Comments on my 6π paper

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Purposes of this note

This note, which discusses [4], has two purposes: (1) to answer a doubt expressed by Nitsche [6] §A29 about the use of the theorem of Barbosa-do Carmo, and (2) to discuss the last part of the argument, where the finiteness results of [2] and [3] are applied. We show that in the case at hand, these arguments can be simplified-the full power of those difficult results is not really needed.

Regarding (1), Nitsche refers to his §105 where two concerns are expressed: whether the theorem of Barbosa-do Carmo remains applicable to branched minimal surfaces, and the difficulty of defining and using normal variations of a branched minimal surface. These are taken up in the first two sections below. The third section takes up the applications of [2] and [3].

1 Validity of Barbosa-do Carmo

The original *statement* in [1] does not include branched surfaces, but the *proof* does. In that proof, branch points are essentially no different than any other zero of ∇N, i.e. branch points make no more trouble than umbilical points. We can make that clear without giving details of the proof, by simply giving a *statement* of the theorem that does not even refer to minimal surfaces. The proof applies to any analytic map N from the disk to the Riemann sphere, whether it arises as the normal to a minimal surface or not.

Theorem 1 (Barbosa-do Carmo) . *Let N be a complex-analytic map from the closed unit disk D to the Riemann sphere. Suppose that the area of the image $N(D)$ is less than 2π. Then the least eigenvalue λ of the problem $\Delta\phi - \lambda\frac{1}{2}|\nabla|^2\phi = 0$, with $\phi = 0$ on the unit circle, is greater than 2.*

This theorem is usually applied when N is the unit normal to a minimal surface, but that hypothesis is not used in the proof. If a minimal surface has branch points, its normal N extends to the branch points analytically. The zeroes of ∇N arise not only from branch points of the minimal surface but from umbilical points as well.

2 Applicability of Barbosa-do Carmo

Nitsche's second concern is about the difficulty of defining and using normal variations of a branched minimal surface. But in [4], we start with variations in the class of harmonic surfaces usually used in Plateau's problem are used. If k is a "tangent vector" in this space, then one can consider the normal variation $\phi = k \cdot N$. One calculates (as is done in [2]) that ϕ is a member of the kernel of the second variation of area D^2A if k is a member of the kernel of the second variation of Dirichlet's integral, and conversely it is also proved in [2] that every member of the kernel of D^2A arises in this way. Since the kernel of D^2A consists of eigenfunctions of the eigenvalue problem mentioned above for eigenvalue 2, to make Barbosa-do Carmo applicable we need only show that ϕ is not identically zero. That is the main task accomplished in [4], by ruling out "forced Jacobi families."

3 A direct proof to replace dependencies on [2] and [3]

When the boundary curve Γ has total curvature $\leq 6\pi$, then by the Gauss-Bonnet-Sasaki-Nitsche formula, if it bounds a branched minimal surface, the order of the branch point could be at most 1 for an interior branch point, and 2 for a boundary branch point; and there can't be more than one branch point. In [4], before appealing to any finiteness theorems, we prove that if the one-parameter family u^t has a branch point when $t = 0$, then it is immersed for $t > 0$, and has least eigenvalue $\lambda \geq 2$. Then we appeal to [2] (in the case of an interior branch point) or to [3] (for a boundary branch point) to say that is impossible.

We here give direct arguments to complete the proof without reference to the complicated arguments of [2] and [3]. These arguments, of course, have been extracted from the proofs in those papers, but many of the complications that arise there can be avoided in this simple case, where the branch point is of the lowest possible order. This still turned out to be somewhat complicated, but it is much less complicated than [3].

For an interior branch point it is extremely easy to avoid citing [2]: when u^t is immersed, the zeroes of the function f in the Weierstrass representation are double; that is $f(z) = A^2(z)$ for some analytic function A, and as discussed in the first part of [2], A will depend analytically on a rational power of t. Hence when $t = 0$, the roots of f are also double, contradicting the fact that the branch point must have order 1, since the branch points are the common zeros of f and fg^2, so if $f = A^2$ then the branch points are the common zeroes of A and g, each occurring with multiplicity 2.

Now consider the case when u^t has a boundary branch point when $t = 0$, and is immersed for $t > 0$ with least eigenvalue ≥ 2. By the Gauss-Bonnet-Sasaski-Nitsche formula, the branch point has order 2, i.e. $2m$ with $m = 1$. We first claim that it is not the case that some branch point(s) $c_i(t)$ lying outside the parameter domain (for $t > 0$) converge to the boundary branch point (when $t = 0$). Let U be a small neighborhood of the boundary branch point, and let U^+ be the part inside the parameter domain and U^- the part outside. By the Gauss-Bonnet-Sasaki-Nitsche applied to minimal surface u^t over U^+ (or, if you worry about the corners of U^+, of a disk tangent to the boundary at the boundary branch point, small enough to fit inside U), the Gaussian area of u^t over U^+ is approximately 2π more than when $t = 0$ (when t is close to zero). Now apply Gauss-Bonnet-Sasaki-Nitsche to the minimal surface u^t over the domain U. The total curvature of the boundary approaches $(2m+1)2\pi = 6\pi$ since when t goes to zero for U fixed, the surface approaches a branched minimal surface, with the asymptotic form $(\mathrm{Re}(z^{2m+1}), -\mathrm{Im}(z^{2m+1}), O(z^{2m+1+k})$. On the other side of the Gauss-Bonnet-Sasaki-Nitsche formula, for $t > 0$, there is a contribution of 2π from the constant term and 2π from the Gaussian area over U^+. That does not leave room for a contribution of 4π from a branch point $c(t)$ approaching the center of U as t approaches 0. Hence no such branch point exists.[1]

Therefore, as in the case of an interior branch point, the Weierstrass function f has double zeroes and can be written as $(z - a^2)A_0$ with $a = a(t)$ and A_0 analytic in z and t (possibly after replacing t by a rational power of the original t), and $a(t)$ converging to the branch point as $t \to 0$. We

[1]Ruling out branch points $c(t)$ coming from outside the parameter domain was one of the gaps in [3]; this gap is filled in [5] by the "hemispheric covering theorem", which says that the "extra" Gaussian area for $t > 0$ comes in the form of hemispheres, not spheres, and all on the same side of the Riemann sphere. But in case $m = 1$, there is only 2π of extra Gaussian area, so it *obviously* must come as a hemisphere.

have $a(t) = t^\gamma \alpha + O(t^{\gamma+1})$ for some γ. If $\mathrm{Im}(\alpha) > 0$ then as in [2], almost the entire sphere will be covered by the Gauss map on a neighborhood of $a(t)$ still contained in the parameter domain of u, contradicting the fact that the least eigenvalue of u is 2.

Next, we show that u_t is a forced Jacobi direction when $t = 0$. More precisely, $u_t = t^q h$ for some q where h is not identically zero when $t = 0$, and it is h^0 that we claim is forced Jacobi. If not, then $\phi = h \cdot N$ is an eigenfunction, even when $t = 0$; and it has one sign since it is the limit of eigenfunctions for the least eigenvalue 2; but we can show it is $O(|z|^2)$, contradicting the Hopf lemma. This is done much more generally in [3], Theorem 7.2, but in this case we can do it directly and simply. In fact, not only ϕ but h is $O(|z|^2)$. We have

$$\begin{aligned} u_z &= (z-a)^2 A_0 \qquad \text{with } A_0(0,0) \neq 0 \\ u_{zt} &= -2(z-a)a_t A_0 + (z-a)^2 \frac{\partial A_0}{\partial t} \\ &= -2z\alpha\gamma t^{\gamma-1}(A_0^0 + O(t)) + O(z^2) \qquad \text{where } a = \alpha t^\gamma \text{ and } \alpha \neq 0 \end{aligned}$$

Since $u_{zt} = u_{tz} = t^q h_z$, we must have $h_z = -2z\alpha$ and $q = \gamma - 1$. Hence $h = \int_0^z h_z\, dz = O(|z|^2)$. This contradicts the Hopf lemma, as mentioned, and shows that h^0 is a forced Jacobi direction.

The next part of the argument is a special case of the argument on pp. 15–16 of [3], the first part of the proof of Theorem 8.1, but in the case at hand, the argument simplifies considerably, and pp. 17–30 are not needed for the application to the 6π theorem.

Since h is a forced Jacobi direction, we know that h has the form $\mathrm{Re}(\Omega u_z)$, where

$$\Omega(z) = cz^{-J} + O(|z|^{-J+1}),$$

with $c \neq 0$ and $J = 1$ or $J = 2$. We can rule out $J = 2$ since we have assumed $u(0) = 0$ for all t, which implies $h(0) = 0$ for all t. Hence $J = 1$. Therefore

$$\begin{aligned} u_t &= t^q h \\ &= t^q h^0 (1 + O(t)) \\ &= t^q \mathrm{Re}(\Omega u_z)(1 + O(t)) \end{aligned}$$

Writing the coordinates of u as superscripts on the left, as in $u = ({}^1u, {}^2u, {}^3u)$, we have for the function f in the Weierstrass representation, $f_t^0 = t^q(\Omega f)_z$. Here's the proof:

$$\begin{aligned} f_t &= {}^1u_{zt} - i\, {}^2u_{zt} \\ &= ({}^1u_t - i\, {}^2u_t)_z \\ &= t^q({}^1h - i\, {}^2h)_z \\ &= t^q({}^1h^0 - i\, {}^2h^0)_z(1 + O(t)) \\ &= t^q(\Omega({}^1u_z - {}^2u_z)_z(1 + O(t)) \\ &= t^q(\Omega f)_z(1 + O(t)) \qquad \text{as claimed above} \end{aligned}$$

Now putting in the formula for Ω, we have

$$\begin{aligned} f_t(z) &= t^q(cz^{2m-J})_z(1 + O(t) + O(z) \qquad \text{where } m = 1 \text{ and } J = 1 \\ &= t^q(cz)_z(1 + O(t) + O(z) \\ &= ct^q(1 + O(t) + O(z)) \end{aligned}$$

On the other hand, we know that $f = (z - \alpha t^\gamma)^2 A_0(z)(1 + O(t))$, so

$$f_t = -2\alpha t^{\gamma-1}(z - \alpha t^\gamma).$$

These two expressions for f_t must be equal, so we have $\gamma = q+1$ and $c = -2\alpha$. Now, let $w = z/t^\gamma$, so $z = t^\gamma w$. Writing f in terms of w we have

$$\begin{aligned} f(z) &= t^{2\gamma}(w-\alpha)^2(1+O(t^\gamma w)+O(t)) \\ f_t(z) &= -2t^{2\gamma}(w-\alpha)w_t(1+O(t)) + t^{2\gamma}(w-\alpha)^2 O(1) \end{aligned}$$

Since $w_t = (zt^{-\gamma})_t = -\gamma z t^{-\gamma-1} = -\gamma w t^{-1}$, we have

$$f_t(z) = -2t^{2\gamma-1}(w-\alpha)w(1+O(t)) + t^{2\gamma}(w-\alpha)^2 O(1)$$

Comparing this to the previously derived equation $f_t(z) = ct^q(1+O(t)+O(z))$, the lowest exponents of t must match: $q = 2\gamma - 1$, or $q+1 = 2\gamma$. But we already derived $q+1 = \gamma$. This is a contradiction, since $\gamma > 0$. That completes the proof.

References

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