*$; hence $y = w = f(n)$ and $y^* = w^* = h(n)$; here equality is extensional. But since $w = y$ and $w^* = y^*$, we have $y = w$ realized. Hence $\mathrm{Fcn}(f)$ is realized. $\mathrm{Dom}(f) = \omega$ is realized since if $s\ r\ n \in \omega$, then $n = (s)_o$ and

$$\langle 0,\langle n,f(n)\rangle,\langle n,h(n)\rangle\rangle \in f^*, \text{ so } \exists y \in a\ y = f(n)$$

is realized. Finally, $f(0) = x$ is realized, because $h(0) = x^*$ and $f(0) = x$. This completes the verification of the axiom of bounded dependent choices.

J. (induction). The verification of this axiom is standard.

K. (relativized dependent choice RDC). Like bounded dependent choice, except that we use full induction and separation instead of bounded induction and separation.

L. (transfinite induction). Like foundation, except that transfinite induction must be used to make the verification, instead of foundation.

M. (full separation). Like Δ_o-separation.

N. (power set). $\exists x \forall y (y \subseteq a \to y \in x)$. Take $x = \wp(a)$ and

$$x^* = \{\langle e,y,y^*\rangle \in \omega \times \wp(a) \times \wp(\omega \times \mathrm{Rng}(a^*)) :$$

$$y \subseteq a \;\&\; e \ r \ y \subseteq a\}$$

Suppose y and y^* are given and $y \subseteq a$ is realized, say by e, and $y \subseteq a$. Then we have to check that y^* is a subset of $\omega \times \mathrm{Rng}(a^*)$. Suppose $\langle p,z,z^*\rangle \in y^*$. Since $y \subseteq a$ is realized, for some p_o, we have $\langle p_o,z,z^*\rangle \in a^*$; hence $\langle z,z^*\rangle \in \mathrm{Rng}(a^*)$. This completes the verification of power set.

O. (abstraction). For A a Δ_o-formula,

$$\forall x \exists z \forall w (w \in z \leftrightarrow \exists y (y \in x \;\&\; \forall u (u \in w \leftrightarrow A(u,y) \;\&\; u \in x))).$$

Take $z = \{\{u \in x : A(u,y)\} : y \in x\}$, formed by abstraction. We want to prove the existence of z^* such that

$$\langle e,w,w^*\rangle \in z^* \leftrightarrow \exists y,y^* (e \ r \ (y \in x \;\&\; \forall u (u \in w \leftrightarrow A(u,y) \;\&\; u \in x))).$$

To prove that z^* exists, we again need to use the equivalence relation, $w^* \sim v^*$ iff w extensionally $= v$ is realized iff

$$\exists p,q \in \omega \forall e,u,u^* ((\langle e,u,u^*\rangle \in w^* \to \langle \{p\}(e),u,u^*\rangle \in v^*) \;\&$$
$$(\langle e,u,u^*\rangle \in v^* \to \langle \{q\}(e),u,u^*\rangle \in w^*).$$

Note that $\sim$ is given by a Δ_o-formula. Now, by abstraction, we can form the set

$$P = \{\{\langle q,u,u^*\rangle : q \ r (A(u,y) \;\&\; u \in x)\} ; \langle c,y,y\rangle \in \omega \times \mathrm{Rng}(x^*)\}.$$

Now define $z^* = \{\langle e,w,w^*\rangle : \exists v^* \in P (v^* \sim w^* \;\&$

$$\exists \langle y,y^*\rangle \in \mathrm{Rng}(x^*) (e \ r \ (y \in x \;\&\; \forall u (u \in w \to A(u,y) \;\&\; u \in x)))))\}$$

In order to show that the z^* so defined is the z^* we are seeking, we have to prove that if $\exists y,y^* (e \ r (y \in x \;\&\; \forall u (u \in w \leftrightarrow A(u,y) \;\&\; u \in x)))$, then for some v^* in P, $v^* \sim w^*$. Let $v^* = \{\langle q,u,u^*\rangle \in \omega \times \mathrm{Rng}(x^*) : q \ r (A(u,y) \;\&\; u \in x)\}$; then $w^* \sim v^*$. Since $\langle (e)_o,y,y^*\rangle \in x^*$, we have $v^* \in P$. This completes the verification of abstraction, and with it, the proof of the soundness theorem 4.1.

We turn now to the auxiliary theories Ta and Tb, and discuss the notion of realizability appropriate for them. These theories, as defined in §2, depend on a particular definable metric space X, which is a subset of N^N with a metric in "standard form" as discussed in §2, the new constant $\underline{a}$ or $\underline{b}$ stands for an element of X, that is, an element of N^N satisfying an additional condition.

Instead of recursive functions, we use functions recursive in $\underline{a}$ (or $\underline{b}$, as the case may be) to realize the theory Ta (or Tb). The theory of functions recursive in $\underline{b}$ can be formalized in Tb, and the verification that all the set-theoretical axioms are realized proceeds exactly as in theorem 4.1, using $\{e\}^{\underline{b}}$ in place of $\{e\}$ throughout. This leads to

Theorem 4.2

If q-realizability is defined using functions recursive in $\underline{b}$, then the interpretation is sound for Tb; similarly for Ta. (Here T is any non-extensional set theory considered in this paper.)

Proof:

Actually, we first have to give a complete description of the interpretation. We have to explain what $e \ r \ \underline{b} \in x$ and $e \ r \ x \in \underline{b}$ are. We shall take $e \ r \ \underline{b} \in x$ to be $\langle e,\underline{b},\underline{b}^*\rangle \in x^*$ and $e \ r \ x \in \underline{b}$ to be $\langle e,x,x^*\rangle \in \underline{b}^*$. Here $\underline{b}^*$ is a particular set (more precisely, a particular term of our non-extensional set theory Tb); to be explicit, $\underline{b}^* = \{\langle k,\langle n,m\rangle,\langle n,m\rangle\rangle : \underline{b}(n) = m\}$. Thus $(e \ r \ \underline{b}(n) = m) \leftrightarrow b(n) = m$. All the logical and set-theoretical axioms can now be verified exactly as before. It remains only to check the extra axioms involving $\underline{a}$ or $\underline{b}$. First, consider the axiom $\underline{a} \in X$. This has the form $\mathrm{Fcn}(\underline{a}) \;\&\; \forall n \in \omega \exists m \in \omega (\langle n,m\rangle \in a) \;\&\; \forall n,m \in \omega (\rho(a_n,a_m) < 1/m + 1/n)$; here ρ may be taken to be some recursive function, as discussed in §2. The second of these three clauses is realized essentially by $\underline{a}$ itself; of course $\underline{a}$ is extensionally equal to $\{e\}^{\underline{a}}$ for a certain number e.

Next, consider the axioms $\underline{b}(\bar{n})=\bar{m}$ for $b(n)=m$. (Remember that Tb is based on a particular function b, while Ta is not based on any particular a.) We have $(e \ r \ \underline{b}(\bar{n})=\bar{m}) \rightarrow \underline{b}(\bar{n})=\bar{m}$, so that these axioms also are realized in Tb. This completes the proof of Theorem 4.2.

§5. Explicit Definability

In this section we consider the old metamathematical property, if $T \vdash \exists n \in \omega \, P(n)$, then for some n, $T \vdash P(\bar{n})$. We call this the "numerical explicit definability" property. Our goal is to derive various formalized versions of this property for set theories T and the auxiliary theories Ta and Tb, which will suffice to get the desired continuity rules. A few general remarks are in order. The numerical explicit definability property should be compared and contrasted with the set explicit definability property, if $t \vdash \exists x \, P(x)$ then for some explicitly defined $\hat{x}$, $T \vdash P(\hat{x})$. (One might give different meanings to the words "explicitly defined" here; but for example, any set given by a term of our non-extensional set theories is explicitly defined.) These explicit definability properties are already known for certain intuitionistic set theories (See [Fr 3], [M1], and [M3].) These theories have replacement instead of collection. (However, the double-negation interpretation has not been made to work for ZF with replacement, but has been made to work for ZF with collection; see [Fr1].) Friedman and Myhill use a variant of Kleene's "slash", which becomes quite complicated because extensionality is dealt with directly. This realizability is not enough for the needs of the present paper, because it is not recursive, and it is not easily formalized.

Numerical explicit definability results for the auxiliary theories Ta and Tb provide information generalizing what is usually known as "Church's rule", which says that if $\forall n \exists m \, A(n,m)$ is provable, then for some e, $\forall n \, A(n,\{\bar{e}\}(n))$ is provable. If we take the complete separable metric space X to be the integers N, then to say $\forall n \exists m \, A(n,m)$ is provable (the hypothesis of Church's rule) is to say that Ta proves $\exists m \, A(\underline{a},m)$ (the hypothesis of explicit definability for Ta). In the case $X=N^N$ or $X=2^N$ (not to mention the reals or certain function spaces) we get other interesting information. The exact form of these results will be given below. We begin with the most straightforward explicit definability theorem.

<u>Theorem 5.1</u> Let T be one of the non-extensional set theories considered in this paper. If $T \vdash \exists x \in \omega P(x)$ then $T \vdash P(\bar{n})$ for some numeral $\bar{n}$.

<u>Proof</u>: Suppose $T \vdash \exists x \in \omega \, P(x)$. Then, by the soundness of q-realizability for T, there is some e such that $T \vdash \bar{e} \ r \ \exists x \in \omega \, P(x)$. That is, T proves $\exists x, x^*(\bar{e}_0 \ r \ x \in \omega \ \& \ \bar{e}_1 \ r \ P(x) \ \& \ P(x))$, where $e = \langle e_0, e_1 \rangle$. According to the definition of ω^*, T proves $(s \ r \ x \in \omega \rightarrow x=(s)_0)$. Hence T proves $P((\bar{e}_0)_0)$. Now since

T contains arithmetic, T proves $(\bar{e})_0=\bar{n}$, where $n=(e)_0$. Hence T proves $P(\bar{n})$, which completes the proof.

Now consider explicit definability for extensional theories. Our methods yield numerical explicit definability, not for all formulas P, but only for P of the form $x \in Q$, where Q is a specific definable set. We take "definable set" to mean "set given by one of the terms of the non-extensional set theory T". What we would ideally want is a larger system of terms, adequate to prove the set explicit definability theorem. In trying to get such a system of terms, there is a problem in that the choice and collection axioms assert the existence of a set, without there being any obvious definable one. This is why the set explicit definability property is known only for theories with replacement, and not for theories with collection. Although this is an interesting phenomenon, we regard it as a side issue, since our focus here is on continuity rules. We therefore restrict our attention to sets defined by terms. Actually, we could include exponentiation terms as well; if this is done, the definable sets seem to encompass most sets needed for mathematical practice.

<u>Lemma 5.1</u> Let T be one of the extensional set theories considered in this paper; let A^* be the interpretation of A in the non-extensional set theory T-ext, given in Theorem 3.1. Let Q be a definable set in T. Then $T \vdash (x \in Q)^* \leftrightarrow x \in Q$.

<u>Proof</u>: First we must explain precisely what is meant by $(x \in Q)^*$. Here Q is a term, which belongs to T-ext, but not to T; so $x \in Q$ must be interpreted as the formula of T obtained by writing out the definitions of the terms composing Q. Secondly, if A has a free variable x, then A^* has two free variables, x and y, where y is supposed to "witness" that x is a set. (Technically x is interpreted as the pair $\langle x,y \rangle$.) Thus $(x \in Q)^* \leftrightarrow (x \in Q)$ really means $((x \in Q)^*(x,y) \rightarrow x \in Q)$ & $(x \in Q \rightarrow \exists y (x \in Q)^*(x,y))$. Now, the proof proceeds by induction on the complexity of the term Q. For instance, if Q is $\{x \in a: P(x)\}$, where a is a term and P is Δ_0, we have $(x \in Q)^*(x,y) \leftrightarrow S(\langle x,y \rangle) \ \& \ (x \in a)^* \ \& \ P^*(x,y)$, where S is the formula defining the sets of the model of Section 3. According to Theorem 3.1(i), we have $S(\langle x,y \rangle) \rightarrow (P(x,y) \leftrightarrow P^*(x,y))$; so $(x \in Q)^*(x,y) \leftrightarrow S(\langle x,y \rangle) \ \& \ (x \in a)^* \ \& \ P(x,y)$. By induction hypothesis $(x \in a)^* \leftrightarrow x \in a$, since we can prove by induction on terms that the transitive closure of a definable set is definable, so that $S(\langle a,b \rangle)$ for some term b. Thus $(x \in Q)^*(x,y) \leftrightarrow S(\langle x,y \rangle) \ \& \ x \in a \ \& \ P(x,y)$, i.e. $\leftrightarrow S(\langle x,y \rangle) \ \& \ x \in Q$. But $x \in Q$ implies $\exists y \, S(\langle x,y \rangle)$, since TC(x) can be defined if x is known to belong to some transive set, and as we have just mentioned, the transitive closure of Q is definable. Thus $x \in Q \rightarrow \exists y (x \in Q)^*(x,y)$ and $(x \in Q)^*(x,y) \rightarrow x \in Q$. For reasons of space limitation we omit the other cases in the induction on Q.

<u>Theorem 5.2</u> Let T be one of the set theories discussed in this paper, including extensionality. Suppose $T \vdash \exists x \in \omega (x \in Q)$, where Q is a definable set in T. Then for some numeral $\bar{n}$, $T \vdash \bar{n} \in Q$.

<u>Proof</u>: Suppose $T \vdash \exists x \in \omega (x \in Q)$. Then, by Theorem 3.1, $T\text{-ext} \vdash (\exists x \in \omega (x \in Q))^*$. Hence $T\text{-ext} \vdash \exists x \in \omega (x \in Q)^*$, since $x \in \omega$ is equivalent to its *-interpretation, by Theorem 3.1 (ii). By Theorem 5.1, $T\text{-ext} \vdash (\bar{n} \in Q)^*$, for some numeral $\bar{n}$. Hence, by Lemma 5.1, $T \vdash \bar{n} \in Q$. This completes the proof.

Next we turn to explicit definability results for the auxiliary theories Ta and Tb.

<u>Theorem 5.3</u> (i) Suppose $Tb \vdash \exists x \in \omega P(x)$, where T is without extensionality. Then $Tb \vdash P(\bar{n})$, for some numeral $\bar{n}$. If T has extensionality, then the same result holds for P of the form $x \in Q$, where Q is a definable set in T.

(ii) Suppose $Ta \vdash \exists x \in \omega\, P(x)$, where T is without extensionality. Then for some numeral $\bar{e}$, $Ta \vdash \{\bar{e}\}^{\underline{a}}(0) \in \omega$ & $P(\{\bar{e}\}^{\underline{a}}(0))$. If T has extensionality, the same result holds for P of the form $x \in Q$, where Q is a definable set in T.

<u>Proof</u>: Exactly like Theorems 5.1 and 5.2, appealing to the realizability used in Theorem 4.2 instead of 4.1. For (i), we also have to observe that in Tb, if $\{\bar{e}\}^{\underline{b}}(0) \in \omega$ is provable, then for some numeral $\bar{n}$, $\{\bar{e}\}^{\underline{b}}(0)=\bar{n}$ is provable. This is proved just like the corresponding result for T; it consists in observing that the axioms of Tb suffice to formalize the computations of a Turing machine; when a value $b(\bar{n})$ is called for in the course of a computation, one of the axioms of Tb is there to formalize the step in which the "oracle" answers. Of course, this cannot be carried out in Ta, which is why the theorem takes the form it does. This completes the proof of the theorem.

<u>Formalized Explicit Definability</u>

We have to discuss the formalization of the preceding results on explicit definability. They cannot be formalized as they stand (see the general discussion in [B1]), but instead we have to show that there is a sequence of subsystems T_n of each set theory T, such that the explicit definability theorems for T_n can be proved in T, for each fixed integer n. This may not be possible for systems T which have only restricted induction. Here we carry it out for the other theories considered in this paper, which have full induction.

The complexity of a formula of set theory is an integer defined by induction so that prime formulae have complexity zero, and the complexity increases by one at each logical connective and quantifier. We can, for each fixed n, introduce a truth-definition Tr_n (a formula of two free variables; one of which is a number variable, i.e. a variable bound to ω, and prove $T \vdash Tr_n('A',x) \leftrightarrow A(x)$. To be

technically precise, we have to worry about the fact that A can have more than one free variable (x can be a list of variables), and code these variables into the single variable on the left, so that we should actually say

$$T \vdash Tr_n('A',x) \leftrightarrow A((x)_1, \ldots, (x)_m).$$

We neglect this distinction where it is safe to do so. The construction of Tr_n is standard; $Tr_n('A',x)$ is a disjunction, according to the finitely many possible forms of A.

If T is one of our set theories, let T_n be T with all proofs restricted to contain formulae of complexity $\leq n$; and the axioms of T_n are those axioms of T which are of complexity $\leq n$ and occur among the first n axioms of T in some natural enumeration. Thus T_n has finitely many axioms. Note that T_n is not a formal system in the usual sense, since a formula of complexity $\leq n$ might be provable from axioms of complexity $\leq n$, but only through intermediate steps of greater complexity. Nevertheless, T_n is useful for our purposes (chiefly because it saves us from having to use formalized cut-elimination theorems, which in some cases are not even proved yet). By the reflection principle for S, we mean

$$Pr_S('A') \rightarrow A \quad \text{, for all formulae A.}$$

<u>Lemma 5.2</u> Let T be one of the set theories considered in this paper with full induction (i.e. T_2, T_3, T_4, Z, ZF^- + RDC). Then T proves the reflection principle for T_n, for each fixed n.

<u>Proof</u>: We have $Tr_n('A') \leftrightarrow A$; and we prove $Prf_n(j,'P') \rightarrow Tr_n('A')$ by induction on j. It seems that bounded induction will not suffice.

By the 1-consistency of a theory S, (terminology due to Kreisel and Levy) we mean, for A recursive with one free variable,

$$Pr_S(\exists n \in \omega\ A(n)) \rightarrow \exists n \in \omega\ Pr_S('A(\bar{n})').$$

For such A, we have $A(n) \rightarrow Pr_S('A(\bar{n})')$, if S contains a modicum of arithmetic, and indeed this fact itself is provable in any theory which proves that S contains a modicum of arithmetic. Hence 1-consistency follows from the reflection principle for S, for all the S we have reason to consider.

Next we discuss the formalization of the soundness theorems for q-realizability. Let us first discuss what goes wrong with a straightforward attempt to formalize the theorem, $Pr('A') \rightarrow \exists e\ Pr('\bar{e}\ r\ A')$. One would try to prove this by induction on the length of the proof of A; the induction step involves proving

$$Pr('\bar{a}\ r\ (A \rightarrow B)')\ \&\ Pr('\bar{b}\ r\ A') \rightarrow \exists m\ Pr('m\ r\ B').$$

Now from Pr('a r (A → B)') & Pr('b r A'), we easily get Pr('∃n(Tabn & U(n) r B)'). To pass from that to the desired conclusion of the induction step requires the 1-consistency of the theory. However, this is the only obstacle to the straightforward formalization of the proof. In other words, if we have the 1-consistency of a theory S, and we can prove the axioms of S are realized, then we can prove the soundness theorem for S, using nothing more complicated than bounded induction. Thus we obtain

Lemma 5.3 (Formalized realizability). Let T be one of the non-extensional set theories discussed in this paper, and let T_n be as in Lemma 5.2. Then for each fixed n, there is some n* such that

$$T \vdash (Pr_n('A') \to \exists e\, Pr_{n^*}('\bar{e}\ r\ A')),$$

where Pr_n is the provability predicate of T_n.

Remark: By writing Pr('$\bar{e}$ r A'), to be perfectly explicit, we mean Pr_n(Sub(Num(e),'x r A'), where Num is a primitive recursive function producing from e a Gödel number of the numeral $\bar{e}$, and Sub is a function producing a Gödel number of P(t) from Gödel numbers of a term t and a formula P.

Proof: As sketched above, we go by induction on the length of the proof of A, using 1-consistency in the induction step for modus ponens; it is provided by Lemma 5.2. The use of n* on the right in place of n is necessary because the complexity of e r A is usually greater than that of A. We also have to check that for each axiom A of T_n, T proves $\exists e\, Pr_{n^*}(\bar{e}\ r\ A)$. If n* is chosen large enough, this will be a true Σ^o_1 sentence, by Theorem 4.1; therefore provable in arithmetic, hence in T. (Here we use that T_n has only finitely many axioms.) This completes the proof of the lemma.

Lemma 5.4 (Formalized explicit definability). Let T be one of the (non-extensional) set theories discussed in this paper. Let Ta_n and Tb_n be as in Lemma 5.2. For each fixed n, there is an n* such that

(i) $T \vdash (Pr_n('A') \to \exists e\, Pr_{n^*}('\bar{e}\ r\ A'))$, where Pr_n is the provability predicate of T_b.

(ii) $T \vdash (Pr_n('A') \to \exists e\, Pr_{n^*}('\{\bar{e}\}^{\underline{a}}(0)\ r\ A'))$, where Pr_n is the provability predicate of Ta_n.

Proof: Like Lemma 5.3, appealing to Theorem 4.2 instead of Theorem 4.1.

Theorem 5.4 (Formalized explicit definability).

Let T be any of the set theories discussed in this paper. Then T proves numerical explicit definability for T_n, Tb_n, and Ta_n, for each fixed n. To be precise, if Theorems 5.1, 5.2, and 5.3 are altered by changing T, Tb, and Ta to T_n, Tb_n, and Ta_n in the hypothesis and to T_{n^*}, Tb_{n^*}, and Ta_{n^*} in the conclusion, then the resulting statements are provable in T, for some n* depending on n.

Proof: We first choose n* so large that T_{n^*} will prove $s\ r\ x \in \omega \to (s)_o = x$. Now the proof of Theorem 5.1 can be formalized directly, appealing to Lemma 5.3 where the soundness of q-realizability is used. Next, note that the proof of Lemma 5.1 can be formalized in T (with T replaced by T_n in the conclusion), since the statement proved by induction there is Δ_o. The appeal to Theorem 3.1 is all right because, as noted in Theorem 3.1 (iii), the relevant part of Theorem 3.1 can be proved in arithmetic; so if n* is chosen large enough, Theorem 3.1 will be provable in T_{n^*}. Now the proof of Theorem 5.2 can be directly formalized, appealing to Lemma 5.4 where the soundness of q-realizability for Ta and Tb is needed. This completes the proof of Theorem 5.4.

§6. Uniform Continuity and Forcing.

The results of the previous sections are sufficient to establish the derived rules concerning local continuity, but not those concerning local uniform continuity. It is worth reviewing the reasons why the preceding results are not sufficient. What we need to establish is condition (iii) of [B1], which says roughly that each provably recursive function from a compact metric space X to the integers N is provably uniformly continuous (hence provably bounded). Now, as discussed in [B1], we cannot hope to prove all functions from X to N are uniformly continuous in any theory consistent with Church's thesis, because there is a recursive functional defined on all recursive members of 2^N, but not uniformly continuous there. (To compute this functional at an argument y, examine the values $y(0), y(1)\ldots$ until you come to $y(n)$ such that in n steps of computation, you can verify that y cannot be a separation of two fixed recursively inseparable r.e. sets; then set the output equal to $y(n+1)$.) This is our first observation. Our second observation is that any <u>provably</u> recursive functional can be proved to be continuous, by the derived rules which follow from the results already proved; hence, <u>classically</u>, it is uniformly continuous, since X is compact. However, this is not enough; we want to know that it is <u>provably</u> uniformly continuous.

Our solution to this problem lies in using forcing to add a generic real to the universe; any function which is defined on all members of a compact space, including generic ones, will have to be uniformly continuous. We used forcing in [B1] to establish these rules for Feferman's theories; here we apply a similar technique to Friedman's theories. It turns out to be rather complicated to give a suitable definition of forcing that works for the exponentiation axiom, although for theories containing power set it is straightforward.

Suppose the compact space X, whose members are the members x of N^N with $x(n) \le M_n$, for some fixed recursive sequence M_n, is fixed once and for all. Let C be the set of finite sequences of integers $p = \langle p_0, p_1, \ldots p_n \rangle$ such that $p_i \le M_i$. We use the usual notations $(p)_i$ for p_i, $lh(p)$ for $n+1$; and we use the notation (borrowed from forcing) $p \le q$ to mean that q is an initial segment of p (so p gives more information than q). No harm will result from using $\emptyset$ to denote the empty sequence. We use p, q, and r for members of C; thus $\forall p$ means $\forall p \in C$. We are going to assign to each formula A of a set theory $T\dot{a}$ (with an extra constant <u>a</u> for a member of X), a formula $p \Vdash A$ of T (without <u>a</u>), which is read "p forces A". The free variables of $p \Vdash A$ are p together with x', where x are the free variables of A. (The use of x' here is purely for intelligibility; we may technically assume x' is the same variable as x.) We write $p, n \Vdash A$ to abbreviate $\forall q \le p(lh(q) \le n + lh(p) \to q \Vdash A)$. We are now ready to give the clauses defining the forcing interpretation that works for theories containing the power set axiom (below we shall discuss

the modifications needed to treat theories with the exponentiation axiom).

$p \Vdash A \vee B$ is $p \Vdash A \vee p \Vdash B$

$p \Vdash A \& B$ is $p \Vdash A \ \& \ p \Vdash B$

$p \Vdash \exists x A$ is $\exists x' p \Vdash A$

$p \Vdash \forall x A$ is $\forall x' \exists n (p, n \Vdash A)$

$p \Vdash (A \to B)$ is $\forall q \le p (q \Vdash A \to \exists n (q, n \Vdash B))$

$p \Vdash x \in y$ is $\langle p, x' \rangle \in y'$

$p \Vdash x = y$ is $\forall w (w \in x' \leftrightarrow w \in y')$; $p \Vdash \bot$ is $\bot$

These last clauses will also serve to define what it means for p to force an atomic formula containing terms of the non-extensional set theories, once we associate to each such term t another term t' to use in these clauses. As in the soundness proof for realizability, the choice of t' will be apparent in the course of the soundness proof for forcing, and we postpone the definitions of the terms t' until then. We do, however, now give the term <u>a</u>' which is necessary in order that the above clauses should determine what it means to force an atomic formula involving <u>a</u>. Namely, $\underline{a}' = \{\langle p, \langle n, m \rangle \rangle : n < lh(p) \ \& \ m = (p)_n\}$. Note that <u>a</u>' does not involve <u>a</u>, so that generally $p \Vdash A$ is a formula without <u>a</u>.

<u>Remark</u>: We have logic with no negation symbol, and instead a falsum symbol in terms of which negation can be defined. The above definition shows that $p \Vdash \neg A$ iff $\forall q \le p \neg q \Vdash A$, which is the usual clause. Since $\forall$ and $\to$ are classically superfluous, if we use classical logic our definition reduces to the usual notion of forcing.

Our next goal is to give the modification of the above interpretation that will suffice for theories with the exponentiation axiom. We introduce some

<u>Notation</u>: $C_p = \{q \in C : q \le p\}$

$q \le_n p$ means $q \le p \ \& \ lh(q) = n + lh(p)$

$x'/p = \{\langle q, u' \rangle \in x' : q \le p\}$; x'/p is read "x' restricted to p".

Our forcing interpretation will be defined in the following way: we shall first associate to every formula A with free variables x another formula $R_A(p, x')$; we then write $p \Vdash A$ to abbreviate $R_A(p, x'/p)$. In what follows, we make the convention that the variables x' are restricted to so-called "good sets"; that is, $\forall x'$ means $\forall x'(G(x') \to$, and $\exists x'$ means $\exists x'(G(x') \ \&$, where $G(x')$ is a Δ_0 formula defining the good sets, as follows. With ~ for extensional equality, $G(y')$ is $\forall p \forall q \le p \forall w (\langle p, w \rangle \in y' \to \exists v (\langle q, v \rangle \in y' \ \& \ v \sim w/q))$. It is not obvious that any good sets exist; we shall encounter our first ones in Lemma 6.3 below. Now here are the clauses defining the formulae $R_A(p,x')$:

$R_{A\&B}$ is $R_A \& R_B$; $R_{A\vee B}$ is $R_A \vee R_B$; $R_{\exists yA}$ is $\exists y'A$

$R_{x\in y}(p,x',y')$ is $\langle p,x'\rangle \in y'$

$R_{x=y}(p,x',y')$ is $x'=y'$; $R_\bot(p)$ is $\bot$

$R_{\forall y}(p,x')$ is $\forall y' \exists n \forall q \leq_n p\ R_A(q,x'/q)$

$R_{A\to B}(p,x')$ is $\forall q \leq p(R_A(q,x'/q) \to \exists n \forall r \leq_n q\ R_B(r,x'/r))$

Now, using the abbreviation $p \Vdash A$ for $R_A(p,x'/p)$, the clauses for implication and universal quantification can be rewritten in exactly the form we gave for the simpler version of forcing!

Lemma 6.1 If $p \Vdash A$ and $q \leq p$ then $q \Vdash A$.

Proof: A straightforward induction on the complexity of A, using crucially that the primed variables are restricted to good sets, which in fact is built into the definition just to make the atomic case of this lemma work.

Lemma 6.2 If $p,j \Vdash A$ and $p,m \Vdash (A\to B)$, then for some k, we have $p,k \Vdash B$.

Proof: Let $n = \max(k,j)$; by Lemma 6.1, $p,n \Vdash A$ and $p,n \Vdash (A\to B)$. So for each $q \leq_n p$, $q \Vdash (A\to B)$. Hence, for each $q \leq_n p$, $\exists i(q,i \Vdash B)$. Now there are only finitely many $q \leq_n p$. Let k_o be larger than any of the values of i which work for these finitely many q. Then $q \leq p \to q,k_o \Vdash B$. Set $k = n+k_o$. Then $p,k \Vdash B$.

Lemma 6.3 $p \Vdash n \in \omega$ iff $n \in \omega$; more precisely, there is for each n a term n' such that $p \Vdash n \in \omega$ implies $n \in \omega$, and $n \in \omega$ implies $p \Vdash n \in \omega$ with n' substitituted for the corresponding free variable of the formula $p \Vdash n$.

Remark: This is the analogue of saying $n \in \omega$ is "self-realizing". We might call a formula with this property "self-forcing".

Proof: We first define a function n' of n for use in the lemma, by induction: $0'$ is $\emptyset$, and $(n+1)'$ is $\{\langle p,u'/p\rangle : u \leq n\}$. By restricted induction on n, one proves that for all integers n, n' is a good set. We now define ω' (which we promised to do in order to complete the definition of forcing) as $\{\langle p,n'/p\rangle : n\in\omega\}$. The assertions of the lemma may now be proved by a straightforward induction on n.

Lemma 6.4 Let A be an arithmetical predicate. Then for $x \in \omega$, $p \Vdash A(x)$ iff $A(x)$.

Proof: By induction on the complexity of A. The basis case consists of the relations $x=y+z$, $y=x\cdot z$, and successor. These relations have their set-theoretical definitions, so matters are technically complicated. Consider how to prove $(p \Vdash x=y+1$ iff $y=y+1)$. This is done by induction on x, first proving $p \Vdash 0 \in z$ by induction on z. Then we proceed to + and $\cdot$, just as in the set-theoretical development of arithmetic. (This can all be done in $\underset{\sim}{B}$).

Next we do the induction step of the lemma, in which A is, for instance, $\forall z \in \omega B(x,z)$. Suppose $p \Vdash A(x)$; then for some n, we have $p,n \Vdash (z \in \omega \to B(x,z))$. Let $z \in \omega$ be given; using z' produced in Lemma 6.3, we have $p \Vdash z \in \omega$; hence for

some j, we have $p,j \Vdash B(x,z)$. Hence $B(x,z)$, by the induction hypothesis. Since z was arbitrary, we have $A(x)$. Conversely, suppose $A(x)$. We will show $\emptyset \Vdash A(x)$; that is, $\emptyset \Vdash \forall z \in \omega\, B(x,z)$. We claim $q \Vdash z \in \omega$ implies $q \Vdash B(x,z)$. Indeed, if $q \Vdash z \in \omega$ then $z \in \omega$, so $B(x,z)$; hence, by induction hypothesis, $\emptyset \Vdash B(x,z)$.

Lemma 6.5 If y' is substituted for x' in the formula $p \Vdash A(x)$, the result is logically equivalent to $p \Vdash A(y)$. In other words $R_{A(x)}(p,x'/p)$ is equivalent to $R_{A(y)}(p,y'/p)$.

Proof: By induction on the complexity of A.

We are now ready to state the soundness theorem for forcing. Let Ta be the auxiliary theory described in §2, with a constant a for an element of the compact space X.

Theorem 6.1 Let T be any of the non-extensional set theories discussed in this paper, except B-ext. Thus T can be (non-extensional) T_1, T_2, T_3, T_4, Z, or ZF^-+RDC. Then $Ta \vdash A$ implies $T \vdash \exists n(\emptyset,n \Vdash A)$.

Proof: By induction on the length of the proof of A. We have to check the logical axioms and rules, then the set-theoretical axioms. We begin with modus ponens. Suppose $\emptyset,n \Vdash A$ and $\emptyset,m \Vdash (A\to B)$; then by Lemma 6.2, $\emptyset,k \Vdash B$ for some k. (Note that Lemma 6.2 was proved within $\underset{\sim}{B}$.) We leave the reader to check the other propositional axioms and rules (using e.g. the list on page 3 of [Tr]). We turn to the quantifier axioms and rules. Consider the axiom $(\forall xAx \to At)$, for some term t. Consider first the case when t is a variable. Suppose $q \Vdash \forall xAx$; that is, $\forall x' \exists n \forall p \leq_n q\ R_A(p,x'/p)$. Substitute t' for x'; then for some n we have $q,n \Vdash A(t)$, using Lemma 6.5. The case in which t is a term other than a variable is handled the same way, provided we have at hand a corresponding term t' with the property of Lemma 6.5. We shall give, in the course of verifying the set-theoretical axioms, such a term t' for each term t. The other quantifier axioms and rules can be treated similarly.

We now turn to the non-logical axioms, beginning with the axiom $\underline{a} \in X$, which has the following form when written out:

$$\forall n \in \omega \exists m \in \omega(\langle n,m\rangle \in \underline{a}\ \&\ m \leq M_n)\ \&\ \forall x \in a \exists n,m \in \omega(x=\langle n,m\rangle)\ \&$$

$$\forall n,m(\langle n,m\rangle \in a\ \&\ \langle n,r\rangle \in a \to m=r)\ \&\ \forall n,m(\rho(\underline{a}(n),\underline{a}(m)) \leq 1/(n+1)+1/(m+1))$$

where ρ is some recursive function and M_n is some recursive sequence (see §2). First we show $\emptyset \Vdash \forall n \in \omega \exists m \in \omega(\langle n,m\rangle \in \underline{a}\ \&\ m \leq M_n)$. Let n and p be given; we claim $p \Vdash n \in \omega \to \exists m \in \omega(\langle n,m\rangle \in \underline{a}\ \&\ m \leq M_n)$. Suppose $q \leq p$ has $q \Vdash n \in \omega$; then $n \in \omega$, by Lemma 6.3; we must show $q,j \Vdash \exists m \in \omega(\langle n,m\rangle \in \underline{a}\ \&\ m \leq M_n)$ for some j. Take $j=M_n+1+n$. Then $r \Vdash \langle n,m\rangle \in \underline{a}$. Also by Lemma 6.4, $r \Vdash m \leq M_n$, since by definition of C, $m \leq M_n$ is true.

Next consider the conjunct $x \in \underline{a} \rightarrow \exists n,m \in \omega(x=\langle n,m\rangle)$. Suppose $p \Vdash x \in \underline{a}$; then $x=\langle n,m\rangle$ where $n<lh(p)$ & $m=(p)_n$. Thus $p \Vdash n \in \omega$ & $m \in \omega$ & $(x=\langle n,m\rangle)$ by Lemmas 6.3 and 6.4. Hence $\emptyset \Vdash (x \in \underline{a} \rightarrow \exists n,m \in \omega(x=\langle n,m\rangle))$. The last two conjuncts can be verified similarly; this completes the verification of the axiom $\underline{a} \in X$.

We now turn to the set-theoretical axioms. Consider an axiom of the form $\exists x \forall y(y \in x \leftrightarrow A(y))$. Pairing, union, separation, exponentiation, and power set are all of this form. If we can form $x' = \{\langle p,y'/p\rangle: p \Vdash A(y)\}$, then this axiom will be forced by $\emptyset$, as is easily checked. (If we had not been so careful in our definition, we would have to form $\{\langle p,z'\rangle: p \Vdash A(y)\}$; which cannot be done for the exponentiation axiom.) We now check the axioms of this form one by one.

Separation: Here we have to form $x' = \{\langle p,y'/p\rangle: p \Vdash (B(y)\ \&\ y \in a)\}$; that is, $\{\langle p,y'/p\rangle \in a' : p \Vdash (B(y)\ \&\ y \in a)\}$. This can be formed using separation and abstraction. To check Δ_o-separation, we have to prove that $p \Vdash B(y)$ is a Δ_o formula if B is; this is a simple induction on the complexity of B.

The definition of x' just given also determines the term t' corresponding to the term t associated with this separation axiom. It has to be checked that x' as just defined is a good set. Generally if x' is defined as $\{\langle p,y'/p\rangle: p \Vdash C\}$ then x' is good, since if $q \leq p$ and p is in x', then q also forces C, so $\langle q,y'/q\rangle$ is in x', by definition of x'; but $\langle q,y'/q\rangle$ is exactly $\langle q,(y'/p)/q\rangle$, which is what we must prove is in x' in order to show x' is good. All the terms t' which we shall exhibit in verifying the set-theoretical axioms have this form, so we need not repeat the argument in each case.

Union: Here we have to form $x' = \{\langle p,y'/p\rangle : p \Vdash \exists z(y \in z\ \&\ z \in a)\}$; that is, we want x' to consist of all $\langle p,y'/p\rangle$ such that for some z', we have $\langle p,y'/p\rangle \in z'$ and $\langle p,z'/p\rangle \in a'$. Now $\langle p,y'/p\rangle \in z'$ is equivalent to $\langle p,y'/p\rangle \in z'/p$. Hence we want x' to consist of all $\langle p,y'/p\rangle$ belonging to some u with $\langle p,u\rangle \in a'$. This set can be formed in B, using the union axiom to take a union over Rng(a).

Pairing: Take $x' = \bigcup_{q \in C} \{a'/q,b'/q\}$. Thus $\langle q,y'/q\rangle \in x'$ iff $q \Vdash (y=a \vee y=b)$.

Exponentiation: This is the most difficult axiom to verify. Here we have to show how to form $x'=\{\langle p,y'/p\rangle: p \Vdash (Fcn(y)\ \&\ Dom(y)=a\ \&\ Rng(y) \subseteq b\}$. The problem is to give in advance a set to which y'/p must belong, so that x' can be formed by separation. Suppose $\emptyset \Vdash (Fcn(y)\ \&\ Dom(y)=a\ \&\ Rng(y) \subseteq b)$; then where must y' lie? (Remember, we do not have power set available.)

Introduce an equivalence relation $\approx$ on Rng(b') by defining $z' \approx w'$ iff $\emptyset,n \Vdash z=w$ for some n. Then let $[z']$ be the equivalence class of z' under this relation; $[z']$ can be formed using Δ_o-separation. Let S_2 be the set of all $[z']$ for z' in Rng(b') (which exists by abstraction) and let S_o be the set of all finite subsets of S_2 (using exponentiation, union, and abstraction). Let S_1 be the set of all functions from a' to S_o. If f is in S_1, let F(f) be $\{\langle q,\langle w',z'\rangle/q\rangle: \langle q,w'/q\rangle \in a'\ \&\ \forall z' \forall x'([z] \in f(q,w'/q)\ \&\ [x] \in f(q,w'/q) \rightarrow z' \approx x')\}$. (We can form this set by the Δ_o-separation axiom, since the equivalence relation $\approx$ is defined by a Δ_o formula.)

It is tempting to think that if $\emptyset \Vdash Fcn(y)\ \&\ Dom(y)=a\ \&\ Rng(y) \subseteq b$, as we have assumed, then y' must be (extensionally) F(f) for some f in S_1. Now, we can check that $y' \subseteq F(f)$, where

$$f(\langle q,w'/q\rangle) = \{ [z'] : z' \in Rng(b')\ \&\ \exists r \leq q\ r \Vdash \langle w,z\rangle \in y\}.$$

(This set is finite, because $\emptyset,n \Vdash \exists! z \in b(\langle w,z\rangle \in y)$ for some n; and it can be formed using abstraction and separation. We note that "finite" means to be the range of some function defined on some integer, in the intuitionistic context; i.e. to be of bounded size.) However, possibly some $\langle q,\langle w',z'\rangle/q\rangle \in F(f)$ may not have $q \Vdash \langle w,z\rangle \in y$, although z is unique such that $q,n \Vdash \langle w,z\rangle \in y$. To solve this problem, let us say $y'_o \sim_{n,w} y'_1$ iff

$$\forall z' \in Rng(b') \forall q \leq_n \emptyset (q \Vdash \langle w,z\rangle \in y_o \leftrightarrow q \Vdash \langle w,z\rangle \in y_1).$$

So $\forall w' \in Rng(a') \exists n(y' \sim_{n,w} F(f))$. Now, if $y'_o \sim_{n,w} y'_1$, then

$$y'_o = \{\langle q,\langle w',z'\rangle/q\rangle: q \leq_n \emptyset\ \&\ \langle q,\langle w',z'\rangle/q\rangle \in y'_1 \vee q \in S\}$$

for some finite subset S of C. Since the set of all finite subsets of C exists, we can form (by Δ_o-separation) $\{y'_o: y'_o \sim_{n,w} y'_1\}$ for each n,w',y'_1. Thus

$$\forall w' \in a'(y' \in \bigcup_{n \in \omega} \{y': y' \sim_{n,w} F(f)\}), \text{ that is,}$$

$$y' \in \bigcap_{w' \in a'} \bigcup_{n \in \omega} \{y': y' \sim_{n,w} F(f)\} \quad \text{(using abstraction and union).}$$

Thus, if we form by the collection axiom the set

$$S_3 = \{ \bigcap_{w' \in a'} \bigcup_{n \in \omega} \{y':y' \sim_{n,w} F(f)\}: f \in S_1\}, \text{ then}$$

$\emptyset \Vdash (Fcn(y)\ \&\ Dom(y)=a\ \&\ Rng(y) \subseteq b)$ implies $y' \in S_3$. Similarly, we can construct a set S^p_3 such that $p \Vdash (Fcn(y)\ \&\ Dom(y)=a\ \&\ Rng(y) \subseteq b)$ implies $y'/p \in S^p_3$. Using collection again, set $S = \bigcup_{p \in C} S^p_3$, and set

$$x' = \{\langle p,y'/p\rangle \in C \times S: p \Vdash (Fcn(y)\ \&\ Dom(y)=a\ \&\ Rng(y) \subseteq b\}$$

This completes the verification of the exponentiation axiom. There seems to be no hope of eliminating the need for collection in forming S_3; abstraction is designed for collecting sets formed by separation, but here we have to collect sets formed by union. It is worth remarking that replacement would suffice in place of collection.

<u>Infinity</u>. We have already given the constant ω its associated term ω', and we have defined a function $n \mapsto n'$, in the proof of Lemma 6.3., with whose aid the formula $n \in \omega$ was proved to be "self-forcing". Let $P(z)$ be the property, $0 \in z \;\&\; \forall x(x \in z \to x+1 \in z)$, where $x+1$ is the set-theoretic successor function. Using n' and the definition of ω', we easily see that $\emptyset \Vdash P(\omega)$. Moreover, $P(z) \to n \in z$ is also forced, for if $q \Vdash P(z)$ then one proves by induction that for every integer n, some extension of q forces $n \in z$. We note that only restricted induction is required.

<u>Foundation</u>. We first note some facts about well-founded relations. We say $(R,\prec)$ is well-founded if TI holds on $(R,\prec)$ for sets (not formulae): thus the foundation axiom says that $(W,\in)$ is well-founded, for each transitive set W. Suppose (W,Q) is a well-founded relation, and $(R,\prec)$ is a relation such that for some function $F: R \to W$, we have $a \prec b \to Q(F(a),F(b))$. Then $(R,\prec)$ is well-founded. Next, let (W,Q) be a well-founded relation, and define $R_0 = Q$, R_{n+1} by $a\, R_{n+1}\, b$ iff $\exists x \in W(a\, R_n\, x \;\&\; x\, R_n\, b)$. Then each R_n (as well as their union) is a well-founded relation on W. A special case of this is when Q is $\in$; then, for example, if $\langle x,y\rangle \in z$ $y\, R_3\, z$.

Now consider the relation R on any subset of $C \times A$, defined by $\langle r,v\rangle R \langle q,u\rangle$ iff $\langle r,v/r\rangle \in u/q$. Define $F(\langle r,v\rangle) = v/r$. Then $\langle r,v\rangle R\langle q,u\rangle \to \langle r,F(\langle r,v\rangle)\rangle \in F(\langle q,u\rangle)$. Hence $\langle r,v\rangle R \langle q,u\rangle \to F(\langle r,v\rangle)\; R_3\; F(\langle q,u\rangle)$. Hence, by the general facts discussed above, R is a well-founded relation.

We now prove that the foundation axiom is forced. Suppose $p \Vdash \mathrm{Trans}(W) \;\&$ $\forall y(y \in W \;\&\; y \subseteq z \to y \in z)$. We must prove $p \Vdash \forall u(u \in W \to u \in z)$. Define for each set z', the set $z'_0 = \{\langle p,y\rangle : \exists n\, \forall q \leq_n p\, \langle q,y/q\rangle \in z'\}$. Then $p \Vdash y \subseteq z$ can be written $\forall u\, \forall q \leq p(\langle p,u/q\rangle \in y' \to \langle q,u/q\rangle \in z'_0)$, which is equivalent to $\forall u\, \forall q \leq p(\langle q,u/q\rangle \in y'_0 \to \langle q,u/q\rangle \in z'_0)$, which is equivalent to $\forall \langle u,q\rangle R\langle y'_0,p\rangle(\langle u,q\rangle R\langle z'_0,p\rangle)$. Thus $p \Vdash \forall y(y \in W \;\&\; y \subseteq z \to y \in z)$ is equivalent to $\langle y',p\rangle \in W'_0 \;\&\; \forall \langle u,q\rangle\, R\, \langle y'_0,p\rangle(\langle u,q\rangle R\langle z'_0,p\rangle) \to \langle y',p\rangle R\langle z'_0,p\rangle$. Since R is a well-founded relation, and since R is defined by a Δ_0 formula, we conclude $\forall \langle y',p\rangle \in W'_0(\langle y',p\rangle R\langle z'_0,p\rangle)$. That is equivalent to $p \Vdash \forall y(y \in W \to y \in z)$, which was what we had to prove. This completes the verification of the foundation axiom.

<u>Strong Collection</u>. Suppose $p \Vdash \forall x \in a\, \exists y\, A(x,y)$. That is,

$\forall \langle q,x'\rangle \in a'/p\, \exists y',n(q,n \Vdash A(x,y))$. Applying collection, we get some W containing a y' such that $\exists n(q,n \Vdash A(x,y))$ for each $\langle q,x'\rangle \in a'/p$, and such that each y' in W arises this way. Put $z' = \{\langle q,y'\rangle : y' \in W \;\&\; \exists x' \in \mathrm{Rng}(y')(\langle q,x'\rangle \in a \;\&\; q,n \Vdash A(x,y))\}$. Then $p \Vdash (\forall x \in a\, \exists y \in z\, A(x,y) \;\&\; \forall y \in z\, \exists x \in a\, A(x,y))$, so $\emptyset$ forces the strong collection axiom.

<u>Bounded Dependent Choice</u>. Suppose $p \Vdash \forall x \in a\, \exists y \in a\, Q(x,y) \;\&\; x \in a$. We will produce z' such that $p \Vdash (\mathrm{Fcn}(y) \;\&\; \mathrm{Dom}(z)=\omega \;\&\; z(0)=x \;\&\; \forall n \in \omega\, Q(z(n),z(n+1)))$. From the hypothesis about p, we get

$\forall \langle q,x'\rangle \in a'\, \exists y' \in \mathrm{Rng}(y')\, \exists r \leq_n q(r \Vdash y \in a \;\&\; Q(x,y))$; that is

$\forall \langle q,x'\rangle \in a'\, \exists \langle r,y'\rangle \in a'\; r \Vdash Q(x,y)$. Applying dependent choice, we get a sequence $\langle q_n,x'_n\rangle \in a'$, with $q_0 = p$ and $x'_0 = x'$, and $q_{n+1} \Vdash Q(x_n,x_{n+1})$. Now we define

$z' = \{\langle p,\langle n,x_n\rangle'/p\rangle : p \leq q_n\}$. Here n' is as defined in Lemma 6.3, and $\langle n,x_n\rangle'$ is a term built from n' and x'_n as discussed in the verification of the pairing axiom. The rest of the argument is routine.

Now only four axioms remain to be checked: (numerical) induction, power set, relativized dependent choices, and transfinite induction. We omit these verifications, since the proofs for RDC and TI are exactly like the proofs for DC and foundation, and since the proof for numerical induction is completely straightforward. As regards power set: when considering theories with power set, there is no need to use the complicated forcing interpretation given here; instead, one should return to the simpler definition first given. With that definition, the verification of power set is also completely straightforward. This completes the proof of Theorem 6.1.

<u>Lemma 6.6</u> Let A be arithmetic in $\underline{a}$, but not containing $\to$ (in its formulation as a formula of second order arithmetic). Then $p \Vdash A(\underline{a})$ if and only if $\forall f \in X(\bar{f}(\mathrm{lh}(p))=p \to A(f))$. For each fixed A this is provable in B.

Proof: By induction on the complexity of A, like Lemma 6.4.

<u>Theorem 6.2</u> Let T be any of the set theories discussed in this paper except B; T may be either extensional or non-extensional. Suppose $T\underline{a} \vdash \{\bar{e}\}^{\underline{a}}(0) \in \omega$. Then, for some m_0 and k, $T\underline{a} \vdash (\{\bar{e}\}^{\underline{a}}(0) \leq \bar{k} \;\&\; \bar{\underline{a}}(\bar{m}_0)$ determines $\{e\}^{\underline{a}}(0))$.

Remark: The phrase in the last line of the theorem means that the Turing machine computing $\{e\}^{\underline{a}}(0)$ halts using only the first m_0 values of a, and yields a value $\leq k$. This can be expressed in arithmetic using the T-predicate, without mentioning the constant $\underline{a}$, which seems to appear in the formula. Note that, in this case, m will be a modulus of uniform continuity for $\{\bar{e}\}^a(0)$ regarded as a function of a.

Proof: Suppose $Ta \vdash \{\bar{e}\}^{\underline{a}}(0) \in \omega$. By the results of Section 3, if T is extensional we can replace T by the non-extensional version, and still Ta will prove $\{\bar{e}\}^{\underline{a}}(0) \in \omega$. Hence, by Theorem 6.1, arguing in Ta, for some n, we have $\emptyset, n \Vdash \exists i, m \in \omega\, (i = \{\bar{e}\}^{\underline{a}}(0)$ is determined by $\bar{a}(m))$. Since this statement does not involve $\underline{a}$, as discussed above, it is provable in T, not just Ta. Now argue in T: Let $p \leq_n \emptyset$. Then (using Lemma 6.3), for some i and m, we have for some j,

$p, j \Vdash (i = \{\bar{e}\}^{\underline{a}}(0)$ is determined by $\underline{a}(m))$. If we choose j large enough, the same j will work for all $p \leq_n \emptyset$, so that by increasing n, we may assume that for $p \leq_n \emptyset$ we have some i and m, depending on p, such that $p \Vdash (i = \{\bar{e}\}^{\underline{a}}(0)$ is determined by $\underline{a}(m))$. Let m_o be the largest of these values of m, over all $p \leq_n \emptyset$, and let k be the largest of the values of i. Then if $p \leq_n \emptyset$, we have

$p, j \Vdash (\{\bar{e}\}^{\underline{a}}(0)$ is determined by $\underline{a}(m)$ & $\{\bar{e}\}^{\underline{a}}(0) \leq k)$, for some value of j depending on p, since the formula in the last line is a consequence of the one forced by p. Taking the maximum of these values of j, we have

$\emptyset, n+j \Vdash (\{\bar{e}\}^{\underline{a}}(0)$ is determined by $\underline{a}(m_o)$ & $\{\bar{e}\}^{\underline{a}}(0) \leq k)$.

By Lemma 6.6, this last formula is true, since it is forced by every condition of length n+j. (Note that it can be expressed without using implication.) Remembering that we have been arguing in T, we have just proved that $T \vdash \exists k \exists m_o (\{\bar{e}\}^{\underline{a}}(0)$ is determined by $\underline{a}(m_o)$ & $\{\bar{e}\}(0) \leq k)$. Applying explicit definability for T, we get $T \vdash (\{\bar{e}\}^{\underline{a}}(0)$ is determined by $\underline{a}(\bar{m}_o)$ & $\{\bar{e}\}^{\underline{a}}(0) \leq \bar{k})$ for some numerals $\bar{m}_o$ and $\bar{k}$. This completes the proof of the theorem.

7. The Main Theorems about Continuity

In the introduction, we have discussed the various derived rules related to continuity and local continuity, which form the focus of our work. In this section, we plan to establish results for intuitionistic set theories analogous to those obtained for Feferman's theories in [B1]. Those results are of two kinds: derived rules, and consistency or independence results. In the preceding sections, we have done all the necessary work to establish the general metamathematical properties which were shown in [B1] to be sufficient for the derived rules of continuity to hold. For the consistency results, we have something yet to prove in this section.

Before we give our main results, we shall fulfill the promise made in the introduction to explain further the Principle of Local Uniform Continuity. In formulating such a principle, we wish to state something like the Principle of Local Continuity, except that we want δ to depend only on ε. The most curious thing about the Principle of Local Uniform Continuity is that we cannot express exactly what we mean in the usual predicate calculus. What we really mean is

$$\left.\begin{array}{l}\forall a \in X\, \exists b \in Y \\ \forall \varepsilon > 0\, \exists \delta > 0\end{array}\right\} \forall a_o \text{ within } \delta \text{ of } a\ \exists b_o \text{ within } \varepsilon \text{ of } a \ldots$$

That is, b depends only on a, and δ only on ε. Of course, one can say something in the usual predicate calculus which is equivalent to this under some axiom of choice, but that is beside the point. We have not pursued this matter further, but instead have formulated a weaker version of Local Uniform Continuity, which nevertheless has all the interesting consequences mentioned in the introduction. When we refer to Local Uniform Continuity in the rest of this paper, what we mean is this version, called LUC(X,Y) in [B1]. We now state our main theorem on derived rules.

Theorem 7.1 Let T be T_2 (similar to Myhill's CST), T_3, T_4, Z, or ZF^- + RDC, or the non-extensional version of any of these theories. Then T is closed under the rules of local continuity, continuous choice, local uniform continuity, Heine-Borel's rule, and all the rules discussed in [B1].

Corollary If the hypothesis of one of these rules is provable in $\underset{\sim}{B}$, the conclusion is provable in T_2, i.e. requires at most some instances of induction and collection beyond $\underset{\sim}{B}$.

Proof: The necessary conditions laid out in [B1] have been derived for these set theories in Theorem 5.4, Lemma 5.2, and Theorem 6.2.

Remark: In case T is a non-extensional theory, we can allow an arbitrary formula in place of a definable set in the rule of local continuity, local uniform continuity, and continuous choice.

We now shall obtain some consistency results complementing these results on derived rules. These results concern the principles corresponding to the derived rules we have already studied; their statements are obtained from the rules in the obvious way, namely: if a rule says, from A infer B, then the corresponding principle is A → B. Foremost among the principles we study are the Principle of Local Continuity, the Principle of Local Uniform Continuity, and the Principle of Continuous Choice. Note that the Principle of Local Continuity implies the simpler Principle of Continuity, that every function from a complete separable metric space to a separable metric space is continuous; and the Principle of Local Uniform Continuity implies that every function from a compact metric space to a separable metric space is uniformly continuous. Thus our results include the conjectures of [Fr 2] concerning functions from N^N to N and from 2^N to N. The Principle of Local Uniform Continuity also implies Heine-Borel's theorem.

At the end of [M1], there is a "postscript" announcing certain theorems of Friedman, which include special cases of some of the consistency results in this section; see also [B3,§3] for related results. The theorem announced for Friedman in the last line of [M1], about the axiom of choice for the reals with the prefix $\forall x \exists y\, A(x,y)$ instead of $\forall x \exists! y\, A(x,y)$, is false; the axiom is refutable, as discussed in [B1].

We are going to prove our consistency theorems using realizability. The key to these proofs lies in the construction in [B1] of a so-called "weak BRFT" in which all operations on N^N are continuous. To explain what this means: Let S be a set, and suppose C is a class of partial functions from S to S, of several variables, including a pairing function and a binary "universal function" $\emptyset(e,x)$ such that each unary function f in C is $\lambda x \emptyset(e,x)$ for some e. If (S,C) satisfies some other, less important conditions spelled out in [B1], then it is called a "weak BRFT". The use of such structures is that the functions in C can be used in place of recursive functions for realizability. As mentioned, in [B1] a specific weak BRFT is constructed in which all operations from N^N to N are continuous. (Each weak BRFT contains a copy of the integers, calling some element O and using p(x,O) for successor, where p is the pairing function. Thus it makes sense to speak of operations from N^N to N.) As a matter of fact, two specific weak BRFT's are constructed with this property: in one of them, all operations on 2^N to N are uniformly continuous, and in the other, there is a continuous, but not uniformly continuous function on 2^N to N. Call these weak BRFT's S_o and

S_1 respectively.

Theorem 7.2 ZF^- + RDC is consistent with Church's thesis CT plus "All functions from a complete separable space X to a separable space Y are continuous".

Remark: Continuity cannot be improved to uniform continuity without dropping Church's thesis.

Theorem 7.3 ZF^- + RDC is consistent with the Principle of Local Continuity plus "There is a non-uniformly continuous, continuous function from 2^N to N."

Theorem 7.4 ZF^- + RDC is consistent with the Principle of Local Uniform Continuity.

Corollary: ZF^- + RDC is consistent with "All functions from a compact metric space to a separable metric space are uniformly continuous."

Proof: The idea of all the proofs is to use realizability to prove the theorem for ZF^- + RDC − ext, and then use the ideas of Theorem 3.1 to prove it for ZF^- + RDC. We first show how to use realizability to prove the theorems for T-ext, where for simplicity we write T for ZF^- + RDC.

Generally speaking, realizability interpretations can be either formal or informal; that is, e r A can be either a formula of the formal system, or an informal predicate. For instance, Kleene's original interpretation for arithmetic can be taken either way. The q-realizability given earlier in this paper for set theory is necessarily formal, however, since it is not clear how to interpret $\in$ informally. Of course we can also do (formal) "1945-realizability" for set theory, which is analogous to Kleene's original "1945-realizability" (as it has come to be called) for arithmetic. Here are the clauses defining this interpretation:

$e\ r\ x \in y$	is	$\langle e,x \rangle \in y$
$e\ r\ (A \ \&\ B)$	is	$(e)_o\ r\ A \ \&\ (e)_1\ r\ B$
$e\ r\ (A \vee B)$	is	$((e)_o = O \rightarrow (e)_1\ r\ A)\ \&$ $((e)_o \neq O \rightarrow (e)_1\ r\ B))$
$e\ r\ (A \rightarrow B)$	is	$\forall a(a\ r\ A \rightarrow \{e\}(a)\ r\ B)$
$e\ r\ \exists x\, A$	is	$\exists x\ e\ r\ A$
$e\ r\ \forall x\, A$	is	$\forall x\ e\ r\ A$

We then have the soundness theorem, T-ext $\vdash$ A implies T-ext $\vdash \bar{e}\ r\ A$ for some numeral $\bar{e}$. The proof of the soundness theorem is so similar to the soundness theorem for q-realizability that we do not write it out here.

Now Kreisel-Lacombe-Schoenfield's theorem asserts that every effective operation from N^N to N is continuous; and the same theorem is true for complete separable metric spaces in place of N^N and separable spaces in place of N. (Kreisel-Lacombe-Schoenfield's theorem is proved, for instance, in Rogers [R, p. 362]; the reader will have no difficulty making the extension mentioned.) Moreover, if X and Y are complete separable and separable metric spaces, respectively, with X in standard form, then KLS(X,Y) (in obvious notation) is 1945-realized, as is proved in [B2]. Now, it is easy to see that (1) Church's thesis is realized using 1945-realizability, and (2) with Church's thesis, KLS(X,Y) is equivalent to the assertion that all functions from X to Y are continuous. It follows that "all functions from X to Y are continuous" is realized, and hence consistent with T-ext, by the soundness theorem for 1945-realizability. Thus Theorem 7.2 is proved with T-ext in place of T.

Now we turn to the proofs of Theorems 7.3 and 7.4. In the definition of realizability given above, there is nothing particularly sacred about the recursive functions. If we have any weak BRFT which can be defined and proved to be a weak BRFT in T-ext, we can use it for (formal) 1945-realizability. That is, instead of using $\{e\}(a)=y$ we use $\emptyset(e,a)=y$, where $\emptyset$ is the universal function of the weak BRFT. To be precise, instead of using $\exists n(T(e,a,n)\ \&\ U(n)=y)$ we use the formula defining $\emptyset(e,a)=y$ in T-ext. A priori, it is possible that we might have a weak BRFT which could be proved to be a weak BRFT without having a definable universal function, but that possibility doesn't occur here. Also, one should add to the definition of e r $(A \vee B)$ a proviso that $(e)_o$ is an integer of the BRFT (each weak BRFT contains a copy of the integers).

One can verify by reading the construction of S_o and S_1 in [B1] that their universal functions are definable, and that they can be proved in a very weak non-extensional set theory to be weak BRFT's. To verify that S_o has the property that all functions from 2^N to N are uniformly continuous requires something like König's lemma, which goes beyond intuitionistic systems, but that doesn't affect the usefulness of S_o for formal realizability, which only requires that we be able to prove that it is a weak BRFT.

It follows from the above discussion, and from a soundness proof inessentially different from the one give for q-realizability, that we can assign to each formula A another formula e r_i A, for e realizes A in S_i, and prove that T-ext $\vdash$ A implies T-ext $\vdash \exists e \in S_i (e\ r_i\ A)$.

If A is a sentence, then "$\exists e(e\ r_o\ A)$ is true" can be regarded as an informal realizability interpretation of A. This interpretation can play the same role that "$M \models e(e\ r\ A)$" does for Feferman's theories in [B1]. If one reads the arguments of §3.2 of [B1], making the substitution just mentioned, one finds that Theorem 7.3 is proved, with T-ext in place of T. Similarly, if one reads the arguments of §3.3 of B1, substituting "$\exists e(e\ r_1\ A)$" for "$N \models e(e\ r\ A)$", one finds that Theorem 7.4 is proved, with T-ext in place of T. That is, the proofs that local continuity and related properties are or are not realized are not dependent on the particular theory in detail, but only on the existence of a sound realizability interpretation in some weak BRFT with the properties of S_o or S_1, respectively.

Thus Theorems 7.2, 7.3, and 7.4 are proved for T-ext. Now we discuss how to improve this result to T instead of T-ext. We have to consider the interpretation * from T to T-ext given in Theorem 3.1. Suppose for simplicity that we are working with a two-sorted version of T, with variables for integers and variables for sets.[1] Recall that two sets are $\sim$ if each member of one is $\sim$ some member of the other, and $a \varepsilon b$ if $a \sim$ some $c \in b$. The interpretation for two-sorted T leaves numbers alone. It is then easy to check that two functions from N to N (as sets of ordered pairs) are $\sim$ if and only if they have the same values. Thus if X is any "extensional" subset of N^N, i.e. whenever $a \in X$ and b has the same values as a then $b \in X$, we have $a \varepsilon X$ iff $a \in X$. In particular, any complete separable metric space in standard form (see §2) is such an extensional subset of N^N. Similarly, if X and Y are complete separable spaces and P is an extensional subset of $X \times Y$ in the sense of §2, then $x \varepsilon P$ iff $x \in P$. We shall now prove the following: Let A be an instance of the Principle of Local Continuity. Then T-ext $\vdash (A \leftrightarrow A^*)$. We consider the conjuncts of the hypothesis of Local Continuity one-by-one.

Making use of the "standard form" of X, we may suppose that all references to X in the Principle of Local Continuity are implicit: that is, "$a \in X$" actually is "$a \in N^N\ \&\ \mathrm{Conv}(a)$", where Conv is a formula expressing the convergence conditions on a, i.e. $\mathrm{Conv}(a) \leftrightarrow \forall n,m \in N(\sigma(a_n,a_m) < (1/m)+(1/n))$, where σ is a certain recursive function. Hence, the hypothesis "X is a complete separable metric space" no longer needs to actually occur. Similarly for Y. Consider the hypothesis "P is extensional", which says $d(a,a')=0\ \&\ d'(b,b')=0\ \&\ P(a,b) \rightarrow P(a',b')$, where d and d' are the metrics of X and Y respectively. We have seen already that P is equivalent to P*; by a similar argument it follows that $(d(a,a')=0)$ is equivalent to $(d(a,a')=0)^*$; hence, "P is extensional" is equivalent to its * interpretation.

Consider the hypothesis, "$\forall a \in X$ (b: P(a,b) & $b \in Y$) is closed". We do not have to check this one, since the special case Y=N implies the general case, provably in a very weak theory plus a simple axiom of choice AC_N which is realized, as is shown in [B1]. However, the reader who wishes can verify directly that this hypothesis, too, is equivalent to its * interpretation.

Finally, consider $\forall a \in X \exists b \in Y$ P(a,b). The interpretation of this is $\forall a \in X \exists b \in Y$ P*(a,b); which we have seen is just the original formula again. The conclusion of Local Continuity can be dealt with similarly. Hence, each instance A of the Principle of Local Continuity is provably equivalent to A*.

Now we prove the consistency of T + LC(X,Y). If it is not consistent, then some conjunction of instances of LC(X,Y), say B, implies 0=1 in T. Then, by Theorem 3.1, B* implies 0=1 in T-ext. But B* is provably equivalent to B, in T-ext. Hence LC(X,Y) is inconsistent with T-ext, which possibility we have already ruled out by realizability.

The proofs of Theorems 7.2, 7.3, and 7.4 can be completed by checking that the other statements involved are also equivalent to their * interpretations. The basic reason why this works seems to be that none of these statements mention objects of type higher than functions from N^N to N. The use of standard form for complete separable spaces reduces everything to low types. We check, for instance, that "All functions from N^N to N are continuous" is equivalent to its * interpretation. Now "F: $N^N \to N$" is Fcn(F) & $\forall a,b(a \in N^N$ & $\langle a,b\rangle \in F \to b \in N)$. Now Fcn(F)* says that if $\langle a,b\rangle \in F$ and $\langle a,c\rangle \in F$ then $b \sim c$. But $\langle a,b\rangle \in F$ says a $\sim$some a' and b=F(a'). But then $a \in N^N$ and so b=F(a). Hence Fcn(F)* is equivalent to Fcn(F). The argument shows also that $\langle a,b\rangle \varepsilon F$ iff $\langle a,b\rangle \in F$. Together with what we have already proved, this suffices to complete the proof that "F: $N^N \to N$" is equivalent to its interpretation. Next, "m is a modulus for F at y" is $\forall y \in N^N(\forall i \leq m(z(i)=y(i)) \to F(z)=F(y)$, which is equivalent to its own interpretation in view of the fact that $\langle a,b\rangle \varepsilon F$ iff $\langle a,b\rangle \in F$, as proved above. Hence, "All functions from N^N to N are continuous" is equivalent to its own interpretation, as claimed. The rest of the statements in Theorems 7.2, 7.3, and 7.4 can be treated similarly. This completes the proof.

Footnote 1: We have formulated our set theories T with a constant ω for the von Neumann integers. Alternately one may use a two-sorted theory T^2 with one sort of variables for numbers and one sort for sets (or equivalently, one can use two unary predicates.) At first glance it may seem that T and T^2 are trivially equivalent, but the problem is more subtle. T^2 can be easily interpreted in T. But the converse is more difficult, since T^2 does not necessarily prove the existence of the von Neumann integers. However, if T contains collection, then T^2 does prove the existence of the von Neumann integers, and T can easily be interpreted in T^2. In particular, the application we make of T^2 in the consistency proofs of §7 fall under this remark.

Errata: (1) In Lemma 0.2 of [B1], page 260, the hypothesis should state that for i, j ≤ k, we have $\rho(a_i,a_j) < (1/4i) + (1/4j)$. When the lemma is applied on page 298, we may assume that b satisfies this hypothesis, by replacing b_n by b_{4n}.
(2) Theorem 2.4 of [B1], p. 303, is correct, but something must be added to the proof at line 26, for as it stands, the proof applies only if X' is provably compact, which we could only assure in general if X is locally compact. (This is related to a defect of Bishop's definition of continuity pointed out by Hayashi: continuity on compact sets does not guarantee pointwise continuity unless the space is locally compact). To correct the proof, we appeal at line 26 to the rule of local uniform continuity with a parameter X' for a compact subspace of X. Note that a compact subset X' of X can be coded as a function from N to N, since it is given by a function assigning to each rational $\varepsilon > 0$ a finite ε-approximation to X', and since X is in standard form, each member of X is a sequence of integers. Thus, in the notation of Section 2.8 of [B1], $Q(e) \leftrightarrow$ e codes a compact subset of X is an allowable choice of a set of parameters. In Section 2.8, the rule of local continuity with parameters is derived; the rule of local uniform continuity with parameter may be similarly treated.

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