CAVENDISH WEIGHS THE EARTH

Newton's law of gravitation tells us that any two bodies attract each other–not just the Earth and an apple, or the Earth and the Moon, but also two apples! We don't feel the attraction between two apples if we hold one in each hand, as we do the attraction of two magnets, but according to Newton's law, the two apples should attract each other. If that is really true, it might perhaps be possible to directly observe the force of attraction between two objects in a laboratory. The "great moment" when that was done came on August 5, 1797, in a garden shed in suburban London. The experimenter was Lord Henry Cavendish. Cavendish had inherited a fortune, and was therefore free to follow his inclinations, which turned out to be scientific. In addition to the famous experiment described here, he was also the discoverer of "inflammable air", now known as hydrogen. In those days, science in England was often pursued by private citizens of means, who communicated through the Royal Society. Although Cavendish studied at Cambridge for three years, the universities were not yet centers of scientific research.

Let us turn now to a calculation. How much should the force of gravity between two apples actually amount to? Since the attraction between two objects is proportional to the product of their masses, the attraction between the apples should be the weight of an apple times the ratio of the mass of an apple to the mass of the Earth. The weight of an apple is, according to Newton, the force of attraction between the apple and the Earth:

$$W = \frac{GMm}{R^2}$$

where G is the "gravitational constant", M the mass of the Earth, m the mass of an apple, and R the radius of the Earth. On the other hand the attraction between the two apples is

$$F = \frac{Gm^2}{r^2}$$

where r is the distance between the centers of the two apples. Dividing this by the weight of an apple we get for the ratio

$$\frac{F}{W} = \frac{Gm^2/r^2}{GMm/R^2} = \frac{m/r^2}{M/R^2}$$

This is good in that the unknown constant G canceled out. But it's bad, because we have no idea what M is. (Remember, Bouguer failed to weigh the Earth in 1739.) If we multiply and divide by r/R and do a little algebra, we can capture M in the expression M/R^3, which is almost the expression for the density of the Earth:

$$\frac{F}{W} = \frac{m/r^2}{M/R^2} = \frac{m/r^3}{M/R^3}\frac{r}{R}$$

Now the two fractions on the right make more sense: the first one is the ratio of densities, and the second one the ratio of two known distances:

$$\frac{m/r^3}{M/R^3} = \frac{\text{density of apple}}{\text{density of Earth}}$$

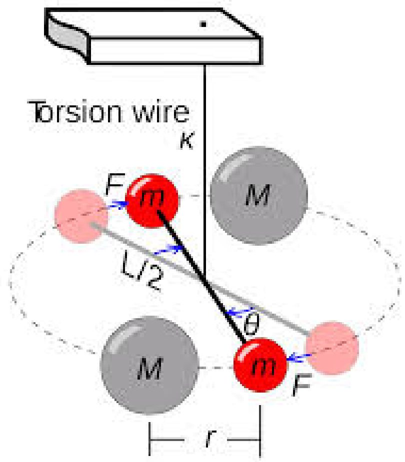
Now, we don't really know the density of the Earth, but it's probably not more than 10 times denser than an apple. And the other factor, r/R, is tiny: r is the distance between the centers of two apples (which would be the diameter if the apples were spherical), so for an order of magnitude estimate, let's take $r = 0.1$ meters and $R = 6378$ km, so $r/R = 0.1/6478000 = 1.6 \times 10^{-8}$. The force between the two apples is about sixty million times smaller than the weight of an apple, if the density of Earth is the same as that of an apple; or if Earth is ten times that dense, maybe "only" six million times smaller.

Could a force that small actually be measured? Remember Bouguer's plan had been to use a lead weight and a mountain instead of two apples; but that didn't work out as planned; the reasons for Bouguer's failure involved geology. What if we just use two big lead weights? Using lead instead of apples should help, because it makes the force we have to measure larger; but it only helps by the ratio of the density of lead to the density of an apple, which isn't as much as a factor of 100, so still leaves a tiny force to measure.

The key idea to make this measurement is to use a "torsion balance". Here is Cavendish's own description of his torsion balance:

> The apparatus is very simple; it consists of a wooden arm, 6 feet long, made so as to unite great strength with little weight. This arm is suspended in a horizontal position, by a slender wire 40 inches long, and to each extremity is hung a leaden ball, about 2 inches in diameter, and the whole is inclosed [sic] in a wooden case, to defend it from the wind.
>
> As no more force is required to make this arm turn round on the center, than what is necessary to twist the suspending wire, it is plain that if the wire is sufficiently slender, the most minute force, such as the attraction of a leaden weight a few inches in diameter, will be sufficient to draw the arm sensibly aside.

Here is a schematic illustration of how a torsion balance works:



This key idea is not due to Cavendish; it is due [6] to Charles-Augustin de Coulomb, who invented it in 1777, and used it to study electrostatic forces, not gravity. But it was invented again independently, sometime before 1783, the geologist John Michell, who had the idea to use it to study gravity. Michell built a torsion balance, but he died before he could begin the experiment. Cavendish inherited the idea, and a first version of the experimental apparatus, from Michell. The apparatus was sent in crates to Cavendish, who completed the experiment in 1797–1798, and published the results [2]. Thus Cavendish was the experimentalist, not the theoretician. Cavendish's paper is also available online at [**?**].

So why is this experiment known as "the Cavendish experiment" and not the "Michell-Cavendish experiment"? Because it was far more difficult to make this delicate measurement than Michell had anticipated. Cavendish had to rebuild the equipment from scratch, and the apparatus he used in the end was far more elaborate! True, it contained a torsion balance with lead weights, as Michell had envisioned. But the list of difficulties Cavendish overcame is long.

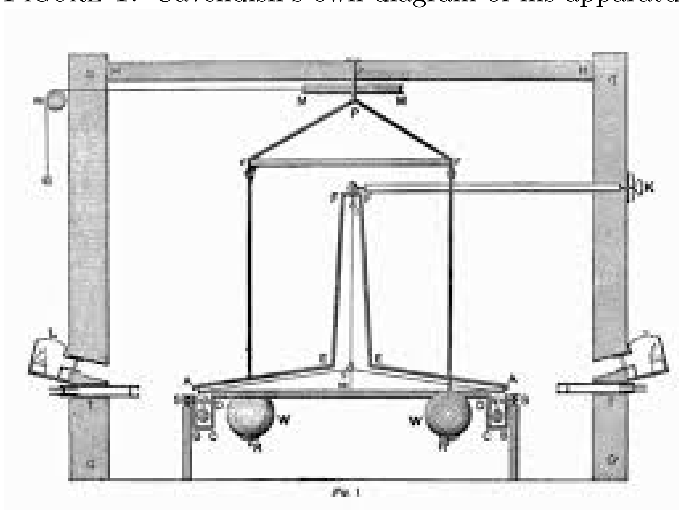
First, Michell had planned to use 8 pound lead weights. Cavendish ended up with 350 pound lead weights. As you can see from the equations, that makes the forces to be measured $350/8 = 44$ times larger, and hence easier to measure. Second, even with such massive lead weights, the slightest breath of air would move the torsion balance enough to make the displacement due to the weights unmeasurable. Therefore, as mentioned in Cavendish's paragraph above, the entire apparatus was enclosed in a box.

The experimental plan was to place the two lead weights near the ends of the torsion balance on opposite sides, so they would both make it twist clockwise, and then move them to the other side, so they would both make it twist counterclockwise. They should be pushed as close as possible to the weights hanging from the balance (to minimize r in the equation for the force). Cavendish says that Michell "seems to have intended to move them by hand". The trouble with that plan (aside from the difficulty of moving the new 350-pound weights by hand) is that any movements at all in the room stirred up air currents, and caused temperature changes. Temperature changes of any kind were "the disturbing force most difficult to guard against", because if any part of the air in the room became warmer than the rest, it would rise, and air elsewhere would fall, ruining the experiment. Cavendish therefore

> resolved to place the apparatus in a room which should remain constantly shut, and to observe the motion of the arm from without, by means of a telescope, and to suspend the leaden weights in such a manner that I could move them without entering the room.

Here we are not talking about a "room" in a house. We are talking about a fairly large shed, custom-built to house this experiment. Cavendish had inherited a sizable fortune from his father; he had a house in town (London) and also a house in Clapham Common, to the south of London. The London house contained the bulk of his library, while he kept most of his instruments at Clapham Common, where he carried out most of his experiments. The shed for this experiment was constructed in the garden of his Clapham Common house. Figure 1 gives Cavendish's own illustration.

FIGURE 1. Cavendish's own diagram of his apparatus



The figure only shows the outline of the box (within the shed) that contained the torsion arm and the weights, to keep out moving air and try to equalize the temperature. Its outline is indicated by AEF in the figure. It could not be completely sealed up because one had to be able to look inside to observe the displacement of the torsion arm. To do that, one could look through one of the two telescopes installed in the shed walls. You can see that the line of sight passed through a hole in the inner box at A. The telescopes focus on small ivory strips marked off in tiny divisions. One needed light to see them; that was provided by flames burning lamp oil. But those had to be located outside the shed, behind the telescopes, so that the flame would not create air currents inside the shed. These illuminating flames are visible in the illustration.

Moving air was a major problem. It had to be possible to move the large 350 pound weights from outside the shed without creating air currents. You can see that there is a pulley at the upper left of the diagram. That apparently allowed the rotation of the weights (without raising or lowering them) to the other position. In other words, the weight shown on the right in front of the torsion bar, will be moved to be on the left in front of the torsion bar, while the weight shown on the left behind the bar will revolve around to be on the right behind the bar. It is not quite clear from the figure exactly how the pulley mechanism worked.

Another difficulty presented itself:

> Suppose the arm to be at rest, and its position to be observed, let the weights be then moved, the arm will not only be drawn aside thereby, but it will be made to vibrate, and the vibrations will continue a great while.

The period of these "vibrations" was about twenty minutes, so they really were ponderous oscillations. Moreover, one couldn't just wait for them to settle down, because "notwithstanding the pains taken to prevent any disturbing force, the arm will seldom remain at rest for an hour together." Therefore measurements were taken while the arm was oscillating. How should that be done? "As the vibrations are constantly diminishing, it is evident that the mean between two extreme points will not give the true point of rest." Instead Cavendish followed this method: "I observe three successive extreme points of a vibration, and take the mean between the first and third as the extreme point of vibration in one direction, and assume the mean between this and the second extreme, as the point of rest."[1]

Thus Cavendish measured the displacement of the torsion arm due to the lead weights. But he wanted to measure the force of attraction between the lead weights and the smaller weights attached to the torsion arm. That force could be calculated from the fact that, at the extremes of oscillation, the torsion force (exerted by the twisted wire) had to equal the force attraction between the weights. Let θ be the angle by which the torsion arm is twisted from its equilibrium position. The torque exerted by the twisted wire, for small angular displacements θ, is equal to $k\theta$, where k is the "torsion constant" (a physical characteristic depending on the material and thickness of the wire). Cavendish measured (the extremes of) θ, so to calculate the force he also needed to know k. For that he used the (by then very well-known) connection between the period of oscillation (the time for one full oscillation) and the constant k. The bigger the constant k, the shorter the period. The period also depends on the moment of inertia (resistance to twisting force) of the torsion arm.[2]

[1]Was that a good plan? Suppose the vibrations are dying off as $e^{-x}\cos \pi x$. Then the first and third extremes would be e^{-2n} and $e^{-2(n+1)}$, and the second would be $-e^{-(2n+1)}$. So the measurement would be $\frac{1}{2}(e{-}2n + e^{-2n-2} - 2e^{-2n-1})$. For example when $n = 1$ we have $(e^{-2} + e^{-4} - 2e^{-3})/2 = (0.135 + 0.018 - 2 \cdot 0.0498)/2 = 0.027$, while the true midpoint is of course 0. In Cavendish's data, the decrease from one oscillation to the next is not as great as this, so the error is smaller. He analyzes the possible errors due to air resistance and "motion of the point of rest" and concludes that this method is legitimate and those errors are negligible.

[2]For rotating motion, Newton's law $F = ma$ gives rise to an equation stating that torque is equal to moment of inertia times angular acceleration. Hence the differential equation that the oscillating torsion

Using this method, Cavendish calculated the attraction between the balls from the period of oscillation of the torsion balance, and then he used this value to calculate the density of the Earth. Cavendish found that the Earth's average density is 5.48 times greater than that of water. John Henry Poynting later noted that the data should have led to a value of 5.448 (Cavendish made an arithmetical mistake in analyzing his data!). Indeed Poynting's number is the average value of the twenty-nine determinations Cavendish included in his paper.

As for the Great Moment itself–that occurred sometime on August 5, 1797, when Cavendish's experiment succeeded for the first time. In fact, it worked a little too well–the torsion arm twisted too much and hit the walls of the box. So the first actual numerical measurement took place the next day, after replacing the torsion wire with a thicker one. But on August 5, Cavendish saw the torsion arm moving under the gravitational force of the lead weights, so he had verified that indeed any two objects attract each other.

The result that Cavendish obtained for the density of the Earth is within 1 percent of the currently accepted figure. From the density of the earth, and the radius R of the Earth (known since the measurement by Eratosthanes), we can calculate the mass M of the earth. Now consider the gravitational acceleration g at Earth's surface. First, we can use the formula $s = \frac{1}{2}gt^2$ for the distance fallen in time t to calculate g, dropping (say) a marble from a height s of one meter and measuring t with a stopwatch. We get the value $g = 9.8\text{m/s}^2$. Second, Newton's law of universal gravitation tells us $g = GM/R^2$. Putting in the value $g = 9.8$, and the values found for R by Eratosthenes and M by Cavendish, we can calculate the universal gravitational constant G. This is still the only method by which the numerical value of G is known.

Cavendish did not explicitly derive (at least, not in his paper) the value of G. In fact, G is not even mentioned in his paper. The reason for that is that in 1798, the modern distinction between weight and force had not been fully reflected in the system of units. The metric system had just been invented and was not in use in English science, so both mass and force were measured in pounds. The constant G was apparently not defined until 1894, although of course it was implicit all along. Historians of science are still arguing about whether Cavendish determined the value of G or "only" the density of the Earth. Since the two are linked by Newton's equation and the radius of Earth (determined by Eratosthenes two millennia before Cavendish), most physicists are willing to give him the credit, even if he never mentioned the letter G.

Why would we care about the value of G? Once G is known, we can use Newton's law of gravitation again to calculate the mass of the Sun, provided we knew what the force of attraction between the Earth and Sun is, and the distance from Earth to Sun. In the first approximation, we treat the Earth's orbit as circular. Then the orbital equation tells us that the centripetal force of gravity is equal to mv^2/R, where now R is the distance from

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arm obeys is

$$k\theta = -I\frac{d^2\theta}{dt^2}$$

and the period of the oscillation is $2\pi\sqrt{I/k}$. This is part of the classical theory of harmonic oscillators. See [6] for this particular formula.

Earth to Sun, and m is the mass of the Earth. Let M be the mass of the Sun, and v the orbital speed of Earth in its orbit. We have

$$\frac{GMm}{R^2} = \frac{mv^2}{R}.$$

The mass of the Earth cancels out, as does one factor of R. Solving for M we have

$$M = \frac{v^2 R}{G}$$

The orbital speed v is such that the Earth covers the circumference $2\pi R$ in one year. Curiously enough, one year is about $\pi \cdot 10^7$ seconds, so we have (in meters/sec)

$$v = \frac{2\pi R}{\pi \cdot 10^7} = 2R \times 10^{-7}.$$

That gives us

$$M = \frac{v^2 R}{G} = \frac{4R^3 \cdot 10^{-14}}{G}$$

Since Cavendish showed us how to calculate G, we only need the Earth-Sun distance R to finish calculating the mass of the Sun. But the Earth-Sun distance (the so-called "astronomical unit") was calculated from measurements of the transit of Venus in 1769, as described in another lecture in this "Great Moments in Science" series. So the necessary values were known 29 years before Cavendish's publication.

The mass of the Sun, once determined, had other interesting consequences. At that time, nuclear and thermonuclear energy were not understood, so the only apparent source of the Sun's energy was ordinary combustion. Suppose the Sun were made of extremely combustible coal. Now that the mass was known, one could compute how long it could burn. The answer was closer to the Biblical age of the Earth (four thousand years) than to the geological age of the Earth. But that is another story.

In 2005, Prof. Norman Scheinberg of City College of New York built a "torsion pendulum" in his basement and measured the density of the Earth. He used a plexiglass cover to eliminate air currents, and 32 kg. lead weights (which he apparently cast himself, one hemisphere at a time, in a cooking pot!). Photos and videos of his experiment are available online at [4].

References

[1] Bryson, B., *A Short History of Nearly Everything*, pp. 59–62.

[2] Cavendish, Henry, Experiments to determine the density of Earth, *Philosophical Transcations of the Royal Society of London* **88**, 469–526 (1798).

[3] Sir Isaac Newton, Bouguer (M., Pierre), and Henry Cavendish, *The Laws of Gravitation: Memoirs by Newton, Bouguer and Cavendish, Together with Abstracts of Other Important Memoirs*, American Book Company (1900). Available free online at https://play.google.com/store/books/details?id=O58mAAAAMAAJ&rdid=book-O58mAAAAMAAJ&rdot=1

[4] Sideways Gravity in the Basement: Norman Scheinberg's Cavendish Experiment, http://web.archive.org/web/20080508011932/http://www.sas.org/tcs/weeklyIssues_2005/2005-07-01/feature1/index.html.

[5] Shamos, Morris H., *Great Experiments in Physics: Firsthand accounts from Galileo to Einstein*, Dover (1959).
[6] Wikipedia article on *Torsion Spring*, section on Torsion Balance.