Large solutions of the Pell equation

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We are concerned with the fundamental (smallest x for given squarefree d) solution of

$$x^2 - dy^2 = 1.$$

A Pell maximum is a d for which x is larger than for any previous value of d. The Pell maxima grow rapidly:

Table: The first few Pell maxima

d	x
3	2
5	9
10	19
13	649
29	9801
46	24335
53	66249
61	1766319049
109	158070671986249
181	2469645423824185801

Kelley (2014) and Silva (2013) independently (empirically) observed that the Pell maxima have these properties:

- d is prime (except 10 and 46)
- $d \equiv 1$ mod 4, and with two exceptions, $d \equiv 5$ mod 8.

Silva also observed that

- Asymptotically, $\log x = c\sqrt{d}\,(\log\log d)$ for some constant c.

Kelley also observed that

- The negative Pell equation $x^2 - dy^2 = -1$ is solvable
- The fundamental unit u of $\mathbb{Q}(\sqrt{d})$ does not have integer coefficients, i.e., does not lie in $\mathbb{Z}[\sqrt{d}]$, except for $d = 409$ and $d = 24049$.
- $h = 1$ (the class number of $\mathbb{Q}(\sqrt{d})$).

Because it was surprising that the Pell maxima are primes, Kelley christened them the "Pell primes."

We can't prove any of those things, but we can prove some implications between them.

For example: The primality of d follows from $h = 1$ and negative Pell solvable, whether or not d is a Pell maximum; so those two properties are "more mysterious" than the primality.

But it's a famous open problem whether there are infinitely many real quadratic fields with $h = 1$, so we don't expect to prove all the Pell maxima have $h = 1$.

Figure: $\log u$ compared to $c\sqrt{d}\,\log\log d$ with $c = 0.4255$, $d \leq 10^{15}$. Red dots are the Pell maxima. Vertical scale is 10^7.

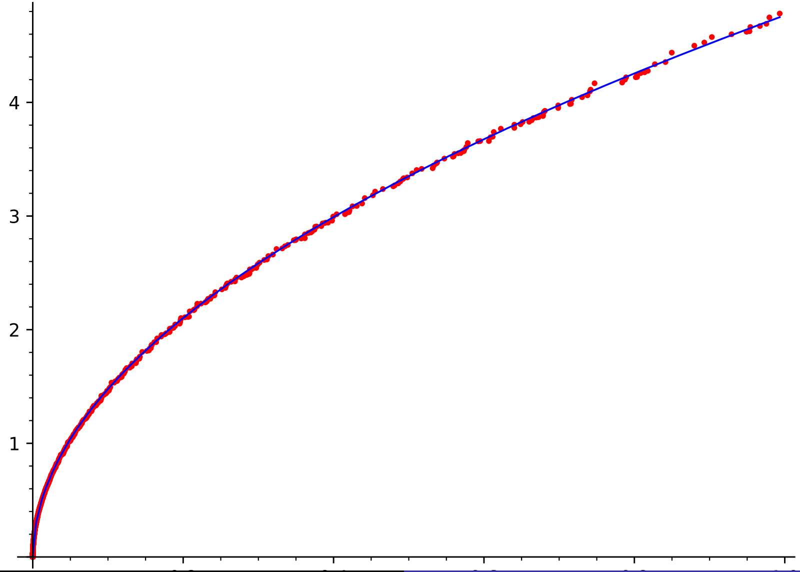
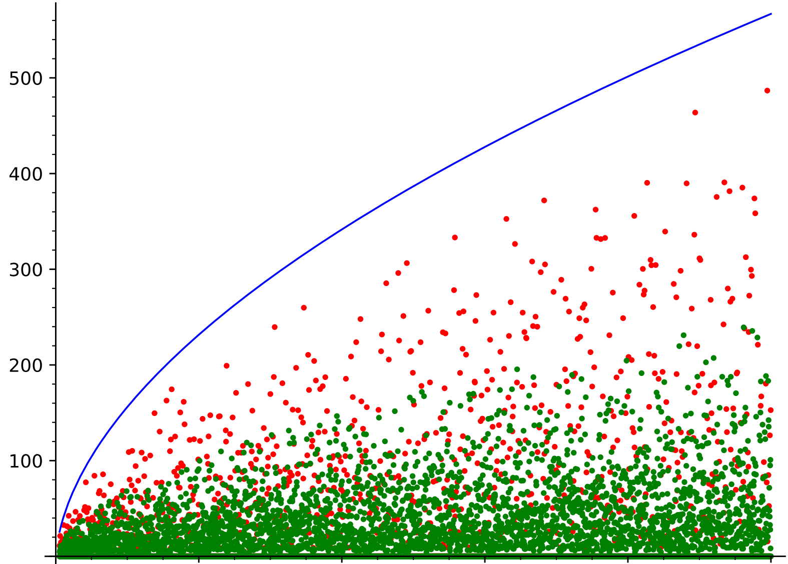


Figure: $\log x$ for all solutions (x, y) of the Pell equation for $d \equiv 1 \bmod 4$. Red means negative norm, green means negative norm.



The class number formula for the field $\mathbb{Q}(\sqrt{d})$, with discriminant D, is

$$h \log u = \frac{1}{2}\sqrt{D}\, L(1, \chi_D)$$

where u is the fundamental unit, and L is the L-series of the quadratic field. Under the assumptions that $h = 1$ and $d \equiv 1$ mod 4 (so $D = d$), the class number formula can be written

$$L(1, \chi_d) = 2\frac{\log u}{\sqrt{d}}.$$

Since those assumptions (empirically) hold for the Pell primes, the conjectured asymptotic form of $\log x = c\sqrt{d}\, \log \log d$ is equivalent to

$$L(1, \chi_d) = 2c\, \log \log d.$$

Figure: $L(1,\chi_d)/(\log\log d)$ seems close to $0.85\ldots$

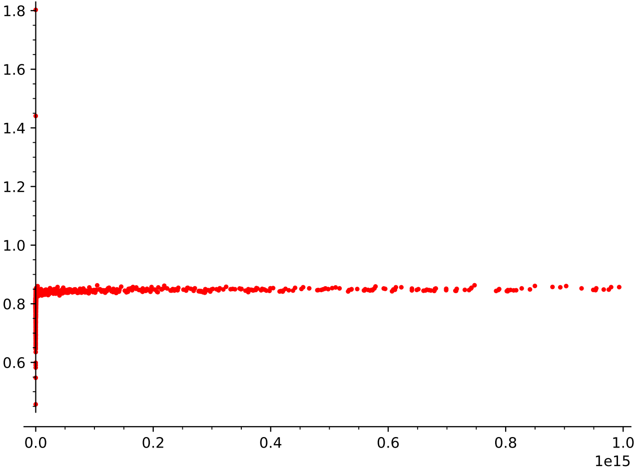


Figure: The log of the n-th Pell prime, versus n.

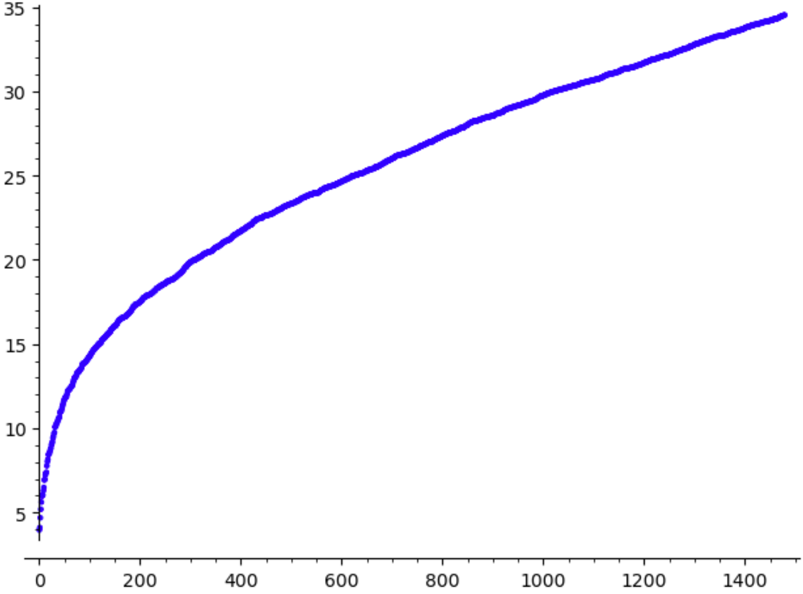
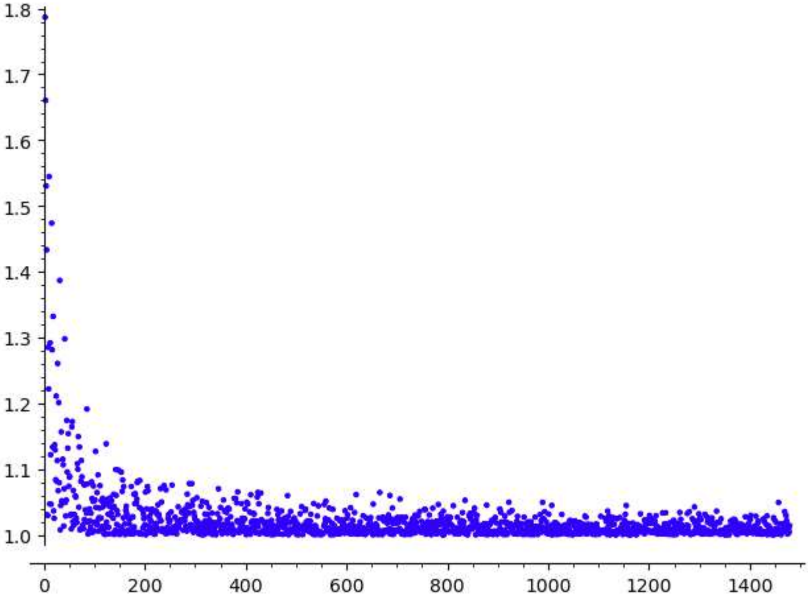


Figure: The ratio of the n-th Pell prime to the previous one



Some textbook facts

(Apologies to the experts). The fundamental unit u of $Q(\sqrt{d})$ is related to the fundamental solution $Z = x + y\sqrt{d}$ of Pell's equation as follows: $Z = u$ or u^2 or u^3 or u^6, where the exponent gets a factor of 2 if negative Pell ($x^2 - dy^2 = -1$) is solvable, and a factor of 3 if $u \notin \mathbb{Z}[\sqrt{d}]$.

The integers of $\mathbb{Q}(\sqrt{d})$ have fractional coefficients when $d \equiv 5$ mod 8, but u can sometimes be in $\mathbb{Z}[\sqrt{d}]$ even then, and no simple criterion is known for when $u \in \mathbb{Z}[\sqrt{d}]$.

Negative Pell is solvable if and only if the norm of the fundamental unit is negative, if and only if the period of the continued fraction of $\sqrt{d}$ is odd. There is no way to tell that except by either computing the continued fraction, or the L-series, e.g. by Dirichlet (1837).

$$h \log u = \frac{1}{2}\sqrt{D}\, L(1, \chi_D).$$

Here $D = d$ when $d \equiv 1$ mod 4, else $D = 4d$.

As d increases, the $\sqrt{D}$ term will tend to give you a new Pell max, if L doesn't decrease too much; i.e., if it's not too big to start with, and doesn't get too small. If h should fail to stay 1, that makes it harder to get a new Pell max. If the starting Pell max is u^6, it will be harder to get the next Pell max if the next one is only u or u^2 or u^3 instead of u^6.

Therefore it's important to study $L(1, \chi_D)$, and to study it relative to the conditions that control whether the Pell solution is u, u^2, u^3, or u^6, namely whether negative Pell is solvable and whether $u \in \mathbb{Z}[\sqrt{d}]$ or not.

We study the asymptotic behavior of

$$\hat{L}(d) := \frac{L(1,\chi_d)}{\log\log d}.$$

Littlewood (1928) assuming GRH, improved by a later analysis of the error term,

$$\hat{L} \quad \leq \quad 3.56 \qquad \text{see Granville(2003)}$$

Littlewood also proved that $\hat{L}$ infinitely often exceeds half that maximum value, or 1.78. We call those "large L-values."

There is a lot of space between 1.78 and 3.56. What really happens?

Languasco (2020) computed $L(1, \chi_d)$ out to $d =$ ten million. For these d, he found that Littlewood's bound can be improved by a factor of 0.62:

$$\begin{aligned} \hat{L}(d) &< 0.62 \cdot 3.56 \\ &= 2.21 \end{aligned}$$

But ten million (seven digits) is a very small number, compared to the known Pell maxima that run to sixteen digits.

Question: What is the least c such that for d sufficiently large, we have $\hat{L}(d) < c$?

Odd and even d

I conjecture that

$$d \text{ even } \rightarrow \hat{L}(d) < 1.24.$$

(or perhaps with a slightly larger constant). I have verified it for d up to 32 million. This is dramatically lower that Languasca's 2.21. It is even less that half Littlewood's bound, which is 1.78. Thus it looks like the "large L-values" are never even.

Table: Some $\hat{L}$ new maxima for even d

d	$\hat{L}(d)$
48	0.561743
52	0.723509
58	0.991081
...	
8440006	1.232641
12317386	1.234604
29268814	1.237712

Odd and even period

Let m be the period of the continued fraction of $\sqrt{d}$. Recall that m is odd if and only if the negative Pell equation $x^2 - dy^2 = -1$ is solvable. I computed, out to six-digit D, whether (and how much) Languasca's bound can be improved for d with odd period.

It is not the case that every new max of $\hat{L}$ has odd period (for example 503281 has period 1574), but most of them do, and so it appears that Languasca's bound cannot be improved under the assumption that negative Pell is solvable.

Questions: What is the least c such that for d sufficiently large and $x^2 - dy^2 = -1$ solvable, we have $\hat{L}(d) < c$? And is c the same with and without the restriction about negative Pell?

Whether $u \in \mathbb{Z}[\sqrt{d}]$ or not

I computed the new maxima of $\hat{L}(d)$ subject to the restriction that $u \notin \mathbb{Z}[\sqrt{d}]$. I computed these values out to d just shy of twenty million. At that point the value of $\hat{L}(d)$ was 0.8126. Most of these d have norm -1, but a few, such as 8942389, have norm 1.

I also computed it for the Pell maxima found by Al Kelley, out to magnitude 10^{16}; by then it increases to 0.85, as shown in an earlier graph.
This value is dramatically less than Littlewood's bound. I conjecture that for some constant c not too different from 0.86, and d sufficiently large,

$$u \notin \mathbb{Z}[\sqrt{d}] \;\rightarrow\; \hat{L}(d) < c.$$

Possibly the same bound holds under the weaker assumption $d \equiv 5$ mod 8.

The double-d conjecture

For $d > 2$: Let (x, y) be the least positive solution of $x^2 - dy^2 = 1$. Then for some $e \leq 2d$, the least positive solution of $X^2 - eY^2 = 1$ has $X > x$.

That is, to find a larger Pell solution, we at most have to double d. This conjecture is supported by the experimental data that the ratio of two successive Pell primes is (much) less than 2; for then, doubling d brings us past the next Pell prime after d, and hence to a larger Pell solution.

As far as I know, there is *no known estimate at all* on the size of a larger Pell solution. Zapponi (2015) notwithstanding.

Things for the experts to prove

- The Pell solutions computationally are all of the form u^6, i.e. negative Pell solvable and $u \notin \mathbb{Z}(d)$.
- The Pell solutions lie on a nice graph: $\hat{L}(d)$ approaches a constant for Pell maxima d.
- The large Pell primes do not correspond to large $\hat{L}$ values, presumably because their other properties (negative Pell, $u \notin \mathbb{Z}(d)$) prevent it.
- Computationally these properties do restrict the values of $\hat{L}$, and several conjectures above are based on those computational results.
- An estimate on how much d must increase to get a larger Pell solution, given one for d, is not known.
- And of course, the class number of the Pell maxima is always 1, so far.