Boundary Regularity in Plateau's Problem

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Abstract

Let Γ be a real-analytic Jordan curve in R^3. Let u be a minimal surface (of the topological type of the disk) bounded by Γ, having a boundary branch point of order $2m$. Suppose $n \leq m$. Then u does not furnish a C^n relative minimum of Dirichlet's integral or area. In particular, the absolute minimum of Dirichlet's integral, and indeed any relative minimum in the C^1 norm, among disk-type surfaces bounded by Γ, has no boundary branch points.

More generally, Γ can be a $C^{N,\mu}$ Jordan curve (with $N \geq 3$) in R^3, such that at each point, each component of Γ has a nonzero j-th derivative for some j. Then we call Γ "nowhere planar." In that case we obtain the same conclusion provided $n \leq N$ as well as $n \leq m$. Again it follows that the absolute minimum of Dirichlet's integral, and indeed any relative minimum in the C^1 norm, among disk-type surfaces bounded by Γ, has no boundary branch points.

These theorems are proved by explicit construction of a family $\tilde{u}$ of harmonic surfaces and an explicit calculation of the Dirichlet integral $E[\tilde{u}] = E[u] - ct^{4m+3} + O(t^{4m+4})$. The technique is called "splitting the branch point" because it is based on replacing z^m with a product of terms $z - a_i(t)$.

Introduction

The history of the problem is somewhat convoluted. Before discussing it, note the following issues: interior branch points *vs.* boundary branch points; true branch points *vs.* false branch points; real-analytic boundary *vs.* smooth boundary; absolute minimum of area or Dirichlet's integral *vs.* C^0 relative minimum *vs.* C^1 relative minimum *vs.* C^n relative minimum; and explicit construction of a family that decreases Dirichlet's integral, or no explicit construction.

Choosing one from each category there arise many possible theorems concerning the absence of branch points from certain minimal surfaces, some of which have been proved in the past, some of which are proved here, and some of which are still open. The main contributions already in the literature are the following: Osserman [19] proved that the absolute minimum of area has no true interior branch points. R. Gulliver [11] and H. W. Alt [1] independently proved that the absolute minimum of area has no false interior branch points. Gulliver, Osserman, and Royden [13] proved that a disk-type minimal surface spanning an analytic Jordan curve has no false branch points in the interior or on the boundary. Gulliver-Lesley [12] proved that the absolute minimum of area has no true boundary branch points, if the boundary is real-analytic, by adapting Osserman's method to the boundary case. These proofs should actually apply to C^0 relative minima as well as the absolute minima. Beeson [7] gave an explicit construction that decreases Dirichlet's integral near an interior branch point; the family can be made C^1 smooth (not C^n as claimed in the paper), so C^1 relative minima have no interior branch points.

In this paper we give a (quite different) explicit construction that decreases Dirichlet's integral near a boundary branch point, and again the family can be made C^1 smooth–in general it can be made C^m smooth, where $2m$ is the order of the branch point–but since m might be 1, the best general result is C^1.

This applies first to any real-analytic boundary, but by quoting known representation formulae for a minimal surface in the neighborhood of a boundary branch point, we are able to extend the construction to the case of a C^3 boundary, provided the boundary is "nowhere planar." It is worth noting that we do not need to appeal to previous results eliminating false branch points–they are eliminated by the calculation, along with true branch points, using the hypothesis that the boundary is nowhere planar.

We leave open two questions: can the boundary regularity theorem be proved for C^3 boundaries without the hypothesis that the boundary curve is nowhere planar? and, can either the boundary or interior regularity be proved for C^n relative minima as well as for C^1 relative minima? So far as we can prove at present, there might be a disk-type minimal surface with an interior or boundary branch point of order 2 whose energy cannot be decreased along a C^2 one-parameter family of harmonic surfaces (that is, a one-parameter family in the C^2 metric) with the same boundary, although (after this paper) we know how to decrease it along a C^1 family.

Preliminaries and Definition of the Variation

We deal with harmonic mappings u from a subset of the plane to R^3. The components will be indicated by ${}^1u$, ${}^2u$, and ${}^3u$. Although the use of a superscript in this position is not standard, it makes it possible to denote (partial) differentiation by a subscript, for example ${}^1u_y$, when otherwise we would have to write the more cumbersome

$$\frac{du_1}{dy}.$$

It is also convenient, when the components of a vector are given by long formulae, to use a column-matrix format for a vector. We omit an explicit dot for the dot product, writing for example uu_y instead of $u \cdot u_y$, as no other interpretation is possible. Otherwise our notation is standard.

All the minimal surfaces considered in this paper are of the topological type of the unit disk. At the very beginning, we consider a disk-type minimal surface parametrized in the disk of radius 1 centered at $(0, 1)$, with a boundary branch point at $(0, -1)$. Our first step, however, is to make a conformal transformation of the parameter domain to the upper half plane H^+, taking the branch point to the origin. We need to consider this transformation in order to be able to speak of the "harmonic extension" of a function defined on the real line, and bounded at infinity. This harmonic extension can be defined by pulling back the boundary values to the disk, using the Poisson integral formula, and then mapping back to the upper half plane. In particular if we consider a 3-vector of such boundary values that traces out a Jordan curve Γ monotonically except for one point, then the harmonic extension is a harmonic surface spanning Γ.

From now on, we consider a minimal surface $u(z)$ parametrized in the upper half plane, with a branch point at the origin, and bounded by a C^n Jordan curve Γ defined on the unit interval, which does not lie in a plane. For the time being, we suppose that Γ is real-analytic, and later consider the case when Γ is only C^n. We suppose that Γ passes through the origin tangent to the X-axis, and that u takes the portion of the real axis near origin onto Γ, with $u(0) = 0$. Then since Γ is a Jordan curve, it is not contained in a line. We still are free to orient the Y and Z axes. We do this in such a way that the normal at the branch point (which is well-defined) points in the negative Z-direction. With Γ oriented in this way, there will be two integers p and q such that Γ has a parametrization in the form

$$\Gamma(\tau) = \begin{bmatrix} \tau \\ \frac{C_1 \tau^{q+1}}{q+1} + O(\tau^{q+2}) \\ \frac{C_2 \tau^{p+1}}{p+1} + O(\tau^{p+2}) \end{bmatrix}$$

for some nonzero real constants C_1 and C_2. We therefore have

$$\Gamma'(\tau) = \begin{bmatrix} 1 \\ C_1 \tau^q + O(\tau^{q+1}) \\ C_2 \tau^p + O(\tau^{p+1}) \end{bmatrix}$$

We write $u(z) = (X(z), Y(z), Z(z))$. We make use of the Enneper-Weierstrass representation of u (see e.g. [8], p. 108)

$$u(z) = Re \left[\begin{array}{l} \frac{1}{2}\int f - fg^2 dz \\ \frac{i}{2}\int f + fg^2 dz \\ \int fg dz \end{array} \right]$$

where f is analytic and g is meromorphic in the upper half-disk. By the boundary regularity theorem of Lewy (see [9], p. 38), u can be extended analytically into some neighborhood of the origin, and hence f can also be so extended, and g can be extended meromorphically.

Definition 1 *The* order *of the branch point is the order of the zero of f. The* index *of the branch point is the order of the zero of g.*

The order of a boundary branch point must be even (since the boundary is taken on monotonically). It is customary to write it as $2m$, and to use the letter k for the index. Thus $f(z) = z^m + O(z^{m+1})$ and $g(z) = cz^k + O(z^{k+1})$ for some constant c.

Let $\tau(z) = X(z) = \text{Re}\frac{1}{2}\int f - fg^2 dz$. Then on the boundary we have

$$u(z) \quad = \quad \Gamma(\tau(z))$$

This parametrization and function $\tau(z)$ were inspired by Lewy's equation (see [9], p. 38).

We suppose for the time being that Γ is real-analytic; it follows from Lewy's theorem *op. cit.* that u is also real-analytic up to the boundary. Hence f and g are also analytic at the origin. We have $f(z) = \sum e_i z^i$ and $g(z) = \sum c_i z^i$. Note that the first nonzero e_i is e_M, where M is the order of the branch point. If u is monotonic on the boundary, as is the case in Plateau's problem, then M must be even, and we have $M = 2m$; but it is not necessary to assume that yet, so we continue to write M for the order. Because of the orientation of Γ, we have e_M real. The first nonzero c_i is c_k.

If not all the coefficients of g are pure imaginary, define δ to be the least integer such that $c_{k+\delta}$ has a nonzero real part. If not all the coefficients of f are real, we define ν to be the least integer such that $e_{M+\nu}$ has a nonzero imaginary part. It cannot be that the coefficients of g are all imaginary and the coefficients of f are all real, for in that case the coefficients of fg^2 would be all real, and so $dY/dx = -\frac{1}{2}\text{Im}(f + fg^2)$ would be identically zero on the real axis, so Y would be constant on the real axis and Γ would lie in a plane. Since $e_{2m} = 1$ is real, we have $\nu > 0$ if ν is defined, but we may have $\delta = 0$ if c_k has a nonzero real part. The imaginary part of $e_{M+\nu}$ will occur in many of our equations, so it will be useful to give it a short name:

Definition 2 $E :=$ Im $e_{M+\nu}$ *is the imaginary part of the coefficient of the lowest power of z in f that has a nonzero imaginary part.* $G :=$ Re $c_{k+\delta}$ *is the real part of the coefficient of the lowest power of z in g that has a nonzero real part.*

We have

$$\begin{aligned}\tau(z) &= X(z) \\ &= \text{Re}\ \frac{1}{2}\int f - fg^2 dz \\ &= \frac{a_M}{M+1}\text{Re} z^{M+1} + O(z^{M+2})\end{aligned}$$

Plugging this into the equation for Γ' above, we have for real z:

$$\Gamma'(\tau(z)) = \begin{bmatrix} 1 \\ C_1 a_M^q z^{(M+1)q} + O(z^{(M+1)q+1}) \\ C_2 a_M^p z^{(M+1)p} + O(z^{(M+1)p+1}) \end{bmatrix}$$

Since $z^{-m}f(z)$ is analytic and nonzero at origin, we can write

$$f(z) = A_0^2(z)z^{2m}$$

where $A_0(z)$ is analytic and does not vanish at origin. By scaling the entire minimal surface and curve Γ we can assume $A_0(0) = 1$. Similarly we can find an analytic function B_0, such that

$$fg^2(z) = B_0^2 z^{2m+2k}.$$

We do not know anything about $B_0(0)$ except that it is nonzero; in particular it might have a nonzero imaginary part for all we know. If we expand A_0 in a power series $\sum \gamma_i z^i$, the first γ_i with a nonzero imaginary part will contribute a power z^{2m+i} to $f = z^{2m}A_0^2$; therefore by the definition of ν we have $i = \nu$.

We choose some distinct complex constants $\alpha_1, \ldots, \alpha_m$. About the α_i we assume, for $m > 1$,
(i) at least one of the α_i is nonzero.
(ii) If α_i is not real and not zero, then its complex conjugate $\bar{\alpha}_i$ occurs as another α_j. In particular each nonzero real α_i (if any) occurs an even number of times in the list $\alpha_1, \ldots, \alpha_n$.

It is always possible to choose α_i satisfying these conditions, when $m > 1$. The case $m = 1$ has to be treated differently; this will be done in a subsequent section.

These conditions given above are enough for our main computations about the Dirichlet integral. But to ensure that the family $\tilde{u}$ takes the boundary monotonically, we need a third assumption:
(iii) $\text{Re}(\alpha_i) \geq 0$, and for at least one i we have $\text{Re}(\alpha_i) > 0$.

Henceforth, we assume that the α_i are chosen to satisfy (i), (ii), and (iii).

We also need numbers $\hat{a}_i$ for $i = 1, \ldots, m$, which we shall eventually specify to be 0, 1 or -1. About the $\hat{a}_i$ we assume the following:
(iv) $\hat{a}_i$ is nonzero if and only if α_i is nonzero.

(v) If $\bar{\alpha}_i = \alpha_j$ then $\hat{a}_i = \hat{a}_j$.

We then define

$$
\begin{aligned}
a_i(t) &= \alpha_i t + \hat{a}_i t^2 \\
A(t,z) &= A_0(t) \prod_{i=0}^{m} (z - a_i(t)) \\
B(z) &= B_0 z^{m+k}
\end{aligned}
$$

Thus $fg^2 = B^2$, and B does not depend on t. Fix a number R much larger than all the $|\alpha_i|$. We will choose R large enough that certain terms are positive. Specifically, we require that $R \geq 2|\alpha_i|$ for each i, and

$$
R \geq 2^{3m+1}, \tag{1}
$$

and two more conditions on R. One of them is given in equation (31) near the end of the paper.

Let N be the number of nonzero complex-conjugate pairs among the α_i. We can index the α_i in such a way that $\alpha_i, \ldots, \alpha_{2N}$ are the nonzero α_i and $\bar{\alpha}_i = \alpha_{N+i}$ for $i \leq N$. The following equations define two polynomials h and p:

$$
\begin{aligned}
h(\xi) &= \prod_{i=1}^{N} |\xi - \alpha_i|^4 - \xi^{4N} \\
p(w) &= \int_0^w \xi^{2m-4N} h(\xi)\, d\xi
\end{aligned}
$$

The last condition on R is that the following inequality is satisfied:

$$
p(w) < 0 \qquad \text{for } w \geq R \tag{2}
$$

We now prove that it is possible to choose R so that this condition is satisfied:

Lemma 1 *For sufficiently large R, (2) is satisfied.*

Proof. We begin by showing that the leading coefficient of h is negative. h is of degree at most $4N-1$, since the term in ξ^{4N} evidently cancels out to zero. The coefficient of ξ^{2N-1} is $-4\sum_{i=1}^{N} \mathrm{Re}(\alpha_i)$, which is negative by assumption (iii). Thus h has degree $4N-1$ and a negative leading coefficient. Then $p(w) = \int_0^w \xi^{2m-4N} h(\xi)\, d\xi$ is a polynomial in w of degree $4N$, with negative leading coefficient. Hence, for large positive w, $h(w)$ is negative. That completes the proof of the lemma.

We will always assume that t is small enough that A_0 and B_0 are analytic in the disk of radius Rt. Let ϕ_1 be a C^n real functions of a real variable x such that

$$\begin{array}{rcll} \phi_1(x) & = & 0 & \text{for } x \le -t^2R \\ \phi_1'(x) & \ge & 0 & \text{for } -t^2R \le x \le 0 \\ \phi_1(x) & = & 1 & \text{for } 0 \le x \le Rt \\ \phi_1'(x) & \le & 0 & \text{for } Rt \le x \le (1+t)Rt \\ \phi_1(x) & = & 0 & \text{for } x \ge (1+t)Rt \end{array}$$

It follows from the stated conditions that $0 \le \phi_1(x) \le 1$ for all x. Define $\phi_2(x) = 1 - \phi_1(x)$. Thus $\phi_1 + \phi_2 = 1$ and hence $\phi_1' + \phi_2' = 0$. We can obtain ϕ_1 by translating and scaling a similar functions which varies on $[0,1]$ instead of $[Rt,(1+t)Rt]$, and joining it with another such function that increases from 0 to 1 on $[0,1]$ instead of $[-Rt^2,0]$. We will need an estimate on ϕ_1'. Since ϕ_1 must change by 1 in an interval of length t^2R, the magnitude of ϕ_1' must be somewhere at least $1/(t^2R)$, but we can arrange that its magnitude is at most a constant times that. For reasons that will become clear below we choose the constant to be e^3.

$$|\phi_1'(x)| \le \frac{e^3}{t^2R} \tag{3}$$

That will follow if the unscaled function which varies on $[0,1]$ instead of $[Rt,(1+t)Rt]$ has derivative bounded by e^3, which is easy to arrange. We shall eventually need to control the n-th derivatives of ϕ_1 as well. If we start with a C^N function ψ varying on $[0,1]$ and scale it to vary on $[0,t^2]$ instead, by defining $\phi(x) = \psi(t^2x)$, then $\phi^{(n)}(x) = t^{2n}\psi^{(n)}(x)$. It follows that we can construct ϕ_1 to satisfy

$$\frac{d^n\phi_1}{dx^n} = O(t^{-2n}). \tag{4}$$

It is easy to see by induction that the exponent in an estimate of this form cannot be improved.

However, we will need a sharper estimate on ϕ_1' in order to prove monotonicity. We need this only on the interval $[-Rt^2,0]$. Essentially, the following lemma says that we can choose ϕ_1 to have its steepest slope near the origin, so that when z is near $-Rt^2$, ϕ_1' is not so large. The previous estimates could be satisfied by a (smoothed-out) linear ϕ_1, but the sharper estimate requires a (smoothed-out) logarithmic ϕ_1.

Lemma 2 *We can construct ϕ_1 to satisfy the properties stated above, including (3) and (4), and also*

$$|\phi_1'(x)| \le \frac{1}{2|x|} \tag{5}$$

when $-Rt^2 \le x \le 0$.

Proof: As above, we could re-scale by a factor of t^2, and then assume $t = 1$. However, it is not difficult to retain the factors of t, and we do retain them for the benefit of any readers who may doubt that it is sufficient to assume $t = 1$. Define

$$\psi(x) := -\frac{1}{3} \ln \frac{|z|}{Rt^2}$$

for $-Rt^2 \leq x \leq -\frac{Rt^2}{e^3}$. Then $\psi(x)$ increases monotonically from 0 to 1 over that interval, and

$$|\psi'(x)| \leq \frac{1}{3|x|}.$$

The maximum value of ψ' is at the right-hand end of this interval, namely $x = -Rt^2/e^3$, and hence

$$|\psi'| \quad \leq \quad \frac{e^3}{3Rt^2}. \tag{6}$$

Extend ψ to be defined for all x by making $\psi(x) = 0$ for $x \leq -Rt^2$ and $\psi(x) = 1$ for $-Rt^2/e^3 \leq x \leq Rt$. Then for $Rt \leq x \leq (1+t)Rt$, $\psi(x)$ decreases from 1 to 0. On this interval a linear decrease will suffice, as we do not need the sharper bound on ψ' that would result from a logarithmic decrease.

Then we could take $\phi_1 = \psi$ except for the fact that ψ is not smooth at the "corners", $x = -Rt^2$ and $x = -Rt^2/e^3$. We will define ϕ_1 by "rounding off the corners" of ψ, so as to make ϕ_1 a C^n function. Specifically we define ϕ_1 as a convolution integral with a "mollifier" Υ. Define Υ to be a positive smooth function with support in $[-t^3, t^3]$, and integral 1, and define

$$\phi_1(x) = \int \Upsilon(x-\xi)\psi(\xi)\, d\xi.$$

The limits of integration are unimportant because both ψ and Υ have bounded support; we may take the limits of integration to be -1 and 1 for example. We can differentiate this equation, differentiating under the integral sign on the right and then integrating by parts to obtain

$$\phi_1'(x) = \int \Upsilon(x-\xi)\psi'(\xi)\, d\xi.$$

Hence

$$\begin{aligned} |\phi_1'(x)| &\leq \sup_{|x-\xi| \leq t^3} |\psi'(\xi)| \\ &\leq \sup |\psi'| \end{aligned}$$

That proves (3), since $|\psi'| \leq e^3/3Rt^2$ by (6).

For x between $-Rt^2 - t^3$ and $-Rt^2/e^3 - t^3$, we have

$$\begin{aligned} |\phi_1'(x)| &\leq \psi'(x+t^3) \\ &= \frac{1}{|3x+t^3|} \\ &= \frac{1}{3|x|-t^3} \\ &\leq \frac{1}{2|x|} \end{aligned}$$

For x within t^3 of $-Rt^2/e^3$, we have

$$\begin{aligned} |\phi_1'(x) &\leq \frac{e^3}{3Rt^2} \\ &= \frac{1}{3\frac{Rt^2}{e^3}} \\ &\leq \frac{1}{3(|x|+O(t^3))} \\ &\leq \frac{1}{2|x|} \end{aligned}$$

for sufficiently small t. Thus we have established the desired bound on ϕ_1' for x between $-Rt^2 - t^3$ and $-Rt^2/e^3 + t^3$. But outside these regions, for negative x, ϕ_1 is constant; so the bound is established for negative x. Then ϕ_1 decreases in the interval $[Rt, (1+t)Rt]$. Here we do not need a pointwise bound, and a (mollified) linear decrease from 1 to 0 would suffice, but the shortest way to finish the proof is simply to let ϕ_1 on this interval of length R^2 be defined as a reflection and translation of ϕ_1 on the interval $[-Rt^2, 0]$; since reflections and translations preserve the magnitude of the derivative, the bounds we established on $[-Rt^2, 0]$ apply to ϕ_1 so defined.

The function ϕ_1 is as many times differentiable as Υ; in particular it is C^n. We do not claim that $\phi_1^{(n)}$ is given by convolution of $\psi^{(n)}$ with Υ, because after the first derivative there may be boundary terms if we integrate by parts, since ψ' is not continuous. But we do not need to integrate by parts to establish that ϕ_1 is in C^n, and we do not need an explicit bound on any derivative above the first; we only need the $O(t^{-2n})$ bound given in (4). This completes the proof of the lemma.

We define $\tilde{X}(t,x) :=$

$$\frac{1}{2}\mathrm{Re}\left[\phi_1 \int_0^x A^2(t,x) + B^2(x)\,dx + \phi_2 \int_0^x A^2(0,x) + B^2(x)\,dx\right]$$

and finally we define $\tilde{u}(t,\cdot)$ to be the harmonic extension of the boundary values $\Gamma(\tilde{X}(t,\cdot))$ to the parameter domain of u. $\tilde{u}$ is a one-parameter family of harmonic

surfaces bounded by Γ. The fact that for each fixed positive (sufficiently small) t, $\tilde{u}$ takes the boundary monotonically, will be established in Lemma 10 below. Note that when $t = 0$ we have ϕ_1 identically zero and ϕ_2 identically 1, and $\tilde{u}$ then coincides with u.
Remark. Originally it seemed natural to make ϕ_1 an even function, taking the value 1 on the interval $[-Rt, Rt]$, but then the family $\tilde{u}$ does not take the boundary monotonically on both sides of the origin. It is necessary to make the variation essentially only on one side of the origin, by letting ϕ_1 be 1 only on $[0, Rt]$. We let ϕ_1 return to zero as near the origin as possible for negative x. This works against monotonicity, but not enough to destroy monotonicity, as there are other terms that outweigh the contribution from this interval of length only $t^2 R$.

Geometric bounds on the index

Theorem 1 Geometric bounds on the index. *Let u have a boundary branch point of order M and index k on a real-analytic boundary segment Γ parametrized as $\Gamma' = (1, C_1\tau^q + \ldots, C_2\tau^p + \ldots)$. Then $k \le (M+1)p$.*

Remark: Wienholtz has shown (private communication) that this bound is best-possible, by constructing examples of branch points bounded by real analytic arcs using the solution of Bjorling's problem given in [8].
Proof: We have $f(z) = \sum e_i z^i$ and $g(z) = \sum c_i z^i$. Note that the first nonzero e_i is a_M, where M is the order of the branch point. If u is monotonic on the boundary, as is the case in Plateau's problem, then M must be even, and we have $M = 2m$; but it is not necessary to assume that in this section, so we continue to write M for the order. We have defined δ and ν in the previous section.

We have

$$\begin{aligned} \tau(z) &= X(z) \\ &= \text{Re}\frac{1}{2}\int f - fg^2 dz \\ &= \frac{a_M}{M+1}\text{Re } z^{M+1} + O(z^{M+2}) \end{aligned}$$

Plugging this into the equation for Γ' above, we have for real z:

$$\Gamma'(\tau(z)) = \begin{bmatrix} 1 \\ C_1 a_M^q z^{(M+1)q} + O(z^{(M+1)q+1}) \\ C_2 a_M^p z^{(M+1)p} + O(z^{(M+1)p+1}) \end{bmatrix}$$

We have $u(z) = \Gamma(\tau(z))$ for real z. To emphasize that we are considering real z we write x instead of z. Differentiating with respect to x we have $du/dx =$

$\Gamma'(\tau(x))\tau'(x)$. Since $\tau(z) = X(z)$ we have

$$du/dx = \Gamma'(X(x))dX/dx$$

Writing out the third component of this vector equation, we have (on the real axis)

$$\begin{aligned} dZ/dz &= dZ/dx = \text{Re } fg \\ &= (C_2 a_M^p z^{(M+1)p} + \ldots)dX/dz \\ &= (C_2 a_M^p z^{(M+1)p} + \ldots)(a_M z^M + O(z^{M+1})) \\ &= C_2 a_M^{p+1} z^{(M+1)p+M} + \ldots \end{aligned}$$

Working now on the left-hand side, the real part of fg begins either with the term in $z^{M+k+\delta}$, or with the term $z^{M+k+\nu}$, depending on which is smaller, δ or ν; or conceivably, these two terms could cancel and the first term would be an even higher power of z. If one of δ or ν is not defined because the coefficients of f are all real or those of g are all imaginary, then the real part of fg begins with whichever term does exist. Thus we have one of the following alternatives: Either

$$(M + 1)p = k + \nu$$

or

$$(M + 1)p = k + \delta$$

or $\nu = \delta$ and $k+\nu < (M+1)p$. Note that if all the coefficients of f are real, so ν is not defined, only the second alternative is possible, while if all the coefficients of g are imaginary, only the first alternative is possible. In all three cases, we have the desired conclusion $k \le (M + 1)p$.

Another geometric lemma

Recall that the number ν is the least exponent such that the coefficient E of $z^{2m+\nu}$ in the Weierstrass function f of u has a nonzero imaginary part. Also recall the definition of δ: $c_{k+\delta}x^{k+\delta}$ is the first term in g whose coefficient has a nonzero real part. (It is possible *a priori* that $\delta = 0$.)

Lemma 3 *Let u be a minimal surface parametrized in the upper half plane, with a boundary branch point at origin. Suppose that u spans a C^2 Jordan curve Γ such that at the branch point, Γ passes through the origin tangent to the X-axis. Suppose that near the origin Γ has a parametrization as discussed above, so that $\Gamma'(\tau) = \langle 0, C_1\tau^q, C_2\tau^p \rangle$ for some p and q. Then with ν as defined above, we have $\nu > 1$.*

Proof. We have by the Weierstrass representation, for real x,

$$\begin{aligned} {}^2u_x &= -\mathrm{Im}(f + fg^2) \\ &= Ex^{2m+\nu} + Ex^{2m+2k+\nu} + O(x^{2m+k+\delta}) + \dots \\ &= Ex^{2m+\nu} + O(x^{2m+2k+\delta}) + \dots \end{aligned}$$

where ... stands for terms involving higher powers of x than those shown. Suppose that $\nu = 1$. Then we have

$$\begin{aligned} {}^2u_x &= Ex^{2m+1} + O(x^{2m+2k+\delta}) + \dots \\ &= Ex^{2m+1} + O(x^{2m+2}) \end{aligned} \tag{7}$$

On the other hand, from the parametrization of Γ and the fact that u spans Γ, we have

$$u_2 = \frac{C_2 s^{q+1}}{q+1} + O(s^{q+2})$$

Therefore

$${}^2u_x = [C_2 s^q + O(s^{q+1})]s_x$$

Arc length s is given by

$$s = \frac{1}{2(2m+1)}x^{2m+1} + O(x^{2m+2})$$

and substituting this expression into the previous equation, we have, with $C_6 = C_2/(4(2(2m+1))^q)$,

$$\begin{aligned} {}^2u_x &= [2C_6 x^{(2m+1)q} + O(x^{(2m+1)q+1})]s_x \\ &= C_6 x^{(2m+1)q} x^{2m} + O(x^{(2m+1)q+2m+1}) \\ &= C_6 x^{(2m+1)(q+1)-1} + O(x^{(2m+1)(q+1)}) \end{aligned}$$

Comparing this with (7), we find $2m+1 = (2m+1)(q+1) - 1$. Subtracting $(2m)$ from both sides we find $1 = (2m+1)q$, which is impossible since $q \geq 1$ and $m \geq 1$. Since $\nu \geq 1$ by definition, that completes the proof of the lemma.

A potential-theoretic lemma

We will need to bound the normal derivatives ${}^2\tilde{u}_y$ and ${}^3\tilde{u}_y$, or at least their differences from ${}^2u_y$ and ${}^3u_y$, in terms of the boundary values of $\tilde{u}$. Let h be a harmonic function in the unit disk. It is well-known (and easily proved, see e.g. [9], p. 16) that ∇h is bounded by $\pi/\sqrt{3}$ times a bound for $d^2h(e^{i\theta})/d\theta^2$. We apply that theorem to prove the following lemma.

Lemma 4 *Suppose that for some function $K(t)$ we have*

$$|{}^i\tilde{u}_{xx} - {}^i u_{xx}| \le K(t) \qquad \text{on } -(1+t)Rt \le x \le (1+t)Rt.$$

Then $|{}^i\tilde{u}_y - {}^i u_y| \le C_7 K(t)$ for all x, for some constant C_7 depending only on R, and all sufficiently small t.

Proof. Let ζ be a conformal mapping from the unit disk to the upper half plane. Define $h(z) = {}^i\tilde{u}(\zeta(z)) - {}^i u(\zeta(z))$. Since $\tilde{u}(x) = u(x)$ for real x outside $[-tR, tR]$, we have $d^2h(e^{i\theta})/d\theta^2 = 0$ outside $\zeta^{-1}([-tR, tR])$. We define

$$H(x) = {}^i\tilde{u}(x) - {}^i u(x).$$

We need to check that a bound on the second derivative H_{xx} translates to a bound on the second derivative $d^2h(e^{i\theta})/d\theta^2$. For this computation we need an explicit expression for $\zeta(z)$. We can take

$$\zeta(z) = i\frac{z+i}{z-i}.$$

Observe that $\zeta(-i) = 0$, $\zeta(i) = \infty$, and $\zeta(1) = -i(1-i)/(1+i) = -1$; since ζ is a linear fractional transformation, it takes circles onto circles or lines, so it takes the unit circle onto the x-axis. Since it takes 0 to i, it takes the unit disk onto the upper half-plane. We have

$$\begin{aligned}
\frac{d\zeta(e^{i\theta})}{d\theta} &= \frac{2ie^{i\theta}}{(e^{i\theta}-i)^2} \\
\frac{d^2}{d\theta^2}\zeta(e^{i\theta}) &= \frac{-2e^{i\theta}}{(e^{i\theta}-i)^2} - \frac{2ie^{2i\theta}}{(e^{i\theta}-i)^4} \\
\frac{d^2}{d\theta^2}h(e^{i\theta}) &= \frac{d^2}{d\theta^2}H(\zeta(e^{i\theta})) \\
&= \frac{d}{d\theta}\left[H_\theta(\zeta(e^{i\theta}))\frac{d\zeta(e^{i\theta})}{d\theta}\right] \\
&= H_{\theta\theta}(\zeta(e^{i\theta}))\left(\frac{d\zeta(e^{i\theta})}{d\theta}\right)^2 + H_\theta(\zeta(e^{i\theta}))\frac{d^2}{d\theta^2}\zeta(e^{i\theta}) \\
&= H_{\theta\theta}(\zeta(e^{i\theta}))\left(\frac{2ie^{i\theta}}{(e^{i\theta}-i)^2}\right)^2 + H_\theta(\zeta(e^{i\theta}))\left[\frac{-2e^{i\theta}}{(e^{i\theta}-i)^2} - \frac{2ie^{2i\theta}}{(e^{i\theta}-i)^4}\right]
\end{aligned}$$

Now, for $-Rt \le x \le Rt$, assuming $t < 1/R$ and $x = \zeta(e^{i\theta})$, we have $e^{i\theta}$ in the lower half-plane, and hence $|e^{i\theta} - i| \ge 1$. Hence

$$\left|\frac{d^2h(e^{i\theta})}{d\theta^2}\right| \le 4\sup|H''| + 4|H'|.$$

Since H is nonzero only on $[-(1+t)Rt, (1+t)Rt]$, H_θ is bounded by $4Rt$ times any bound for $H_{\theta\theta}$. We have assumed $|H_{\theta\theta}(x)| \leq K(t)$. Putting in the bounds for the derivatives of H, we have

$$\left|\frac{d^2 h(e^{i\theta})}{d\theta^2}\right| \leq 4K(t) + 16RtK(t)$$

For sufficiently small t, the second term is dominated by the first and we have

$$\left|\frac{d^2 h(e^{i\theta})}{d\theta^2}\right| \leq 8K.$$

Now we are in a position to apply the theorem cited at the beginning of the section. It yields the desired conclusion of the lemma, with $C_7 = 8\pi\sqrt{3}$. That completes the proof of the lemma.

The real and imaginary parts of A^2

We define $w = z/t$. Most of our calculations will take place in the w-plane. Recall that $A(t,z) = A_0(z)\prod_{i=0}^m (z - a_i(t))$ and that $a_i(t) = t\alpha_i + t^2 i\hat{a}_i$. We define

$$\begin{aligned}
\tilde{A}^0(w) &= A_0(tw) \\
\tilde{A}(w) &= A(tw) \\
\mathbf{A}(w) &= \prod_{i=0}^{m}(w - \alpha_i)
\end{aligned}$$

Observe that

$$\begin{aligned}
A &= t^m\mathbf{A} + O(t) \\
\int_0^x A^2 dx &= t^{2m+1}\int_0^w \mathbf{A}^2 dw + O(t^{2m+2}) \\
\tilde{A}_0(w) &= 1 + \text{Re } O(t) + iEt^\nu w^\nu + \text{Im } O(t^{\nu+1}w^{\nu+1})
\end{aligned}$$

Lemma 5 *We have, for real w,*

$$\begin{aligned}
\text{Re } A^2 &= t^{2m}\mathbf{A}^2 + O(t^{2m+1}w^{2m-2N}) \\
\text{Im } A^2 &= -4t^{2m+1}\mathbf{A}^2\sum_{i=1}^{N}\frac{\hat{a}_i(w - \text{Re } \alpha_i)}{|w-\alpha_i|^2} + O(t^{2m+2}w^{2m-2N})
\end{aligned}$$

where there are N pairs of non-real, complex-conjugate α_i.

Proof. Without loss of generality we may suppose that the nonzero α_i with positive imaginary part are $\alpha_1, \ldots, \alpha_N$, and that $\bar{\alpha}_i = \alpha_{N+i}$ for $i = 1, \ldots, N$. That is, the complex conjugates of $\alpha_1, \ldots, \alpha_N$ are $\alpha_{N+1}, \ldots, \alpha_{2N}$. The first imaginary term in A_0^2 is the z^ν term, and the imaginary part of its coefficient has already been given the name E. Thus for real w we have

$$\begin{aligned}
A_0^2 &= 1 + \text{Re } O(t) + iEt^\nu w^\nu + O(t^{\nu+1}w^{\nu+1}) \\
A^2 &= A_0^2 \prod_{i=1}^{m} (z - a_i(t))^2 \\
&= \left[(1 + \text{Re } O(t)) + iEt^\nu w^\nu + O(t^{\nu+1}w^{\nu+1})\right] t^{2m} w^{2m-2N} \\
&\quad \prod_{i=1}^{N} \left[(w - \alpha_i - i\hat{a}_i t)(w - \bar{\alpha}_i - i\hat{a}_{N+i} t)\right]^2
\end{aligned}$$

Since $\hat{a}_i = \hat{a}_{N+i}$ we have

$$\begin{aligned}
A^2 &= \left[(1 + \text{Re } O(t)) + iEt^\nu w^\nu + O(t^{\nu+1}w^{\nu+1})\right] t^{2m} w^{2m-2N} \\
&\quad \prod_{i=1}^{N} \left[(w - \alpha_i - i\hat{a}_i t)(w - \bar{\alpha}_i - i\hat{a}_i t)\right]^2
\end{aligned}$$

The terms in the indexed product can be simplified as follows (for real w):

$$\begin{aligned}
&\left[(w - \alpha_i - i\hat{a}_i t)(w - \bar{\alpha}_i - i\hat{a}_i t)\right]^2 \\
= &\left[|w - \alpha_i|^2 - \hat{a}_i t(w - \alpha_i + w - \bar{\alpha}_i) - \hat{a}_i t^2\right]^2 \\
= &\left[|w - \alpha_i|^2 - 2\hat{a}_i t(w - \text{Re } \alpha_i) - \hat{a}_i t^2\right]^2 \\
= &\ |w - \alpha_i|^4 - 4i\hat{a}_i |w - \alpha_i|^2 (w - \text{Re } \alpha_i) + O(t^2)
\end{aligned}$$

Putting this expression into the indexed product again we obtain

$$\begin{aligned}
A^2 &= \left[(1 + \text{Re } O(t)) + iEt^\nu w^\nu + O(t^{\nu+1}w^{\nu+1})\right] t^{2m} w^{2m-2N} \\
&\quad \prod_{i=1}^{N} \left[|w - \alpha_i|^4 - 4i\hat{a}_i |w - \alpha_i|^2 (w - \text{Re } \alpha_i) + O(t^2)\right] \qquad (8)
\end{aligned}$$

Taking the real part, we find

$$\begin{aligned}
\text{Re } A^2 &= (1 + \text{Re } O(t)) t^{2m} w^{2m-2N} \prod \left[|w - \alpha_i|^4 + O(t^2)\right] \\
&= t^{2m} w^{2m-2N} \prod |w - \alpha_i|^4 + O(t^{2m+1} w^{2m-2N}) \\
&= t^{2m} \mathbf{A}^2 + O(t^{2m+1} w^{2m-2N})
\end{aligned}$$

for real w. This is the first formula claimed in the lemma.

Now take the imaginary part of (8). We have

$$\begin{aligned}\text{Im } A^2 &= (1+\text{Re } O(t))t^{2m}w^{2m-2N}\big[Et^\nu w^\nu \prod |w-\alpha_i|^4 \\ &\quad -4\prod(w-\alpha_i)^4 \sum_{i=1}^N \frac{\hat{a}_i t(w-\text{Re } \alpha_i)}{|w-\alpha_i|^2}\big]\end{aligned}$$

Since $\nu > 1$ by Lemma 3, we can drop the term in t^ν. We then have

$$\begin{aligned}\text{Im } A^2 &= -4(1+\text{Re } O(t))t^{2m}w^{2m-2N}\prod(w-\alpha_i)^4 \sum_{i=1}^N \frac{\hat{a}_i t(w-\text{Re } \alpha_i)}{|w-\alpha_i|^2} \\ &= -4t^{2m+1}w^{2m-2N}\prod(w-\alpha_i)^4 \sum_{i=1}^N \frac{\hat{a}_i (w-\text{Re } \alpha_i)}{|w-\alpha_i|^2} \\ &\quad + O(t^{2m+2}w^{2m-2N}) \\ &= -4t^{2m+1}\mathbf{A}^2 \sum_{i=1}^N \frac{\hat{a}_i t(w-\text{Re } \alpha_i)}{|w-\alpha_i|^2} + O(t^{2m+2}w^{2m-2N})\end{aligned}$$

That completes the proof of the lemma.

The case $m = 1$

The case $m = 1$ requires a different treatment, since there are only m of the α_i, so in case $m = 1$ there is only one of them, so we cannot have two α_i which are complex conjugates. Instead we will split A^2 as a whole, instead of splitting only A. Recall that the Weierstrass function $f(z)$ of the original surface, given by $f(z) = (d/dz)({}^1u - {}^2u)$, is $A_0^2 z^{2m}$. We choose a complex number α, and a real number $\hat{a}$, which will be 1 or -1; we will specify the sign of $\hat{a}$ later. For z real and $w = z/t$ we have

$$\begin{aligned}A_2 &:= A_0^2(z-\alpha t \ -i\,\hat{a}t^2)(z-\bar{\alpha}t - i\,\hat{a}t^2) \\ \mathbf{A}_2 &:= (w-\alpha)(w-\bar{\alpha}) \\ &= |w^2+\alpha|^2 \\ &= w^2 - 2\text{Re}(\alpha) + |\alpha|^2\end{aligned}$$

Of course, if we were to use this formula in the Weierstrass representation of a minimal surface, we would be introducing $g(z) = \sqrt{A_2}$, which is not analytic, since A_2 has a zero at it so there is a branch cut of the square root function in the parameter domain. But, we will not introduce A_2 into the Weierstrass representation. We use A_2 only to define $\tilde{X}$, and we define, as before, $\tilde{u}$ to be the harmonic extension of $\Gamma \circ \tilde{X}$. Specifically we take

$$\tilde{X} = \frac{1}{2}\int_0^x A_2(x) + B^2(x)dx$$

where as before B^2 does not depend on t but is just the $-fg^2$ of the given surface u. This defines a one-parameter family of harmonic surfaces jointly analytic in t and z.

Lemma 6 *Suppose $m = 1$ and A_2 is as defined above. We have, for real w,*

$$\begin{aligned} \text{Re } A_2 &= t^2\mathbf{A}_2 + O(t^4) \\ \text{Im } A_2 &= -2t^3\mathbf{A}_2\frac{\hat{a}(w - \text{Re}(\alpha)}{|w-\alpha|^2} + O(t^4) \end{aligned}$$

Remark. This is the same formula as in the previous lemma, with $\mathbf{A}_2$ in place of $\mathbf{A}^2$, and $m = N = 1$.
Proof.

$$\begin{aligned} A_2 &= A_0^2(z - \alpha t \ - i\,\hat{a}t^2)(z - \bar{\alpha}t - i\,\hat{a}t^2) \\ &= (1 + O(z))(wt - \alpha t - i\,\hat{a}t^2)(wt - \bar{\alpha}t - i\,\hat{a}t^2) \\ \text{Re}\, A_2 &= t^2(1 + O(tw))(w^2 + |\alpha|^2 - \hat{a}^2t^2)) \\ &= t^2(w^2 - 2\text{Re}(\alpha) + |\alpha|^2) + O(t^4) \end{aligned}$$

proving the first part of the lemma.

The first term with nonzero imaginary part in A_0^2 (for real z) is Ez^ν.

$$\begin{aligned} \text{Im}\, A_2 &= \text{Im}\, A_0t^2(\text{Re}\, A_2 + O(t)) + \text{Re}\, A_0t^2\text{Im}((w - \alpha - i\,\hat{a}t)(w - \bar{\alpha} - i\hat{a}t)) \\ &= t^2(Ez^{\nu+2} + O(z^{\nu+3}))\text{Re}(\mathbf{A}_2) \\ &\quad + t^2(1 + O(z))\text{Im}((w - \alpha - i\,\hat{a}t)(w - \bar{\alpha} - i\hat{a}t)) \\ &= t^2(1 + O(z))\left[O(z^3) + \text{Im}((w - \alpha - i\,\hat{a}t)(w - \bar{\alpha} - i\hat{a}t))\right] \\ &= t^2(1 + O(tw))\left[O(t^3w^3) - 2t(w - \text{Re}(\alpha)\hat{a} - t^2\hat{a}^2\right] \\ &= t^3(1 + O(tw))\left[O(t^2w^3) - 2(w - \text{Re}(\alpha)\hat{a} - t\hat{a}^2\right] \\ &= -2t^3\left[(w - \text{Re}(\alpha)\hat{a} + O(t)\right] + O(t^4w^2) \\ &= -2t^2(w - \text{Re}(\alpha)\hat{a}) + O(t^4) \\ &= -2t^2\mathbf{A}_2\frac{\hat{a}(w - \text{Re}(\alpha))}{|w-\alpha|^2} + O(t^4) \end{aligned}$$

in view of $\mathbf{A}_2 = |w - \alpha|^2$. That completes the proof of the lemma.
Remark: Nothing prevents us from using this method when $m > 1$. In particular when m is odd, if we wanted to avoid having $\alpha_i = 0$, we could split the last zero of f as we have done here for $m = 1$.

Calculation of the Dirichlet Integral

The Dirichlet integral is given by

$$E[u] \quad = \quad \frac{1}{2}\int\int |\nabla u|^2\, dx\, dy$$

$$= \int_0^\infty u\, u_y \, dx$$

The last formula can obtained by conformal transformation back to a disk, then integrating by parts in the disk and then transforming back to the half plane, which is easier than integrating over a semicircle of large radius and worrying about the integral on the circular part.

We divide this integral into pieces as follows: $|x| \le Rt$ is the first piece; $Rt \le |x| \le (1+t)Rt$ are the second two pieces; and $(1+t)Rt \le |x$ are the third and fourth pieces. The first piece includes the branch point when $t = 0$.

Before calculating $E[\tilde{u}]$, we begin by calculating $E[u]$. On $-(1+t)Rt \le x \le (1+t)Rt$, we have

$$\begin{aligned}
u &= \begin{bmatrix} \frac{1}{2}\mathrm{Re}\int f + fg^2\,dx \\ \frac{1}{2}\mathrm{Im}\int f - fg^2\,dx \\ \mathrm{Re}\int fg\,dx \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2(2m+1)}x^{2m+1} + \dots \\ \frac{E}{2(2m+\nu+1)}x^{2m+\nu+1} + \frac{1}{2m+2k+\delta}x^{2m+2k+\delta} + \dots \\ \frac{G}{2m+k+\delta+1}x^{2m+k+\delta+1} + \dots \end{bmatrix} \\
u_y &= -\mathrm{Im}\ u_z \\
&= -\mathrm{Im}\begin{bmatrix} \frac{1}{2}(f + fg^2) \\ \frac{i}{2}(f - fg^2) \\ fg \end{bmatrix} \\
&= -\begin{bmatrix} \frac{E}{2}x^{2m+\nu} + \frac{G\,\mathrm{Im}(c_k)}{2}x^{2m+2k+\delta} + \dots \\ \frac{1}{2}x^{2m} + \dots \\ \mathrm{Im}(c_k)x^{2m+k} + \dots \end{bmatrix}
\end{aligned}$$

We note for future reference the first component of this last equation:

$$^1u_y = -\frac{E}{2}x^{2m+\nu} - \frac{G\,\mathrm{Im}(c_k)}{2}x^{2m+2k+\delta} + \dots \tag{9}$$

Continuing with the calculation of $E[u]$, we have

$$\begin{aligned}
uu_y &= C_3x^{4m+\nu+1} + C_4x^{4m+2k+\delta+1} + C_8x^{4m+2k+\delta+1} + \dots \\
&= C_3x^{4m+\nu+1} + C_5x^{4m+2k+\delta+1} + \dots
\end{aligned}$$

where the C_i are constants whose exact value does not matter. Incidentally, $C_3 \neq 0$, since

$$C_3 = \frac{E}{4(2m+\nu+1)} - \frac{E}{4(2m+1)} \tag{10}$$

but we do not need this fact (and in the non-analytic boundary case, we will not necessarily have such fine control over this term). When we substitute $x = tw$, we get

$$
\begin{aligned}
uu_y &= C_3 t^{4m+\nu+1} w^{4m+\nu+1} + C_8 t^{4m+2k+\delta+1} w^{4m+2k+\delta+1} \\
&\quad + O(t^{4m+\nu+2}) + O(t^{4m+2k+\delta+2})
\end{aligned} \tag{11}
$$

valid in the region $|w| \leq R$.

Now we turn from u to $\tilde{u}$. We have

$$
\tilde{u}\tilde{u}_y = {}^1\tilde{u}\,{}^1\tilde{u}_y + {}^2\tilde{u}\,{}^2\tilde{u}_y + {}^3\tilde{u}\,{}^3\tilde{u}_y
$$

and the plan is to compute each of the three terms separately (on the real axis), then add them up and integrate. We have, for $m > 1$,

$$
\begin{aligned}
{}^1\tilde{u} &= \mathrm{Re} \int_0^x A^2 + B^2\, dz \\
&= \mathrm{Re} \int_0^x A^2\, dz + \mathrm{Re} \int_0^x B^2\, dz
\end{aligned}
$$

For $m = 1$, we must write A_2 in place of A^2. Otherwise the same arguments will apply. We could introduce a new letter, for example A_3, to stand for A^2 if $m > 1$ and A_2 if $m = 1$, but it seems less confusing to simply ask the reader to remember that when $m = 1$, A^2 means A_2. Note that $\mathbf{A}$ never occurs by itself below, but only in the context $\mathbf{A}^2$.

We transform to the w-plane, changing A^2 to $\mathbf{A}^2$. The factor A_0 becomes $1 + O(t)$ and disappears. The B^2 term starts with a higher power of t and is absorbed into the $O(t)$ term. On the real axis, $\mathbf{A}^2$ is real, so we can drop Re.

$$
{}^1\tilde{u} = t^{2m+1}(1 + O(t)) \int_0^w \mathbf{A}^2 dw \tag{12}
$$

Differentiating, we have

$$
\begin{aligned}
{}^1\tilde{u}_y &= -\mathrm{Im}\ {}^1\tilde{u}_z \\
&= -\mathrm{Im}\ \frac{d}{dz} \int_0^z A^2 + B^2\, dz \\
&= -\mathrm{Im}\ (A^2 + B^2) \\
&= -\mathrm{Im}\ A^2 + O(z^{2m+2k}) \\
&= -4t^{2m+1} \mathbf{A}^2 \sum_{i=1}^{N} \frac{\hat{a}_i (w - \mathrm{Re}\ \alpha_i)}{|w - \alpha_i|^2} + O(t^{2m+2} w^{2m-2N})
\end{aligned}
$$

by Lemma 5 if $m > 1$. In case $m = 1$, we take $N = 1$, so the sum contains only one term, and appeal to Lemma 6 to justify the last step. The error term in

Lemma 6 is also correct, namely $O(t^4)$, which is what results in the last equation with $N = 1$ and $m = 1$. Multiplying this by (12) we find (for real w)

$$^1\tilde{u}^1\tilde{u}_y = -t^{4m+2}\mathbf{A}^2 \int_0^w \mathbf{A}^2 dw \left(\sum \frac{\hat{a}_i(w - \text{Re } \alpha_i)}{|w - \alpha_i|^2} \right) + O(t^{4m+3}w^{2m-2N}) \quad (13)$$

Now we turn to the calculation of $^2\tilde{u}$, which is the harmonic extension of $\Gamma_2(\tilde{X})$. On the real axis we have

$$^2\tilde{u} = \frac{C_1\tilde{X}^{q+1}}{q+1} + O(\tilde{X}^{q+2})$$

Substituting $\tilde{X} = \frac{1}{2}\int A^2 + B^2 dx = \frac{1}{2}t^{2m+1}\int \mathbf{A}^2 dw + O(t^{2m+2})$ we have

$$\begin{aligned} ^2\tilde{u} &= \frac{C_1 t^{(2m+1)(q+1)}}{(q+1)2^{q+1}} \left(\int \mathbf{A}^2 dw \right)^{q+1} + O(t^{(2m+1)(q+1)+1}) &(14)\\ &= O(t^{(2m+1)(q+1)}) &(15) \end{aligned}$$

since $q \geq 1$.

Similarly we can calculate $^3\tilde{u}$:

$$\begin{aligned} ^3\tilde{u} &= \frac{C_1\tilde{X}^{p+1}}{p+1} \\ &= \frac{C_2 t^{(2m+1)(p+1)}}{(p+1)2^{p+1}} \left(\int \mathbf{A}^2 dw \right)^{p+1} + O(t^{(2m+1)(p+1)+1}) &(16)\\ &= O(t^{(2m+1)(p+1)}) &(17) \end{aligned}$$

Lemma 7 *On $|z| \leq Rt$ we have*

$$\begin{aligned} |^2\tilde{u}_y - {}^2u_y| &= O(t^{(2m+1)(q-1)+4m}) \\ |^3\tilde{u}_y - {}^3u_y| &= O(t^{(2m+1)(p-1)+4m}) \end{aligned}$$

Proof. We have

$$\begin{aligned} ^2\tilde{u} &= \frac{C_2 s^{q+1}}{q+1} + O(s^{q+2}) \\ ^2\tilde{u}_x &= C_2 s^q s_x + O(s^{q+1}) s_x \\ ^2\tilde{u}_{xx} &= C_2 q s^{q-1} s_x^2 + C_2 s^q s_{xx} + O(s^q) s_x^2 + O(s^{q+1}) s_{xx} \\ s &= \frac{1}{2}\int_0^x |A^2| + |B^2|\, dx \\ s_x &= \frac{1}{2}(|A^2| + |B^2|) = \frac{1}{2}t^{2m}\mathbf{A}^2 + O(t^{2m+1}) \end{aligned}$$

$$
\begin{aligned}
s_{xx} &= \frac{1}{2}A^2\sum_{i=1}^{m}\frac{2}{x-a_i(t)} + O(z^{2m+2k-1}) \\
&= A^2\sum_{i=1}^{m}\frac{1}{x-a_i(t)} + O(z^{2m+2k-1})
\end{aligned}
$$

Changing z to tw, and adding the pairs of terms in the sum corresponding to complex-conjugate pairs α_i and α_{N+i}, and noting that the terms for which $\alpha_i = 0$ all have a factor of w in the denominator, so they can be grouped together, we have

$$
\begin{aligned}
s &= \frac{1}{2}t^{2m+1}\int \mathbf{A}^2\,dw + O(t^{2m+2}) \\
s_{xx} &= t^{2m-1}\mathbf{A}^2\left[\frac{m-N}{w} + 2\sum_{i=1}^{N}\frac{w-\text{Re }\alpha_i}{|w-\alpha_i|^2}\right] + O(t^{2m}) \\
{}^2\tilde{u}_{xx} &= C_2 q s^{q-1}s_x^2 + C_2 s^q s_{xx} + O(s^q)s_x^2 + O(s^{q+1})s_{xx} \\
&= C_2 q t^{(2m+1)(q-1)}\left(\frac{1}{2}\int_0^w \mathbf{A}^2 dw\right)^{q-1}\frac{t^{4m}\mathbf{A}^4}{4} \\
&\quad + C_2 t^{(2m+1)q}\left(\frac{1}{2}\int_0^2 \mathbf{A}^2 dw\right)^{q} t^{2m-1}\mathbf{A}^2\left[\frac{m-N}{w} + 2\sum\frac{w-\text{Re }\alpha_i}{|w-\alpha_i|^2}\right] \\
&\quad + O(t^{(2m+1)q})O(t^{4m}) + O(t^{(2m+1)(q+1)})O(t^{2m-1}) \\
{}^2\tilde{u}_{xx} &= \frac{C_2 q}{2^{q+1}}t^{(2m+1)(q-1)}(\int_0^w \mathbf{A}^2 dw)^{q-1}t^{4m}\mathbf{A}^4 \\
&\quad + \frac{C_2}{2^q}t^{(2m+1)q}\left(\int_0^2 \mathbf{A}^2 dw\right)^{q} t^{2m-1}\mathbf{A}^2\left[\frac{m-N}{w} + 2\sum_{i=1}^{N}\frac{w-\text{Re }\alpha_i}{|w-\alpha_i|^2}\right] \\
&\quad + O(t^{(2m+1)q+4m}) \\
&= \frac{C_2}{2^q}t^{(2m+1)(q-1)+4m}\left(\int_0^w \mathbf{A}^2 dw\right)^{q-1} \\
&\quad \mathbf{A}^2\left[\frac{q\mathbf{A}^2}{2} + \int_0^w \mathbf{A}^2 dw\left[\frac{m-N}{w} + 2\sum_{i=1}^{N}\frac{w-\text{Re }\alpha_i}{|w-\alpha_i|^2}\right]\right] \\
&\quad + O(t^{(2m+1)q+4m})
\end{aligned}
$$

and therefore, for some constant C_{15} depending only on R, we have

$$
|{}^2\tilde{u}_{xx}| \le C_{15}t^{(2m+1)(q-1)+4m} \tag{18}
$$

We can make a similar calculation for u instead of $\tilde{u}$:

$$
{}^2u = \frac{C_2 s^{q+1}}{q+1} + O(s^{q+2})
$$

$$
\begin{aligned}
{}^2u_x &= C_2 s^q s_x + O(s^{q+1}) s_x \\
{}^2u_{xx} &= C_2 q s^{q-1} s_x^2 + C_2 s^q s_{xx} + O(s^q) s_x^2 + O(s^{q+1}) s_{xx} \\
s &= \frac{1}{2}\int_0^x f - fg^2 dx \\
&= \frac{1}{2(2m+1)} t^{2m+1} w^{2m+1} + O(t^{2m+2} w^{2m+2}) \\
s_x &= \frac{1}{2} x^{2m} + O(x^{2m+1}) = \frac{1}{2} t^{2m} w^{2m} + O(t^{2m+1}) \\
s_{xx} &= m x^{2m-1} + O(x^{2m-2}) \\
&= m t^{2m-1} w^{2m-1} + O(t^{2m-2}) \\
{}^2u_{xx} &= \frac{C_2}{2^{q+1}(2m+1)^{q-1}} t^{(2m+1)(q-1)} w^{(2m+1)(q-1)} t^{4m} w^{4m} \\
&\quad + \frac{mC_2}{2^q (2m+1)^q} t^{(2m+1)q} w^{(2m+1)q} t^{2m-1} w^{2m-1} \\
&\quad + O(t^{(2m+1)q+4m}) + O(t^{(2m+1)(q+1)+2m-1}) \\
&= \frac{4m+1}{2^{q+1}(2m+1)^q} C_2 t^{(2m+1)(q-1)+4m} w^{(2m+1)(q-1)+4m} + O(t^{2m+1)q+4m})
\end{aligned}
$$

and therefore, for some constant C_{16} depending only on R, we have

$$
|{}^2u_{xx}| \le C_{16} t^{(2m+1)(q-1)+4m} \tag{19}
$$

Remark: Both second derivatives (of u_2 and $\tilde{u}_2$) have the same asymptotic power of t, but a different function of w as coefficient of that power. A calculation can be made directly from the Weierstrass representation of u, but this involves the unknown powers ν and δ. These powers can be connected to the geometric numbers p and q by a Lewy-style analysis, as above, but this directly geometric analysis is better for our purpose (namely, getting rid of the terms in Dirichlet's integral arising from u_2 and u_3).

By (18),(19) and the triangle inequality, we have

$$
|{}^2\tilde{u}_{xx} - {}^2u_{xx}| \le (C_{15} + C_{16}) t^{(2m+1)(q-1)+4m}.
$$

Applying Lemma 4, we have for some constant C_{17} depending only on R,

$$
|{}^2\tilde{u}_y - {}^2u_y| \le C_{17} t^{(2m+1)(q-1)+4m},
$$

which proves the first claim of the lemma. The bound on $|{}^3\tilde{u}_y - {}^3u_y|$ is proved by the same calculation, changing ${}^2u$ to ${}^3u$ and q to p. That completes the proof of the lemma.

Lemma 8 *For* $|w| \le (1+t)R$ *we have*

$$
\begin{aligned}
&|{}^2\tilde{u}\,{}^2\tilde{u}_y - {}^2u\,{}^2u_y| + |{}^3\tilde{u}\,{}^3\tilde{u}_y - {}^3u\,{}^3u_y| \\
&= O(t^{(2m+1)(q+1)+2m}) + O(t^{(2m+1)(p+1)+2m+k-1})
\end{aligned}
$$

Proof. Consider

$$ {}^2\tilde{u}\,{}^2\tilde{u}_y - {}^2u\,{}^2u_y. $$

Adding and subtracting ${}^2\tilde{u}\,{}^2u_y$ we find

$$ {}^2\tilde{u}\,{}^2\tilde{u}_y - {}^2u\,{}^2u_y = ({}^2\tilde{u}_y - {}^2u_y){}^2\tilde{u} + ({}^2\tilde{u} - {}^2u){}^2u_y $$

Using the triangle inequality, we have

$$ |{}^2\tilde{u}\,{}^2\tilde{u}_y - {}^2u\,{}^2u_y| \leq |{}^2\tilde{u}_y - {}^2u_y||{}^2\tilde{u}| + |{}^2\tilde{u} - {}^2u||{}^2u_y| $$

By Lemma 7 we have

$$ |{}^2\tilde{u}\,{}^2\tilde{u}_y - {}^2u\,{}^2u_y| \leq O(t^{(2m+1)(q-1)+4m})|{}^2\tilde{u}| + |{}^2\tilde{u} - {}^2u||{}^2u_y| $$

By (15) we have $|{}^2\tilde{u}| = O(t^{(2m+1)(q+1)})$. Hence

$$ \begin{aligned} |{}^2\tilde{u}\,{}^2\tilde{u}_y - {}^2u\,{}^2u_y| &\leq O(t^{(2m+1)(q-1)+4m})O(t^{(2m+1)(q+1)}) + |{}^2\tilde{u} - {}^2u||{}^2u_y| \\ &\leq O(t^{2q(2m+1)+4m}) + |{}^2\tilde{u} - {}^2u||{}^2u_y| \qquad (20) \end{aligned} $$

Next we estimate $|{}^2\tilde{u} - {}^2u|$. Let $\tilde{s}$ be arc length along the real axis for $\tilde{u}$, and s be arc length for u. Then

$$ \begin{aligned} {}^2\tilde{u} - {}^2u &= \frac{C_2}{q+1}(\tilde{s}^{q+1} - s^{q+1}) + O(\tilde{s}^{q+2}) + O(s^{q+2}) \\ &= \frac{C_2}{q+1}t^{(2m+1)(q+1)}\left[(\left(\int_0^w \mathbf{A}^2 dw\right)^{q+1} - \frac{w^{2m+1}}{2m+1}\right] \\ &\quad + O(t^{(2m+1)(q+2)}) \\ &= O(t^{(2m+1)(q+1)}) \end{aligned} $$

Putting this result into (20) we have

$$ |{}^2\tilde{u}\,{}^2\tilde{u}_y - {}^2u\,{}^2u_y| \leq O(t^{2q(2m+1)+4m}) + O(t^{(2m+1)(q+1)})|{}^2u_y| \qquad (21) $$

The term ${}^2u_y$ can be explicitly computed from the Weierstrass representation. We have

$$ \begin{aligned} {}^2u_y &= -\mathrm{Im}\ {}^2u_z \\ &= -\mathrm{Im}\left(\frac{i}{2}(f + fg^2)\right) \\ &= -\frac{1}{2}\mathrm{Re}(f + fg^2) \\ &= -\frac{1}{2}\mathrm{Re}(A^2 - B^2) \\ &= -\frac{1}{2}t^{2m}\mathbf{A}^2 + O(t^{2m+1}) \\ &= O(t^{2m}) \end{aligned} $$

Putting this result into (21) we have

$$\begin{aligned}|{}^2\tilde{u}\,{}^2\tilde{u}_y - {}^2u\,{}^2u_y| &\leq O(t^{2q(2m+1)+4m}) + O(t^{(2m+1)(q+1)})O(t^{2m}) \\ &\leq O(t^{2q(2m+1)+4m}) + O(t^{(2m+1)(q+1)+2m})\end{aligned}$$

The first term has a larger exponent than the second, so we can drop it, arriving at

$$|{}^2\tilde{u}\,{}^2\tilde{u}_y - {}^2u\,{}^2u_y| = O(t^{(2m+1)(q+1)+2m}) \tag{22}$$

We must now make a similar calculation for the third components ${}^3\tilde{u}$ and ${}^3u$. The first part of the calculation is exactly the same, except for changing the coordinate index from 2 to 3 and changing q to p. In this way we arrive at

$$|{}^3\tilde{u}\,{}^3\tilde{u}_y - {}^3u\,{}^3u_y| \leq O(t^{2p(2m+1)+4m}) + O(t^{2m+1)(p+1)})|{}^3u_y| \tag{23}$$

The term ${}^3u_y$ can be explicitly computed from the Weierstrass representation. We have

$$\begin{aligned}{}^3u_y &= -\mathrm{Im}\ {}^3u_z \\ &= -\mathrm{Im}(fg) \\ &= -\mathrm{Im}(c_k)z^{2m+k} + O(z^{2m+k+1}) \\ &= O(t^{2m+k})\end{aligned}$$

Putting this result into (23) we have

$$\begin{aligned}|{}^3\tilde{u}\,{}^3\tilde{u}_y - {}^3u\,{}^3u_y| &\leq O(t^{2p(2m+1)+4m}) + O(t^{2m+1)(p+1)})O(t^{2m+k}) \\ &\leq O(t^{2p(2m+1)+4m}) + O(t^{(2m+1)(p+1)+2m+k})\end{aligned}$$

The exponent of the second term can be compared to the exponent of the first term using the bound $k \leq (2m+1)p$ derived in Theorem 1. We have

$$\begin{aligned}(2m+1)(p+1)+2m+k &\leq (2m+1)(p+1)+2m+(2m+1)p \\ &\leq (2m+1)(2p+1)+2m \\ &\leq 2p(2m+1)+4m+1\end{aligned}$$

which is one more than the exponent of the first term. Therefore, if we decrease the exponent of the second term by one, we can drop the first term, arriving at

$$|{}^3\tilde{u}\,{}^3\tilde{u}_y - {}^3u\,{}^3u_y| \leq O(t^{(2m+1)(p+1)+2m+k-1}) \tag{24}$$

Combining equations (22) and (24), we have

$$\begin{aligned}&{}^2\tilde{u}\,{}^2\tilde{u}_y - {}^2u\,{}^2u_y + {}^3\tilde{u}\,{}^3\tilde{u}_y - {}^3u\,{}^3u_y & (25)\\ =\ & O(t^{(2m+1)(q+1)+2m}) + O(t^{(2m+1)(p+1)+2m+k-1}) & (26)\end{aligned}$$

on $|w| \leq (1+t)R$. That completes the proof of the lemma.

Lemma 9 $\int_{-Rt}^{Rt} \tilde{u}\tilde{u}_y\,dx - \int_{-Rt}^{Rt} uu_y\,dx = ct^{4m+3} + O(t^{4m+4})$, *where the sign of* c *can be made positive or negative by choosing the signs of the* $\hat{a}_i$.

Proof.

$$\begin{aligned}
\tilde{u}\tilde{u}_y - uu_y &= {}^1\tilde{u}\,{}^1\tilde{u}_y - {}^1u\,{}^1u_y \qquad (27)\\
&\quad + {}^2\tilde{u}\,{}^2\tilde{u}_y - {}^2u\,{}^2u_y + {}^3u\,{}^3u_y - {}^3u\,{}^3u_y\\
&= {}^1\tilde{u}\,{}^1\tilde{u}_y - {}^1u\,{}^1u_y \qquad (28)\\
&\quad + O(t^{(2m+1)(q+1)+2m}) + O(t^{(2m+1)(p+1)+2m+k-1}) \qquad (29)
\end{aligned}$$

by Lemma 8. We now work on the terms from ${}^1\tilde{u}$ and ${}^1u$. By (13), we have

$${}^1\tilde{u}\,{}^1\tilde{u}_y = -t^{4m+2}\mathbf{A}^2 \int_0^w \mathbf{A}^2 dw \left[\sum_{i=1}^N \frac{\hat{a}_i(w - \text{Re }\alpha_i)}{|w-\alpha_i|^2}\right] + O(t^{4m+3}w^{2m-2N})$$

and by (11) we have

$$\begin{aligned}
uu_y &= C_3 t^{4m+\nu+1} w^{4m+\nu+1} + C_5 t^{4m+2k+\delta+1} w^{4m+2k+\delta+1}\\
&\quad + O(t^{4m+\nu+2}) + O(t^{4m+2k+\delta+2})
\end{aligned}$$

Subtracting the last two equations we have

$$\begin{aligned}
{}^1\tilde{u}\,{}^1\tilde{u}_y - {}^1u\,{}^1u_y &= -t^{4m+2}\mathbf{A}^2 \int_0^w \mathbf{A}^2 dw \left[\sum_{i=1}^N \frac{\hat{a}_i(w - \text{Re }\alpha_i)}{|w-\alpha_i|^2}\right]\\
&\quad + O(t^{4m+3}w^{2m-2N})\\
&\quad - C_3 t^{4m+\nu+1} w^{4m+\nu+1} - C_5 t^{4m+2k+\delta+1} w^{4m+2k+\delta+1}\\
&\quad + O(t^{4m+\nu+2}) + O(t^{4m+2k+\delta+2})\\
&= -t^{4m+2}\mathbf{A}^2 \int_0^w \mathbf{A}^2 dw \sum_{i=1}^N \frac{\hat{a}_i(w - \text{Re }\alpha_i)}{|w-\alpha_i|^2}\\
&\quad + O(t^{4m+3}w^{2m-2N}) + O(t^{4m+\nu+1}) + O(t^{4m+2k+\delta+1})
\end{aligned}$$

By Lemma 3, we have $\nu > 1$; and we also have $2k + \delta + 1 \geq 3$. We arrive at

$${}^1\tilde{u}\,{}^1\tilde{u}_y - {}^1u\,{}^1u_y = -t^{4m+2}\mathbf{A}^2 \int_0^w \mathbf{A}^2 dw \sum_{i=1}^N \frac{\hat{a}_i(w - \text{Re }\alpha_i)}{|w-\alpha_i|^2} + O(t^{4m+3})$$

Substituting this into equation (29) we have

$$\begin{aligned}
\tilde{u}\tilde{u}_y - uu_y &= -t^{4m+2}\mathbf{A}^2 \int_0^w \mathbf{A}^2 dw \sum_{i=1}^N \frac{\hat{a}_i(w - \text{Re }\alpha_i)}{|w-\alpha_i|^2}\\
&\quad + O(t^{4m+3}) + O(t^{(2m+1)(q+1)+2m}) + O(t^{(2m+1)(p+1)+2m+k-1})
\end{aligned}$$

We want to absorb both the last two error terms into the $O(t^{4m+3})$ term. To do that we must show that $(2m+1)(q+1)+2m \geq 4m+3$ and that $(2m+1)(p+1)+2m+k-1 \geq 4m+3$. From $q \geq 1$ we have

$$\begin{aligned}(2m+1)(q+1)+2m &\geq 6m+2 \\ &\geq 4m+4\end{aligned}$$

and from $p \geq 1$ we have

$$\begin{aligned}(2m+1)(p+1)+2m+k-1 &\geq 2(2m+1)+2m+k-1 \\ &\geq 6m+k+1 \\ &\geq 4m+k+3 \\ &\geq 4m+3\end{aligned}$$

Hence the two last error terms in question can indeed be absorbed into the $\mathbf{O}(t^{4m+3})$ term. We thus have

$$\tilde{u}\tilde{u}_y - uu_y = -t^{4m+2}\mathbf{A}^2 \int_0^w \mathbf{A}^2 dw \sum_{i=1}^N \frac{\hat{a}_i(w - \text{Re } \alpha_i)}{|w-\alpha_i|^2} + O(t^{4m+3}) \quad (30)$$

Integrating along the x-axis from $-Rt$ to Rt we find

$$\begin{aligned}&\int_{-Rt}^{Rt} \tilde{u}\tilde{u}_y - uu_y \, dx \\ = \; & -t^{4m+2} \int_{-Rt}^{Rt} \mathbf{A}^2 \int_0^w \mathbf{A}^2(\zeta) d\zeta \sum_{i=1}^N \frac{\hat{a}_i(w - \text{Re } \alpha_i)}{|w-\alpha_i|^2}) dx + \int_{-Rt}^{Rt} O(t^{4m+3}) dx\end{aligned}$$

Changing dx to $t\,dw$, we pick up an extra factor of t:

$$\begin{aligned}= \; & -t^{4m+3} \int_{-R}^{R} \mathbf{A}^2 \int_0^w \mathbf{A}^2(\zeta) d\zeta \sum_{i=1}^N \frac{\hat{a}_i(w - \text{Re } \alpha_i)}{|w-\alpha_i|^2} dw + \int_{-R}^{R} O(t^{4m+4}) dw \\ = \; & -t^{4m+3} \int_{-R}^{R} \mathbf{A}^2 \int_0^w \mathbf{A}^2(\zeta) d\zeta \sum_{i=1}^N \frac{\hat{a}_i(w - \text{Re } \alpha_i)}{|w-\alpha_i|^2} dw + O(t^{4m+4}) \\ = \; & -t^{4m+3} \sum_{i=1}^N \hat{a}_i \left[\int_{-R}^{R} \mathbf{A}^2 \int_0^w \mathbf{A}^2(\zeta) d\zeta \frac{w - \text{Re } \alpha_i}{|w-\alpha_i|^2} dw \right] + O(t^{4m+4})\end{aligned}$$

The summands are not zero, provided R was chosen large enough, since the integrand is asymptotic to w^{4m} for large $|w|$. Specifically, we have to choose R so large that for at least one i we have

$$\int_{-R}^{R} \mathbf{A}^2 \int_0^w \mathbf{A}^2(\zeta) d\zeta \frac{w - \text{Re } \alpha_i}{|w-\alpha_i|^2} dw > 0 \quad (31)$$

Now we define $\hat{a}_i$ to be 1 or -1, with the sign chosen so that the contribution of the i-th term in the sum is negative. That completes the proof of the lemma. *Remarks*: (1) if we choose R so large that the terms mentioned above are *all* positive, we can take all the $\hat{a}_i$ to have the same sign. (2) We could just as well make Dirichlet's integral increase as decrease, by choosing $\hat{a}_i$ to have the opposite sign.

Lemma 10 *$\tilde{u}$ takes the boundary monotonically.*

Proof. Recall that $\tilde{u}$ is the harmonic extension of $\Gamma(\tilde{X}(t,x))$, where $\tilde{X}(t,x) =$

$$\mathrm{Re}\left[\phi_1(x)\int_0^x A^2(t,x) + B^2(x)\,dx + \phi_2(x)\int_0^x A^2(0,x) + B^2(x)\,dx\right]$$

It suffices to show $\tilde{X}_x \geq 0$ on the real axis. We have

$$\begin{aligned} \tilde{X}_x &= \phi_1\,\mathrm{Re}(A^2(t,x) + B^2(x)) + \phi_2\,\mathrm{Re}(A^2(0,x) + B^2(x)) \\ &\quad + \phi_1'\int A^2(t,x) + B^2(x)\,dx + \phi_2'\int A^2(0,x) + B^2(x)\,dx \qquad (32) \end{aligned}$$

Since u takes the boundary monotonically, we have

$$u_x = \mathrm{Re}(A^2(0,x) + B^2(0,x)) \geq 0,$$

so the second term in (32) (which is the only nonzero term for $x \geq (1+t)Rt$) is nonnegative. We next consider the first term, which is the only nonzero term for $0 \leq x \leq Rt$. We will prove

$$\mathrm{Re}(A^2(t,x) + B^2(x)) \quad \geq \quad 0. \qquad (33)$$

By the definitions of $\mathbf{A}$ and N we have

$$\begin{aligned} \mathbf{A} &= \prod_{i=1}^{m}(w - \alpha_i) \\ &= w^{m-2N}\prod_{i=1}^{N}|w - \alpha_i|^2 \end{aligned}$$

Squaring, we have

$$\mathbf{A}^2 = w^{2m-4N}\prod_{i=1}^{N}|w - \alpha_i|^4$$

We have $B^2(x) = O(z^{2m+2k}) = O(t^{2m+2}w^{2m+2})$. By Lemma 5 we then have

$$\begin{aligned} \mathrm{Re}(A^2 + B^2) &= t^{2m}\mathbf{A}^2 + O(t^{2m+1}w^{2m-2N}) \\ &= t^{2m}w^{2m-4N}\prod_{i=1}^{N}|w - \alpha_i|^4 + O(t^{2m+1}w^{2m-4N}) \end{aligned}$$

$$\geq \quad (1+O(t))t^{2m}w^{2m-4N}\prod_{i=1}^{N}|w-\alpha_i|^4 \tag{34}$$
$$\geq \quad 0$$

for t sufficiently small. To treat $m = 1$, we only have to replace A^2 by the special function A_2 used in the definition of $\tilde{u}$ for $m = 1$. The above proof remains valid, appealing to Lemma 6 in place of Lemma 5 for the formula for $\mathrm{Re}(A_2)$. That completes the proof of (33).

On the interval $-Rt \leq w \leq 0$ we shall need a somewhat sharper estimate, as the positive contribution of this term is needed to outweigh a negative contribution from another term. Since we have assumed $\mathrm{Re}(\alpha_i) \geq 0$, it follows that when $w \leq 0$ we have

$$|w-\alpha_i| \geq |\alpha_i|.$$

(If this is not geometrically evident, one can quote the law of cosines in the triangle formed by w, α_i, and the origin to prove it.) It then follows from (34) that for w in $[-Rt, 0]$,

$$\mathrm{Re}(A^2+B^2) \quad \geq \quad (1+O(t))t^{2m}w^{2m-4N}\prod_{i=1}^{N}|\alpha_i|^4 \tag{35}$$

Remark: If none of the α_i are real, then X_x is never zero on the boundary, so $\tilde{u}$ is in that case a path through harmonic surfaces taking the boundary strictly monotonically. Otherwise, ${}^1\tilde{u}$ will vanish at the real α_i. That is OK as far as the present proof goes, but if we needed strict monotonicity, we could just choose the nonzero α_i not to be real.

Outside the intervals $[-Rt, 0]$ and $[Rt, (1+t)Rt]$, we have $\phi_1' = \phi_2' = 0$, and since both ϕ_1 and ϕ_2 are nonnegative, we have proved $\tilde{X}_x \geq 0$ there. We therefore can assume that x lies in one of these two intervals. First assume x is in $[Rt, (1+t)Rt]$. Define

$$c = \frac{1}{2}R^{2m-4N}\prod_{i=1}^{N}|R-\alpha_i|^4$$

and recall that we have assumed $R \geq 2|\alpha_i|$, which makes $|R-\alpha_i| \geq R/2$ and hence $c \geq R^{2m}/2^{4N+1}$. Then by (34) we have

$$\begin{aligned}\mathrm{Re}(A^2(t,x)+B^2(x)) \quad &\geq \quad ct^{2m} \\ &\geq \quad \frac{t^{2m}R^{2m}}{2^{4N+1}}.\end{aligned} \tag{36}$$

Similarly we have for the second term in $\tilde{X}_x$ that

$$\begin{aligned}\mathrm{Re}(A^2(0,x)+B^2(x)) \quad &\geq \quad \frac{1}{2}R^{2m}t^{2m} \\ &\geq \quad \frac{t^{2m}R^{2m}}{2^{4N+1}}\end{aligned} \tag{37}$$

and hence, adding (36) (37), we have (for x in $[Rt, (1+t)Rt]$)

$$\phi_1 \mathrm{Re}(A^2(t,x) + B^2(x)) + \phi_2 \mathrm{Re}(A^2(0,x) + B^2(x)) \geq \frac{t^{2m}R^{2m}}{2^{4N}} \tag{38}$$

in view of $\phi_1 + \phi_2 = 1$.

We now estimate the two terms involving the derivatives of the ϕ_i, in case $m > 1$. Note that $\phi_1 + \phi_2 = 1$, so $\phi_1' + \phi_2' = 0$. We have

$$\begin{aligned}
& \phi_1' \int A^2(t,x) + B^2(x)\,dx + \phi_2' \int A^2(0,x) + B^2(x)\,dx \\
= \quad & \phi_1' \int A^2(t,x) + B^2(x)\,dx + \phi_2' \int A^2(0,x) + B^2(x)\,dx \\
= \quad & \phi_1' \int A^2(t,x)\,dx + \phi_2' \int A^2(0,x)\,dx + (\phi_1' + \phi_2') \int B^2(x)\,dx \\
= \quad & \phi_1' \int A^2(t,x)\,dx + \phi_2' \int A^2(0,x)\,dx
\end{aligned}$$

Adding and subtracting $\phi_1' \int A^2(0,x)dx$ we have

$$\begin{aligned}
& \phi_1' \int A^2(t,x)\,dx - \phi_1' \int A^2(0,x)\,dx + \phi_1' \int A^2(0,x)\,dx + \phi_2' \int A^2(0,x)\,dx \\
= \quad & \phi_1' \int \left[A^2(t,x) - A^2(0,x)\right]\,dx + (\phi_1' + \phi_2') \int A^2 dx \\
= \quad & \phi_1' \int \left[A^2(t,x) - A^2(0,x)\right]\,dx \qquad (39) \\
= \quad & \phi_1' t^{2m+1} \int_0^w \left[w^{2m-4N} \prod_{i=1}^N |w - \alpha_i|^4 - w^{2m} + O(tw^{2m+1}) \right]\,dw \\
= \quad & \phi_1' t^{2m+1} \left(\int_0^w \left[w^{2m-4N} \prod_{i=1}^N |w - \alpha_i|^4 - w^{2m} \right]\,dw + O(tw^{2m+2}) \right) \\
= \quad & \phi_1' t^{2m+1} \left(\int_0^w w^{2m-4N} \left[\prod_{i=1}^N |w - \alpha_i|^4 - w^{4N} \right]\,dw + O(tw^{2m+2}) \right) \\
= \quad & t^{2m+1} \phi_1' \left(\int_0^w w^{2m-4N} \left[\prod_{i=1}^N |w - \alpha_i|^4 - w^{4N} \right]\,dw + O(tw^{2m+2}) \right)
\end{aligned}$$

The integrand is a polynomial of degree at most $2m-1$, so the integral is a polynomial of degree at most $2m$. Hence for small w it is $O(w^{2m})$, and hence large compared to the error term $O(tw^{2m+2})$. Now the argument splits into cases. Case 1: w lies in $[R, (1+t)R]$, and Case 2: w lies in $[-Rt, 0]$. These are the only cases we need consider, since these are the only intervals where ϕ_1' is nonzero.

Case 1: suppose w lies in $[R, (1+t)R]$. By Lemma 1, the integral is negative for $w \geq R$. Since $\phi_1' \leq 0$ for positive w, the product of the two negative terms is non-negative, as desired. We can therefore obtain a non-negative lower bound on the expression in question valid for sufficiently small t, by deleting the error term and putting in a factor of 1/2. The result is

$$\begin{aligned} & \phi_1' \int A^2(t,x) + B^2(x)\, dx + \phi_2' \int A^2(0,x) + B^2(x)\, dx \\ \geq \quad & \frac{1}{2} t^{2m+1} \phi_1' \int_0^w w^{2m-4N} \left[\prod_{i=1}^N |w - \alpha_i|^4 - w^{4N} \right] dw \end{aligned}$$

That completes the proof in case 1, for $m > 1$.

Case 2: Suppose w lies in $[-Rt, 0]$. By (35) we have

$$\mathrm{Re}(A^2 + B^2) \quad \geq \quad (1 + O(t)) t^{2m} w^{2m-4N} \prod_{i=1}^N |\alpha_i|^4$$

and hence it suffices to show that the terms involving ϕ_1' and ϕ_2' have magnitude bounded above by this quantity. In symbols, we have

$$\begin{aligned} \tilde{X}_x \quad & \geq \quad \mathrm{Re}\ \left[A^2(t,x) + B^2(0,x)\right] + \phi_1' \int_0^x \left[A^2(t,x) - A^2(0,x)\right] dx \\ & \geq \quad (1 + O(t)) t^{2m} w^{2m-4N} \prod_{i=1}^N |\alpha_i|^4 \\ & \quad + t^{2m+1} \phi_1' \int_0^w w^{2m-4N} \left[\prod_{i=1}^N |w - \alpha_i|^4 - w^{4N} + O(t) \right] dw \end{aligned}$$

The indexed product is a polynomial in w, whose constant term is $\prod |\alpha_i|^4$. Since we are concerned here with small w we write this polynomial as $\prod |\alpha_i|^4 + O(w)$. Putting this expression into the integrand we have

$$\begin{aligned} X_x \quad & \geq \quad (1 + O(t)) t^{2m} \prod_{i=1}^N |\alpha_i|^4 w^{2m-4N} \\ & \quad + t^{2m+1} \phi_1' \int_0^2 w^{2m-4N} \left[\prod_{i=1}^N |\alpha_i|^4 + O(w) + O(t) \right] dw \end{aligned}$$

Now we integrate the polynomial; note that since the $O(w)$ term represents a polynomial, integrating it produces an error term of one higher degree.

$$\begin{aligned} X_x \quad & \geq \quad (1 + O(t)) t^{2m} \prod_{i=1}^N |\alpha_i|^4 w^{2m-4N} \\ & \quad + \phi_1' \left[t^{2m+1} w^{2m-4N+1} \frac{\prod_{i=1}^N |\alpha_i|^4}{2m - 4N + 1} + O(t^{2m+1} w^{2m-4N+2}) \right] \end{aligned}$$

Since we have assumed w is in $[-Rt, 0]$, we have $\phi_1' \geq 0$, and it is multiplied by an odd power of w, which will be negative on this interval. Therefore the second term is negative. We can therefore replace it by the negative of an upper bound for its magnitude. We have the estimate

$$|\phi_1'| \leq \frac{1}{2|z|} = \frac{1}{2t|w|}$$

from Lemma 2. Putting this in for ϕ_1' we obtain

$$\begin{aligned} X_x &\geq (1 + O(t))t^{2m} \prod_{i=1}^{N} |\alpha_i|^4 w^{2m-4N} \\ &\quad - \frac{1}{2t|w|} \left[t^{2m+1} |w|^{2m-4N+1} \frac{\prod_{i=1}^{N} |\alpha_i|^4}{2m - 4N + 1} + O(t^{2m+1} w^{2m-4N+2}) \right] \\ &\geq t^{2m} \prod_{i=1}^{N} |\alpha_i|^4 w^{2m-4N} \left[1 - \frac{1}{2(2m - 4N + 1)} + O(t) + O(w) \right] \end{aligned}$$

Since on the interval $-Rt \leq w \leq 0$ we have $w = O(t)$, we can drop the $O(w)$ term:

$$X_x \geq t^{2m} \prod_{i=1}^{N} |\alpha_i|^4 w^{2m-4N} \left[1 - \frac{1}{2(2m - 4N + 1)} + O(t) \right]$$

Since $2m - 4N + 1 \geq 1$, the expression in brackets is at least $1/2$ for sufficiently small t; in particular it is positive. That completes the proof for $m > 1$.

We now take up the case $m = 1$, and examine the terms involving ϕ_1' and ϕ_2'. First suppose $Rt \leq w \leq (1 + t)Rt$. Then

$$\begin{aligned} &\phi_1' \int A^2(t, x) + B^2(x)\, dx + \phi_2' \int A^2(0, x) + B^2(x)\, dx \\ &= \phi_1' t^3 \left[\int_0^w |w - \alpha|^2 - w^2\, dw + O(tw^4) \right] \\ &= \phi_1' t^3 \left[\int_0^w -2\mathrm{Re}(\alpha) w + |\alpha|^2\, dw + O(tw^4) \right] \\ &= \phi_1' t^3 \left[(-\mathrm{Re}(\alpha) w^2 + |\alpha|^2 w) + O(tw^4) \right] \end{aligned}$$

If we choose $\mathrm{Re}(\alpha) > 0$, then the leading term of the polynomial in w is negative, and dominates the error term $O(tw^4)$, so for sufficiently small t, and positive w. the entire expression is nonnegative, as desired. That takes care of the interval $R \leq w \leq (1 + t)R$.

Now suppose w lies in $[-Rt, 0]$. With A_2 in place of A^2, we have

$$\tilde{X}_x \;\geq\; \mathrm{Re}\; \left[A_2(t,x) + B^2(0,x)\right] + \phi_1' \int_0^x]A_2(t,x) - A_2(0,x)]\; dx$$

We have

$$\begin{aligned} A_2 &= (x - \alpha t + O(t^2))(x - \bar{\alpha} t + O(t^2)) \\ &= t^2|w - \alpha|^2 + O(t^3) \\ &\geq (1 + O(t))t^2|\alpha|^2 \end{aligned}$$

Since $B^2(0,x) = t^{2m+2k}\mathbf{B}^2(w)$, it is $O(t^3)$, and we have

$$\begin{aligned} X_x &\geq (1 + O(t))t^2|\alpha|^2 + O(t^3) + \phi_1' \int_0^x [A_2(t,x) - A_2(0,x)]\; dx \\ &\geq (1 + O(t))t^2|\alpha|^2 + \phi_1' t^3 \int_0^w |w - \alpha|^2 - w^2\, dw + O(t^3) \\ &\geq (1 + O(t))t^2|\alpha|^2 + \phi_1' t^3 \left[-\mathrm{Re}(\alpha) w^2 + |\alpha|^2 w\right] + O(t^3) \\ &\geq (1 + O(t))t^2|\alpha|^2 + \phi_1' t^3 \left[|\alpha|^2 w + O(w^2)\right] + O(t^3) \end{aligned}$$

Now since w is negative, and ϕ_1' is positive, we may replace ϕ_1' by the upper bound given in Lemma 2:

$$\begin{aligned} \tilde{X}_x &\geq (1 + O(t))t^2|\alpha|^2 - t^3 \frac{1}{2t|w|}\left[|w| + O(w^2)\right] + O(t^3) \\ &\geq t^2|\alpha|^2 \left[1 - \frac{1}{2} + O(w^2)\right] + O(t^3) \\ &\geq \frac{1}{2} t^2 |\alpha|^2 + O(t^3) \end{aligned}$$

where we have dropped the $O(t^2 w^2)$ term since $w = O(t)$ in this interval. For sufficiently small t we have

$$\tilde{X}_x \;\geq\; \frac{1}{3} t^2 |\alpha|^2$$

This completes the proof of the monotonicity lemma.

Theorem 2 *Let Γ be a real-analytic Jordan curve. Let u be a minimal surface bounded by Γ, of the topological type of the disk, with a boundary branch point of order $2m$ on Γ, and suppose $n \leq m$. Then we can construct a C^n-smooth one-parameter family $\tilde{u}$ of minimal surfaces bounded by Γ, containing the original surface u, and converging to u in the C^n norm, such that the Dirichlet integral $E[\tilde{u}]$ is less than $E[u]$; indeed $E[u] - E[\tilde{u}] \geq ct^{4m+3}$ for some positive constant c.*

Corollary 1 *If u is a C^1 relative minimum of area (in particular if it is an absolute minimum of area) among disk-type surfaces bounded by a real-analytic Jordan curve Γ, then u has no boundary branch points.*

Proof. The family $\tilde{u}$ has been defined above as the harmonic extension of $\Gamma(\tilde{(}X(t,x))$, where $\tilde{X}(t,x) =$

$$\mathrm{Re}\left[\phi_1(x)\int_0^x A^2(t,x)+B^2(x)dx+\phi_2(x)\int_0^x A^2(0,x)+B^2(x)dx\right]$$

We have computed

$$\begin{aligned} E_1 &:= \int_0^{Rt} \tilde{u}\tilde{u}_y - uu_y dx \\ &= -ct^{4m+3}+O(t^{4m+4}) \end{aligned}$$

for some constant $c > 0$, and we must now show that integrals over the rest of the real line can be controlled, so as not to swamp the contribution near the origin. Specifically we have

$$E[{}^1\tilde{u}] - E[u] = E_1 + E_2 + E_3 \tag{40}$$

where

$$\begin{aligned} E_2 &:= \int_{-Rt^2}^0 {}^1\tilde{u}^1\tilde{u}_y - {}^1u^1u_y\,dx + \int_{Rt}^{(1+t)Rt} {}^1\tilde{u}^1\tilde{u}_y - {}^1u^1u_y\,dx \\ E_3 &:= \int_{-\infty}^{-Rt^2} {}^1\tilde{u}^1\tilde{u}_y - {}^1u^1u_y\,dx + \int_{(1+t)Rt}^{\infty} {}^1\tilde{u}^1\tilde{u}_y - {}^1u^1u_y\,dx \end{aligned}$$

We will estimate E_2 and E_3 to be $O(t^{4m+4})$. The estimates we used

$${}^i\tilde{u}^i\tilde{u}_y - {}^iu^iu_y$$

for $i = 2$ and $i = 3$ were valid for the entire boundary, since they were based on the global estimate for the gradient of a harmonic function in terms of the boundary values of the second derivative. Specifically we have as in (25),

$$\begin{aligned} {}^2\tilde{u}^2\tilde{u}_y - {}^2u^2u_y dy &= O(t^{(2m+1)(q+1)+2m}) \\ {}^3\tilde{u}^3\tilde{u}_y - {}^3u^3u_y &= O(t^{(2m+1)(p+1)+2m}) \end{aligned}$$

with these equations valid for all x. The error terms are uniform in θ after a conformal mapping of the half-plane to the unit disk, so that we have

$$\int_{-\infty}^{\infty} {}^2\tilde{u}^2\tilde{u}_y - u_2{}^2u_y\,dx = O(t^{(2m+1)(q+1)+2m}) \tag{41}$$

$$\int_{-\infty}^{\infty} {}^3\tilde{u}^3\tilde{u}_y - {}^3u^3u_y\,dx = O(t^{(2m+1)(p+1)+2m}) \tag{42}$$

We take up, for $i = 2$ and $i = 3$,

$$\int_{Rt}^{(1+t)Rt} {}^i\tilde{u}\,{}^i\tilde{u}_y - {}^i u\,{}^i u_y \, dx.$$

Let us define v to be the harmonic extension of $\Gamma \circ \tilde{X}$. This is the formula used to define $\tilde{u}$ for $0 \le z \le Rt$, so we have

$$\tilde{u}(x) = \phi_1(x)v(x) + \phi_2(x)u(x)$$

where ϕ_1 and ϕ_2 are a partition of unity whose properties have been listed when $\tilde{u}$ was defined. Then the various estimates we derived for $\tilde{u}$ on $0 \le w \le R$ are valid for v on $|w| \le (1+t)R$.

$$\begin{aligned} {}^1\tilde{u}\,{}^1\tilde{u}_y - {}^1u\,{}^1u_y &= (\phi_1{}^1v + \phi_2{}^1u)(\phi_1{}^1v_y + \phi_2{}^1u_y) - {}^1u\,{}^1u_y \\ &= (\phi_1^2\,{}^1v + \phi_1\phi_2{}^1u){}^1v_y + (\phi_2^2 - 1){}^1u\,{}^1u_y \\ &\quad + \phi_1\phi_2{}^1v\,{}^1u_y \end{aligned} \tag{43}$$

We have by (11) that

$$\begin{aligned} {}^1u\,{}^1u_y &= O(t^{4m+\nu+1}) \\ &= O(t^{4m+3}) \end{aligned}$$

since $\nu > 1$ by Lemma 3. We have

$${}^1v - {}^1v_y = O(t^{4m+2})$$

by (30). We have

$${}^1u = \int f - fg^2 \, dx = t^{2m+1} \int \mathbf{A}^2 dw + O(t^{2m+2})$$

and hence

$${}^1u = O(t^{2m+1}).$$

We have

$${}^1v_y = O(t^{2m+1})$$

by (12). We have ${}^1u = O(t^{2m+1})$, so

$${}^1u\,{}^1v_y = O(t^{4m+2}).$$

We have

$${}^1v = O(t^{2m+1})$$

and from (9) we have

$$\begin{aligned} {}^1u_y &= \frac{E}{2}x^{2m+\nu} + \frac{G\,\mathrm{Im}(c_k)}{2}x^{2m+2k+\delta} + \dots \\ &= \frac{E}{2}t^{2m+\nu}w^{2m+\nu} + \frac{G\,\mathrm{Im}(c_k)}{2}t^{2m+2k+\delta}w^{2m+2k+\delta} + \dots \\ &= O(t^{2m+\nu}) + O(t^{2m+2k+\delta}) \\ &= O(t^{2m+2}) \end{aligned}$$

since $\nu > 1$.

Putting these estimates into equation (43) we have

$$\begin{aligned} {}^1\tilde{u}\,{}^1\tilde{u}_y - {}^1u\,{}^1u_y &= O(t^{4m+2}) + O(t^{4m+\nu+1}) + O(t^{2m+1})O(t^{2m+2}) \\ &= O(t^{4m+2}) \end{aligned}$$

Now we integrate:

$$\begin{aligned} \int_{Rt}^{(1+t)Rt} {}^1\tilde{u}\,{}^1\tilde{u}_y - {}^1u\,{}^1u_y \; dx &= Rt^2 O(t^{4m+2}) \\ &= O(t^{4m+4}) \end{aligned}$$

and the same estimate is valid for the integral from $-Rt^2$ to 0, since that interval is also of length Rt^2. By (41) and (42), the terms arising from the second and third components of $\tilde{u}$ and u add only $O(t^{(2m+1)(q+1)+2m})$ and $O(t^{(2m+1)(p+1)+2m})$. We have

$$(2m+1)(q+1) + 2m \geq 2(2m+1) + 2 \geq 4m + 4$$

and similarly with p instead of q, so these terms are $O(t^{4m+4})$. When integrated over an interval of length Rt^2, they add only $O(t^{4m+6})$. Therefore

$$\int_{Rt}^{(1+t)Rt} \tilde{u}\tilde{u}_y - uu_y \; dx = O(t^{4m+4})$$

This is smaller than the t^{4m+3} term we found for the integral from $-Rt$ to Rt, whose sign we can control. Again, the same estimate is valid for the integral from $-Rt^2$ to 0.

Now we take up the integral E_3. Let S be the union of the two intervals $(-\infty, -Rt^2)$ and $(Rt(1+t), \infty)$, so that E_3 can be written as an integral over S. Here we have $\tilde{u} = u$, so the integral in question is

$$\begin{aligned} E_3 &= \int_S u(\tilde{u}_y - u_y)\, dx \\ &= \int_S {}^1u({}^1\tilde{u}_y - {}^1u_y)\, dx \end{aligned}$$

$$+ \int_S {}^2u({}^2\tilde{u}_y - {}^2u_y)\,dx$$
$$+ \int_S {}^3u({}^3\tilde{u}_y - {}^3u_y)\,dx$$

To analyze the second and third terms, it will be convenient to make a conformal mapping back to the disk, where we started at the beginning of the paper. Let ζ be the mapping from the disk to the upper half-plane defined earlier. Let $U(z) = u(\zeta(z))$ and $\tilde{U}(z) = \tilde{u}(\zeta(z))$, so that U and $\tilde{U}$ are defined in the disk. Define

$$T := \zeta^{-1}(S).$$

The integral in question becomes

$$\begin{aligned} E_3 &= \int_T U(e^{i\theta})(\tilde{U}_r(e^{i\theta}) - U_r(e^{i\theta}))\,d\theta \\ &= \int_T {}^1U(e^{i\theta})({}^1\tilde{U}_r - {}^1U_r)\,d\theta \\ &\quad \int_T U_2(e^{i\theta})({}^2\tilde{U}_r - {}^2U_r)\,d\theta + \int_T U_2(e^{i\theta})({}^3U_r - {}^3U_r)\,d\theta \end{aligned}$$

We have already discussed the contributions from the second and third components, and found that the two integrands are $O(t^{4m+4})$; this term is uniform in θ, i.e., the integrand is bounded by a constant times t^{4m+4} independently of θ. Upon integrating over the unit circle we still have $O(t^{4m+4})$. Getting this result is the only reason for going back to the unit disk, so we now return to the upper half plane:

$$\begin{aligned} E_3 &= \int_S {}^1u({}^1\tilde{u}_y - {}^1u_y)\,dx + O(t^{4m+4}) \\ &= \int_{-\infty}^{\infty} {}^1u({}^1\tilde{u}_y - {}^1u_y)\,dx - \int_{-Rt^2}^{(1+t)Rt} {}^1u({}^1\tilde{u}_y - {}^1u_y)\,dx \end{aligned}$$

Integrating by parts, and noting that the terms at infinity cancel out since Γ is a Jordan curve, we have

$$\begin{aligned} E_3 &= -\int_{-\infty}^{\infty} {}^1u_y({}^1\tilde{u} - {}^1u)\,dx - \int_{-Rt^2}^{(1+t)Rt} {}^1u({}^1\tilde{u}_y - {}^1u_y)\,dx \\ &= -\int_{-(1+t)Rt}^{(1+t)Rt} {}^1u_y({}^1\tilde{u} - {}^1u)\,dx - \int_{-Rt^2}^{(1+t)Rt} {}^1u({}^1\tilde{u}_y - {}^1u_y)\,dx \end{aligned}$$

since for $z \in S$, we have ${}^1\tilde{u} = {}^1u$. Combining the integrals and multiplying out the integrands, we have

$$E_3 = -\int_{-Rt^2}^{(1+t)Rt} {}^1u_y{}^1\tilde{u} - {}^1u_y{}^1u - {}^1u^1\tilde{u}_y + {}^1u^1u_y\,dx$$

$$
\begin{aligned}
&= -\int_{-Rt^2}^{(1+t)Rt} {}^1u_y {}^1\tilde{u} - {}^1u {}^1\tilde{u}_y \, dx \\
&= -\int_{-Rt^2}^{(1+t)Rt} {}^1u_y {}^1\tilde{u} dx - \int_{-(1+t)Rt}^{(1+t)Rt} {}^1u {}^1\tilde{u}_y \, dx
\end{aligned}
$$

Integrating the second term by parts one more time, it becomes identical to the first term, and we have

$$
E_3 = -2 \int_{-Rt^2}^{(1+t)Rt} {}^1u_y {}^1\tilde{u} \, dx \tag{44}
$$

Recall from equation (9) that

$$
\begin{aligned}
{}^1u_y &= \frac{E}{2} x^{2m+\nu} + \frac{G}{2} x^{2m+2k+\delta} + \ldots \\
&= \frac{E}{2} t^{2m+\nu} w^{2m+\nu} + \frac{G}{2} t^{2m+2k+\delta} w^{2m+2k+\delta} + \ldots
\end{aligned}
$$

Since by Lemma 3 we have $\nu > 1$, we have

$$
{}^1u_y = O(t^{2m+2}) \tag{45}
$$

Recall that

$$
\begin{aligned}
{}^1\tilde{u} &= t^{2m+1} \int \mathbf{A}^2 dw + O(t^{2m+2}) \\
&= O(t^{2m+1})
\end{aligned}
$$

Substituting this result and (45) into equation (44) we have

$$
\begin{aligned}
E_3 &= -2 \int_{-Rt^2}^{(1+t)Rt} O(t^{2m+2}) O(t^{2m+1}) && (46) \\
&= O(t^{4m+4}) && (47)
\end{aligned}
$$

since the integrand is $O(t^{4m+3})$ and the interval of integration is of length $O(t)$. We now have proved

$$
\begin{aligned}
E_2 &= O(t^{4m+4}) \\
E_3 &= O(t^{4m+4}) \\
E_1 &= -ct^{4m+3} + O(t^{4m+4})
\end{aligned}
$$

and it follows that

$$
E[\tilde{u}] - E[u] = -ct^{4m+3} + O(t^{4m+4}).
$$

We still have to study the smoothness of the family $\tilde{u}$. Of course the function $\tilde{u}$ is as smooth as the interpolation functions ϕ_1 and ϕ_2, but what has to be studied is how the higher derivatives of $\tilde{u} - u$ with respect to x converge to zero. In the region $0 < z \leq Rt$, the function $\tilde{u}$ is analytic in z and t, so there is no problem there. In the regions $z \geq (1+t)Rt$ and $z \leq -Rt^2$, we have $\tilde{u} - u$ identically zero (on the boundary), so there is no problem there. The rest of our discussion will focus on the regions $Rt \leq z \leq (1+t)Rt$ and $-Rt^2 \leq z \leq 0$. We have from the Weierstrass representation,

$$\begin{aligned} {}^1v - {}^1u &= \frac{1}{2}t^{2m+1}\left[\int_0^w \mathbf{A}^2(\zeta)d\zeta - \frac{w^{2m+1}}{2m+1}\right] + O(t^{2m+2}) \\ &= O(t^{2m+1}) \end{aligned}$$

since on the region in question, w stays bounded as t goes to zero. (This is no better bound than we have individually on v and u in this region.) For the other two components, we have from the geometry of Γ,

$$\begin{aligned} {}^2v - {}^2u &= O(t^{(2m+1)(q+1)}) \\ {}^3v - {}^3u &= O(t^{(2m+1)(p+1)}) \end{aligned}$$

Hence

$$\begin{aligned} \tilde{u} - u &= O(t^{2m+1}) \\ \frac{d^n}{dx^n}(\tilde{u} - u) &= O(t^{2m+1-n}) \end{aligned}$$

Recall from (4) that we constructed ϕ_1 to satisfy

$$\frac{d^n\phi_1}{dx^n} = O(t^{-2n}).$$

The n-th derivative of $\tilde{u} - u$ is given by

$$\begin{aligned} \frac{d^n}{dx^n}(\tilde{u} - u) &= \sum_{i=0}^{n}\binom{n}{i}\left[\phi_1^{(i)}\tilde{u}^{(n-i)} + \phi_2^{(i)}\tilde{u}^{(n-i)}\right] \\ &= \sum_{i=0}^{n} O(t^{-2i})O(t^{2m+1-(n-i)}) \\ &= \sum_{i=0}^{n} O(t^{2m+1-n-i}) \\ &= O(t^{2m+1-2n}) \end{aligned}$$

Hence the $\tilde{u}$ converges to u in the C^n norm only for $n \leq m$, but for those values of n it does converge. That completes the proof of the theorem.

The case of smooth but not analytic boundary

In this section we investigate how far the assumption that Γ is real-analytic can be relaxed.

Definition 3 *The Jordan curve Γ is called* **nowhere planar** *if at every point each component of Γ has a nonzero n-th derivative for some n.*

If Γ is nowhere planar, then if P is any point on Γ, after a rigid motion which brings P to the origin and makes Γ tangent to the X-axis at origin, we will have the representation of Γ' in terms of p and q which is basic to our argument, for some p and q.

We review the known facts about the behavior of a minimal surface at a boundary branch point on a $C^{2,\mu}$ Jordan arc.
(i) the surface itself is $C^{2,\mu}$ up to the boundary (see [9], p. 33).
(ii) With the branch point at origin, we have the following asymptotic representation (see [9], Theorem 2 on p. 121):

$$u_z = Az^M + o(|z|^M)$$

for some integer M, the order of the branch point. If the boundary is taken on monotonically near the branch point then M must be even, say $M = 2m$. The complex vector A satisfies $A^2 = 0$.
(iii) the unit normal extends continuously to the branch point. Orienting the surface so that the normal points up at the branch point, we then have $g(z) = o(1)$ and

$$^3u_z = f(z)g(z) = o(|z|^{2m}).$$

In the analytic case we would have $g(z) = z^k + O(z^{k+1})$ for some k, the index of the branch point. In the $C^{2,\mu}$ case, we do not know that the g has Taylor coefficients beyond the $2m$-th. Therefore, we define the index k to be the least integer such that we do not have $|g(z)| = o(|z|^k)$. If such an integer k does not exist then $|g(z)| = o(|z|^k)$ for every k; in such a case we say the branch point has infinite index. Since $g(z) = o(1)$ we have $k \geq 1$.

We need the following more thorough analysis of the situation regarding the definition of the index of a branch point and the behavior of the Gauss map near a branch point on a $C^{2,\mu}$ boundary.

Lemma 11 *Let u be a minimal surface with a boundary branch point at origin, bounded by a $C^{2,\mu}$ Jordan arc, and oriented so the normal points up at the branch point. Then there exists an integer k such that*

$$\begin{aligned} fg^2(z) &= cz^{2m+2k} + o(z^{2m+2k}) \\ g(z) &= cz^k + o(z^k) \\ g_z &= ckz^{k-1} + o(z^{k-1}) \end{aligned}$$

In particular the branch point has finite index and g_z is bounded.

.
Proof. We have $fg^2 = {}^2u_z - {}^1u_z$, so fg^2 is $C^{2,\mu}$ up to the boundary, by (i). By (ii) we have

$$fg^2(z) = o(z^{2m})$$

when the normal at the branch point points up. Since the real and imaginary parts of fg^2 are harmonic, and it is C^1 at the boundary, it satisfies hypothesis (2) on page 142 of [9], and hence hypothesis (A2) on the next page. If the minimal surface does not lie in a plane, then fg^2 is not constant. We can therefore apply Theorem 2 on page 143 of [9], with ν in the theorem equal to $2m$ here, and $X = fg^2$. The conclusion is that

$$\lim_{z \to 0} z^{-2m}(fg^2)_z \qquad \text{exists}$$

and there is a least nonnegative integer j such that we do not have $fg^2(z) = o(|z|^j)$, and for that j we have

$$\lim_{z \to 0} z^{-j+1}(fg^2)_z \qquad \text{exists and is not zero.}$$

We have $j = 2m + 2k$ by the definition of k above, and the asymptotic formula $f(z) = z^{2m} + o(z^{2m})$. More precisely, we have proved that k exists, i.e. the case of infinite index is impossible. We have

$$fg^2(z) = cz^{2m+2k} + o(z^{2m+2k})$$

and hence

$$g(z) = cz^k + o(z^k).$$

We have asymptotic expansions of f, fg^2, and fg, since these are defined by

$$\begin{aligned} f &= {}^1u_z - i\,{}^2u_z \\ fg^2 &= {}^1u_z + i\,{}^2u_z \\ fg &= {}^3u_z \end{aligned}$$

and as above, these functions are C^1 up to the boundary by (i) and then the theorem on page 143 of [9] yields the desired asymptotic expansions. Now we compute g_z. We have

$$\begin{aligned} (fg^2)_z &= f_z g^2 + 2fgg_z \\ g_z &= \frac{(fg^2)_z - f_z g^2}{2fg} \\ &= \frac{(fg^2)_z}{2fg} - \frac{f_z g^2}{2fg} \\ &= \frac{(2m+2k)c^2 z^{2m+2k-1} + o(z^{2m+2k-1})}{2cz^{2m+k} + o(z^{m+k})} \end{aligned}$$

$$
\begin{aligned}
& - \frac{(2mz^{2m-1} + o(z^{2m-1}))(c^2 z^{2k} + o(z^{2k}))}{2cz^{2m+k} + o(z^{m+k})} \\
= \; & (m+k)cz^{k-1} + o(z^{k-1}) - (mz^{-1} + o(z^{-1}))(cz^k + o(z^k)) \\
= \; & kcz^{k-1} + o(z^{k-1})
\end{aligned}
$$

Hence g_z also has the desired asymptotic behavior. That completes the proof of the lemma.

Remark: We could not apply the theorem from [9] directly to g, since we do not know *a priori* that g extends in C^1 fashion to the boundary.

The integers ν and δ cannot be defined as in the analytic-boundary case, since the requisite power series may not exist. Instead we make the following definition.

Definition 4 *ν is the least integer such that for small real x we do not have*

$$\mathrm{Im}(|f(x)|) = o(|x|^\nu).$$

δ is the least integer such that we do not have

$$\mathrm{Re}(|g(x)|) = o(|x|^\delta).$$

Theorem 3 *Let Γ be a $C^{N,\mu}$ Jordan curve, where $N \geq 3$ and $\mu > 0$. Suppose Γ is nowhere planar (as defined above). Let u be a minimal surface of the topological type of the disk, bounded by Γ, with a boundary branch point of order $2m$ on Γ, and suppose $n \leq m$ and $n \leq N$. Then we can construct a C^n-smooth one-parameter family $\tilde{u}$ of minimal surfaces bounded by Γ, containing the original tsurface u, and converging to u in the C^n norm, such that the Dirichlet integral $E[\tilde{u}]$ is less than $E[u]$; indeed $E[u] - E[\tilde{u}] \geq ct^{4m+3}$ for some positive constant c.*

Corollary 2 *Let Γ be a $C^{N,\mu}$ Jordan curve, where $N \geq 2$ and $\mu > 0$. Suppose Γ is nowhere planar (as defined above). If u is a C^1 relative minimum of area (in particular if it is an absolute minimum of area) among disk-type surfaces bounded by Γ, then u has no boundary branch points.*

Proof. By the known boundary regularity results ([9], p. 33), u is of class $C^{N,\mu}$ at the boundary, in particular of class C^N. This is needed only at the end, when we prove the convergence in C^n norm of $\tilde{u}$ to u. In the rest of the proof we never took more than two derivatives of u. Define $\Omega_0(t^i)$ to mean a term which is $o(t^j)$ for every $j < i$. Such a term might be $o(t^i)$ or even zero. To imply that the term is definitely not $o(t^i)$ we write $\Omega_1(t^i)$. We now list the changes that need to be made to the proof given above for the analytic case:

(i) Replace every term involving $Et^{\nu+j}$ for some j can be by $\Omega_1(t^{\nu+j})$.
(ii) Replace every term involving $Gt^{\delta+j}$ for some j by $\Omega_1(t^{\delta+j})$.

(iii) Replace every term involving t^{k+j} by $\Omega_0(t^{k+j})$.

(iv) Replace every term with an anonymous constant C_i and an exponent of t involving δ, ν, or k by an Ω_0 term with the same exponent of t.

(v) Replace every error term $O(t^{j+1})$ with $o(t^j)$.

After these replacements, the proof is still valid. This has to be verified by a detailed reading of the proof, but is routine except for the following point. When computing s_{xx}; we must differentiate $B^2 = fg^2$ and get $o(z^{2m})$. By the lemma above, $(fg^2)_z$ has an asymptotic expansion at the branch point, so fg^2 can be differentiated as in the analytic case. Similarly for f and fg.

References

[1] Alt, H. W., Verzweigungspunkte von H-Flächen I. *Math. Zeitschrift* **127**, 333-362 (1972).

[2] Alt, H. W., Verzweigungspunkte von H-Flächen II. *Math. Ann* **201**, 33-55 (1973).

[3] Barbosa, J. L., and do Carmo, M., Stable minimal surfaces, *Bull. Amer. Math. Soc.* **80** (1974) 581-583.

[4] Barbosa, J. L., and do Carmo, M., On the size of a stable minimal surface in R^3, *Amer. J. Math.* **98** (1976), 515-528.

[5] Beeson, M., Some results on finiteness in Plateau's problem, Part I, *Math Zeitschrift* **175** (1980) 103-123.

[6] Beeson, M. Some results on finiteness in Plateau's problem, Part II, *Math Zeitschrift* **181** (1982) 1-30.

[7] On interior branch points of minimal surfaces, *Math Zeitschrift* **171** (1980) 133-154.

[8] Dierkes, U., Hildebrant, S., Küster, A., Wohlrab, O., *Minimal Surfaces I: Boundary Value Problems*, Springer-Verlag, Berlin Heidelberg New York (1992).

[9] Dierkes, U., Hildebrant, S., Küster, A., Wohlrab, O., *Minimal Surfaces II: Boundary Regularity*, Springer-Verlag, Berlin Heidelberg New York (1992).

[10] Gulliver, R. A minimal surface with an atypical boundary branch point, in: B. Lawson and K. Tenenblat (eds.), *Differential Geometry: a Symposium in honor of Manfredo P. do Carmo*, pp. 211-228, Longman, Harlow (1991).

[11] Regularity of minimizing surfaces with prescribed mean curvature vector, *Ann. Math.* **97**, 275-305 (1973).

[12] Gulliver, R., and Lesley, F. D., On boundary branch points of minimizing surfaces, *Arch. Ration. Mech. Anal.* **52**, 20-25 (1973).

[13] Gulliver, R., Osserman, R., and Royden, H. L., A theory of branched immersions of srufaces, *Am. J. Math* **95**, 750-812 (1973).

[14] Kreyszig, E., *Differential Geometry*, Dover, New York (1991). Originally published: University of Toronto Press, Toronto (1959).

[15] Lewy, H. On the boundary behaviour of minimal surfaces, Proc. Natl. Acad. Sci. USA **37**, 102-110 (1951).

[16] Morrey, C., Jr., *Multiple Integrals in the Calculus of Variations*, Springer-Verlag, Berlin/Heidelberg/New York (1966).

[17] , Nitsche, J. C. C.,*Lectures on Minimal Surfaces, Volume 1*, Cambridge University Press, Cambridge (1988).

[18] Osserman, R., *A Survey of Minimal Surfaces*, Dover Publications, New York (1986).

[19] Osserman, R., A proof of the regularity everywhere of the classical solution to Plateau's problem, *Ann. Math* (2) **91**, 550-569 (1970).

[20] Protter, M., and Weinburger, H., *Maximum Principles in Differential Equations*, Prentice-Hall, Englewood Cliffs, New Jersey (1967).

[21] Tomi, F., On the local uniqueness of the problem of least area, Arch. Ration. Mech. Anal. **52**, 12-381 (1978).

[22] Tomi, F., and Tromba, A. J., On the structure of the set of curves bounding minimal surfaces of prescribed degeneracy, J. für die reine und angewandte Mathematik **316** (1980) 31-43.

[23] Tromba, A. J., On the number of simply connected minimal surfaces spanning a curve, Mem. Am. Math. Soc. No. 194, **12** (1977).