BOUGUER FAILS TO WEIGH THE EARTH, 1739

Newton's theory of the motion of the planets was very successful. The law of universal gravitation that he used said that *any two bodies* attract each other: not just moons, planets, and the sun, but any bodies.

$$F = \frac{GMm}{r^2}$$

where M and m are the two masses, r the distance between them, and G the "gravitational constant."

Well then, how about taking the two objects to be a mountain and a small lead weight? A "plumb bob" is just a weight on a thin flexible wire. We could hang a plumb bob next to a mountain, and the mountain should pull it a little sideways, so that it would not point to the center of the earth as it would when not near a mountain. If we just measure that deflection, we could check Newton's law.

FIGURE 1. Chimborazo, the mountain in Ecuador whose gravitational attraction Bouguer tried to measure.

We could do more than just check the law. By measuring the deflection, we could calculate the sideways force acting on the plumb bob. We could then compare that force with the weight of the plumb bob, which is the force of gravity between the Earth and the plumb bob. We know the radius of Earth, and we can estimate the mass of the mountain. We have two equations (Newton's law for the mountain and the bob, and Newton's law for the Earth and the bob). There are two unknown quantities: G and the mass of the Earth. With two equations, we can find them both! (Check the math yourself, below.) Therefore, in a sense, this experiment would weigh the Earth!

That must have been somewhat tantalizing. The man to do it was, in his own opinion, Jean Bouguer. The mountain he chose was Chimborazo (see photograph above).

At that time, Chimborazo was thought to be the highest mountain on Earth, measured from sea level–Mt. Everest hadn't been discovered yet. Actually, the summit of Chimborazo is the farthest point on Earth from the center of Earth. That is because it rests on the "equatorial bulge".

How did a French scientist happen to be in Ecuador (then part of the Territory of Quito, claimed by Spain)? Ah, that also has to do with Newton's law. One of the things Newton's law explains is why the Earth is spherical: Once it must have been liquid, or made of many small solid pieces, and all the parts of it attracted all the other parts according to Newton's law. The result is that the parts packed themselves together as closely as possible: in a sphere. But since the Earth is spinning–or perhaps we should say, *if* the Earth is spinning, as that wasn't quite *proved* yet, but it was by then widely believed–there should be an "equatorial bulge", as matter at the Equator should be subject not only to a gravitational acceleration but to a centrifugal acceleration due to the rotation of the Earth. The Equator should bulge to the height where the decrease in gravity due to the height just balances the centrifugal acceleration. Below you can check for yourself how this calculation goes.

One can measure the equatorial bulge this way: By sighting the North Star at night, one determines latitude. By sighting it in two places separated by one degree (of elevation of the North Star), we find two places where latitude differs by one degree. Then, by surveying the land in between those places carefully (in the daytime!) we measure the distance between them. If the Earth were a perfect sphere, that would be 1/360 of the circumference. But if the Earth has a bulge, then near the equator it should be more than 1/360 of the circumference, and near the poles it should be less.

Moreover, there is a third way to check Newton's law: gravity should less on the top of a mountain than at the base, because the top is farther from the center of the Earth. Say a mountain is 4 miles high; that is 1/1000 of the radius of Earth, so gravity should be less by a factor of $1.001^2 = 1.002$, i.e. less by two-tenths of a percent. That should be measurable!

So there were *three* possible ways to check Newton's law near an equatorial mountain like Chimborazo. Was gravity smaller at the summit? Would it deflect a plumb line? And was the Equator bulging? If so, then a degree of latitude should be longer on the surface.

For these reasons, the French Academy of Sciences sent a large and well-equipped expedition to Peru in 1735; and for good measure, they sent another expedition to Lapland, near the North Pole, to measure a degree of latitude there. Would it be smaller? The Lapland expedition was under the direction of Swedish physicist Celsius (whose name is now used for a temperature scale) and French mathematician Maupertais. They determined, before the equatorial expedition finished its measurements, that the Earth was flattened at the North Pole.

The Equatorial Expedition was quite an adventure. Its members left France in May 1735. They landed on the Caribbean coast of Columbia, then sailed to Panama, traveled overland to the Pacific, and continued by sail to Ecuador. There they split into two groups, traveling overland through rain forests, arriving in Quito in June 1736. Bouguer and his colleagues successfully measured arcs of the Earths curvature on the Equator from the plains near Quito to the southern city of Cuenca. That would have involved sighting the stars at night with astronomical instruments, and surveying overland distances during the day.

These measurements were completed by 1739. It is no coincidence that half a century later, the French Revolution defined the meter as a certain fraction of the distance from the North Pole to the Equator. But the Expedition stayed in Ecuador more than fourteen years! They tried (unsuccessfully) more than once to climb Mount Chimborazo, probably to measure gravity at the top. They discovered rubber, and witnessed two eruptions of the volcano Cotopaxi in 1743 and 1744. (Chimborazo is also a volcano; it erupts about every thousand years.) One of Bouguer's co-directors, Godin, stayed in South America until 1751, serving in Peru as a professor.

Now what about the experiment of "weighing the Earth" by measuring the deflection of a pendulum? First of all, how could we do that? We need some way to measure "straight up" other than the usual way of using a plumb bob, since our aim is to find out that the plumb bob is *not* vertical. The key is to make two measurements on opposite sides of the mountain, at points A and B say, and let O be the center of the Earth.[1] Now you find a star S in plane AOB and use your astronomical instruments to sight S both from A and from B. At least AS and BS are parallel, and we can measure the deflection of the plumb lines from AS and BS. If one plumb line is deflected more than the other that doesn't matter. Just add the deflections. Since we know the radius of the Earth, we can calculate the slightly different directions of OA and AB. Then we have two very small angles (less than one minute of arc, 1/60 of a degree) to subtract; the difference may be several seconds of arc. (A second of arc is 1/3600 of a degree.)

Evidently it was necessary to build a small house around the apparatus to control wind and temperature. Here is a picture of the appartus, from Bouguer's book:

[1]The story of this experiment is told on pages 223–227 of [?]

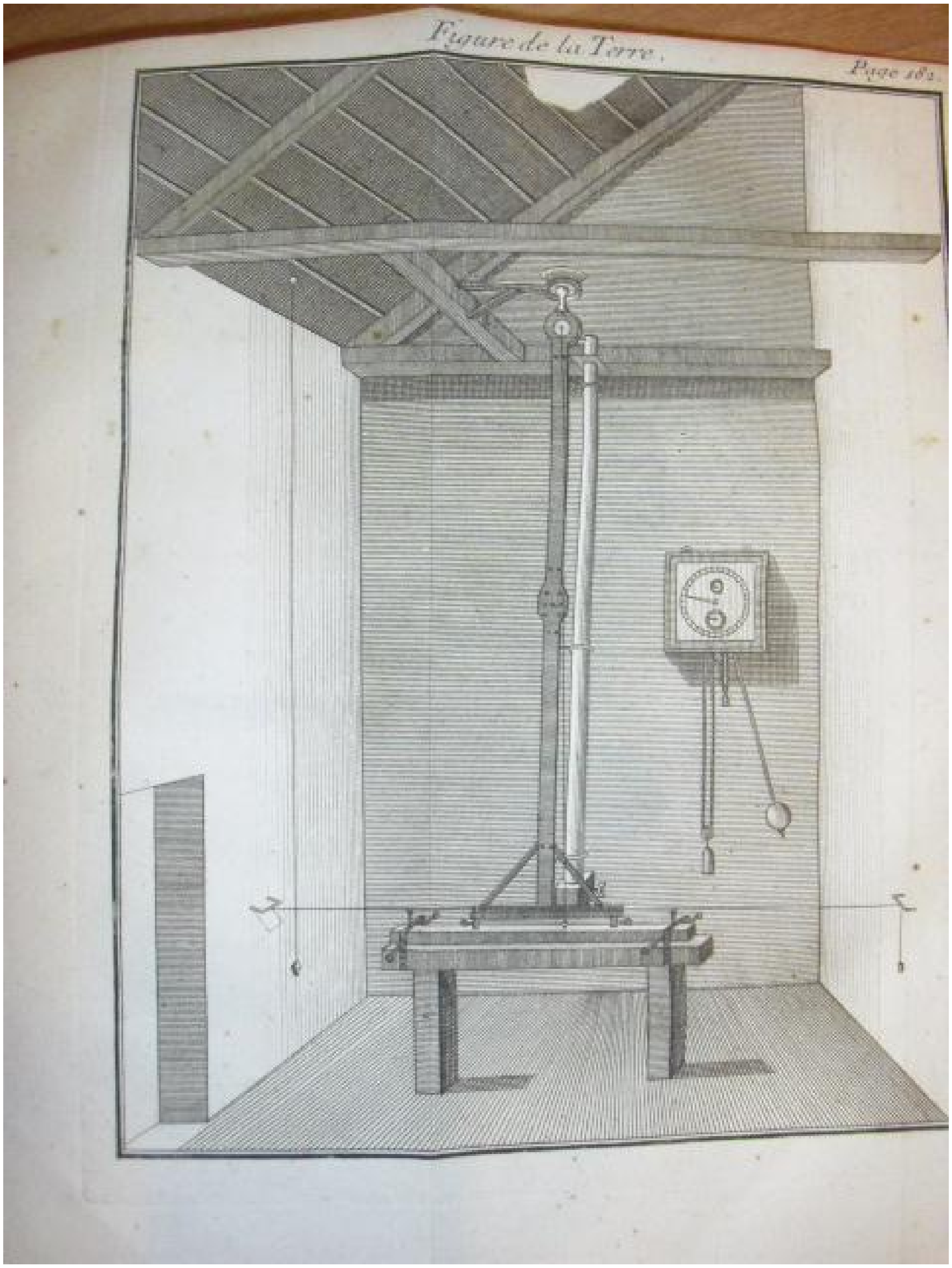


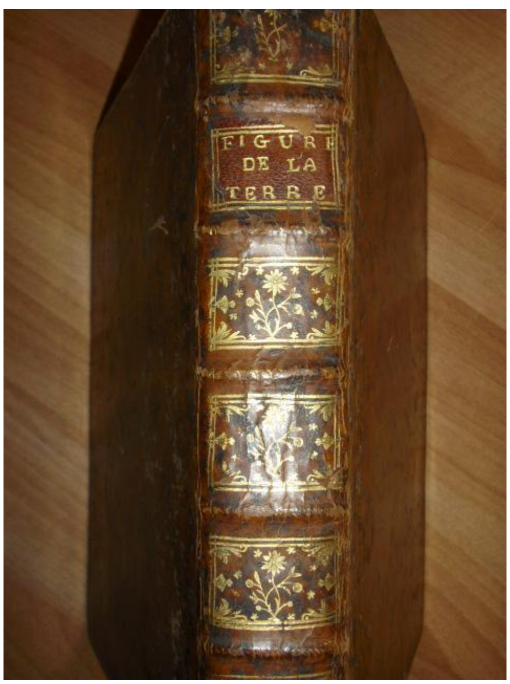
Figure de la Terre.
Page 182.

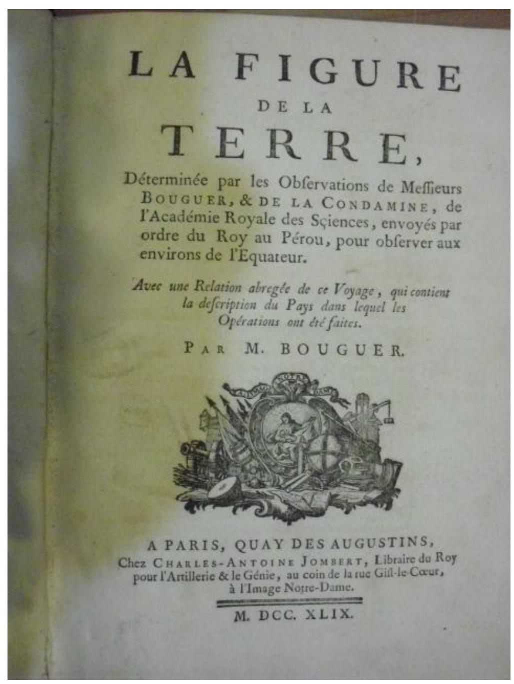
We don't know the exact day of this Great Moment, but at some point Bouguer tried this experiment. The result was completely unexpected: The plumb bobs were deflected slightly *away* from the mountain. This was the opposite of the expected result!

Bouguer was a true scientist: he reported what he found, and did not fudge the results. His book, *La Figure de la Terre*, was published in 1849. Confusion ensued. Maskelyne (Royal Astronomer of England) repeated the experiment in 1774, at Schiehallion, in Perthshire. He got a positive result, giving however a rather low density to the Earth. The experiment was repeated by Petit in 1849, in the Pyrenees, and he, like Bouguer, got a negative result, i.e. the mountain appeared to slightly repel the plumb bob. It was not until 1855 that the modern explanation was put forth by Airy. By this time it was understood that under a crust, the Earth was molten. Airy calculated that mountains were heavy enough to break the crust; therefore, they are essentially floating on whatever is beneath the crust, like icebergs in the ocean. That means that some of whatever is under there is displaced, in equal weight to the mountain. Thus the total mass that is attracting the plumb bob from the mountain side is about the same as the total mass on the side away from the mountain. That is why there isn't much deflection.

A negative result in attempting to verify Newton's law resulted in an advance in our understanding of the internal structure of the Earth. Science doesn't always advance as we think it is going to.

By the way, Bouguer's book, *La Figure de la Terre*, is now a collector's item. The price for a first edition as of October, 2013 is $2932.13. Yet, such are the wonders of twenty-first century technology, that you can order a print-on-demand copy for about sixteen dollars. These photos are of the first edition:





Check the math yourself for weighing the Earth

We will write down Newton's law for the mountain and the plumb bob, and then Newton's law for the Earth and the plumb bob. The letters used in the equations have the following meanings:

- m is the mass of the plumb bob
- M is the mass of the mountain
- r is the distance from the plumb bob to the center of mass of the mountain
- E is the mass of the Earth
- R is the radius of the Earth
- G is the gravitational constant
- mg is the weight of the plumb bob, where $g = 9.8$ m/sec^2 $= 32$ ft/sec^2 is the acceleration of gravity at the Earth's surface.
- F is the force between the mountain and the plumb bob
- θ is the angle of deflection of the plumb bob towards the mountain.

In addition to Newton's law, we need the equation for θ. The plumb bob should hang at a small angle θ to the vertical, such that the force F of its attraction to the mountain is balanced by the weight of the plumb bob. Now, the weight of the plumb bob is pulling straight down, but if the plumb bob is not hanging vertically, we can break that downward pull into a component $mg\cos\theta$ along the wire supporting the bob, and a tangential component $mg\sin\theta$. Since θ is going to be a very small angle, we may neglect the even tinier difference between "horizontal" and "tangential", and equate $mg\sin\theta$ to F. Then our three equations are

Here are the equations:

$$\begin{aligned} mg &= \frac{GEm}{R^2} \\ F &= \frac{GMm}{r^2} \\ F &= mg\sin\theta \end{aligned}$$

The first thing is to eliminate F. Then the second two equations become

$$\frac{GMm}{r^2} = mg\sin\theta$$

Next we replace $\sin\theta$ by θ; for θ as small as we need here (measured in seconds of arc) this is an approximation much better than our accuracy of measurement. We get

$$\frac{GMm}{r^2} = mg\theta$$

Now replace mg on the right by its equivalent from the first equation:

$$\frac{GMm}{r^2} = \frac{GEm}{R^2}\theta.$$

Dividing both sides by MG we have

$$\frac{M}{r^2} = \frac{E}{R^2}$$

Now we can solve for E, the mass of the Earth:

$$E = \left(\frac{R}{r}\right)^2 \frac{M}{\theta}$$

Working at a desk in France in advance of the expedition, what value of θ might we expect? We should calculate that to see if it's even possible to measure such a small angle. Dividing the equation by M we get

$$\frac{E}{M} = \frac{1}{\theta}\left(\frac{R}{r}\right)^2$$

For lack of a better assumption, let's assume that the Earth has the same density as a mountain; and let's think of a mountain as a sphere of radius r; so then we expect E/M to be equal to R^3/r^3. Putting that in we get

$$\left(\frac{R}{r}\right)^3 = \frac{1}{\theta}\left(\frac{R}{r}\right)^2$$

Solving for θ we have

$$\theta = \frac{r}{R}$$

Perhaps r is something like 2 miles, so r/R is $2/4000 = 0.0005$. This is θ in radians. One radian is something like 60° so we get $\theta = 0.03$ degrees, or about half a minute of arc. In practice, the Earth might be denser than your average mountain, and your average mountain might be smaller than a two-mile-sphere, so θ will probably come out small than this number. But it shouldn't be ten times smaller, so we should expect something on the order of ten seconds of arc. Bouguer thought he could measure it!

If he could measure it, he could work the same math backwards to find the density (or the weight) or the Earth, to whatever accuracy he could get for the mass of the mountain and the distance to its center of mass, both a bit difficult to measure accurately, but surely possible to at least one digit. But the true story turned out to be rather more complicated, as described above.

Check the math yourself about the equatorial bulge

Newton's law predicted that the Earth should be flattened at the poles and should bulge at the equator. Our aim here is to calculate the magnitude of the bulge that Newton's law predicts.

Let R be the radius that Earth would have if there were no bulge. Now suppose there is a bulge of height h (above R). Then the force of gravity at the equator on a mass m is

$$F = \frac{GMm}{(R+h)^2}$$

The difference between this force and the force if there were no bulge must be equal to the centrifugal force. The formula for the centrifugal force is $mv^2/(R+h)$, where v is the velocity of a point on the Earth's equator due to the daily rotation of the Earth. Hence the equation to be solved for h is

$$\frac{mv^2}{R+h} = \frac{GMm}{R^2} - \frac{GMm}{(R+h)^2}$$

We can divide both sides by m and multiply by $(R+h)^2$, obtaining

$$\begin{aligned} v^2(R+h) &= GM\left(\frac{R+h}{R}\right)^2 - GM \\ &= GM\frac{(R+h)^2 - R^2)}{R^2} \\ &= GM\frac{R^2 + 2Rh + h^2 - R^2)}{R^2} \\ &= GM\frac{2Rh + h^2}{R^2} \end{aligned}$$

Now the h^2 term can be neglected, because it is much smaller than $2Rh$. So we have

$$\begin{aligned} v^2 &= GM\frac{2Rh}{R^2} \\ &= 2GM\frac{h}{R} \end{aligned}$$

We can get rid of GM/R^2 because that is the acceleration of gravity at the Earth's surface, which is $g = 9.8$ m/sec$^2 = 32$ ft/sec^2. So we have

$$v^2 = 2gh$$

At least the units are right here: we have meters per second^2 on both sides. Then

$$h = \frac{v^2}{2g}$$

To get a numerical value for h, we have to find v in meters per second. In one day, a point on the equator moves 40,000 km, which is the circumference of the Earth. So $v = 4 \times 10^4$ km/day, which is 4×10^7 meters/day. But we need that in seconds, so we have to divide by 3600×24. Thus

$$\begin{aligned} h &= \frac{v^2}{2g} \\ &= \frac{1}{2 \cdot 9.8}\left(\frac{4 \times 10^7}{3600 \times 24}\right)^2 \\ &= 10935 \text{ meters} \\ &= 10.9 \text{ km} \end{aligned}$$

Now to measure this, we will need to survey the overland distance corresponding to one degree of latitude. If there were no bulge, that distance would be $R(2\pi/360)$. But at the equator it will instead be $(R + h)(2\pi/360)$. Taking $2R$ as 12713.5 km, the former 110.946 km and the latter is 111.136 km. That should be measurable–but it would be a challenge in mountainous jungle. The art of surveying in Europe in the eighteenth century was up to it, though! Bouguer's famous 1849 book reported on it in detail.

Did we get the right answer? According to Wikipedia's article on the equatorial bulge, the right answer is that the diameter measured across the equatorial plane is 42.72 km more than measured between the poles. That is more like $4h$ than $2h$, but that is because the Earth is not only bulging at the equator, but flattened at the poles. We assumed R to be the value that the radius would have if the Earth were not rotating (and never had been); so that value is intermediate between the polar radius and the equatorial radius. Therefore our rough calculation did come out right.

Today, these numbers can be checked by observations from satellites. Changes in average sea level can be measured to the fraction of a millimeter! The details of how that is done are also interesting, but that is another story.

References

[1] Jeffries, Sir Harold *The Earth: Its Origin History and Physical Constitution*, Cambridge University Press, Cambridge, England. Sixth Edition, 1976.