Constructivity, Computability, and the Continuum

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Abstract

The nature of the continuum has long been an important issue in the foundations of mathematics. It played an important role in the work of Dedekind, Weyl, and Brouwer, as well as early axiomatic geometers. When recursive function theory was developed, it was immediately applied to the continuum, via Turing's "computable" numbers. But then the construction of "singular covers" showed, it seemed, that the recursive reals had measure zero! Does this mean that we have to choose between constructive reasoning consistent with Church's thesis and a unit interval of positive measure? Bishop's measure theory saved us from the horns of this dilemma–or did it just sweep the difficulty under the carpet? We study the relationship between Bishop's measure theory and the recursive singular covers; as a result of this analysis, we identify a logical principle FP (fullness principle). FP is justified by the informal principle of geometric completeness, and formally refutes Church's thesis CT. We show that FP is a constructive principle in that it is conservative over intuitionistic arithmetic HA and has the numerical existence and disjunction properties. Returning to the original philosophical questions about the nature of the geometric continuum, we ask what the origin of our intuitions is. We show that modern physics supports the view of Helmholtz on this matter: below the Planck length space is not coordinatizable in the usual way.

Introduction

This paper concerns the notions of existence and constructive existence as they apply to real numbers. The following two principles have often been considered:

(Church's thesis) *Every real number can be computed to any desired approximation by an algorithm.*

(Geometric Completeness) *The points on a line segment correspond to real numbers in an interval.*

These two statements seem to be different: the first one expresses an intuition about *particular* real numbers. Each real number that someone might "give" to us is only "given" if we are told how to compute it. The second statement, on the other hand, expresses an intuition about the *totality* of real numbers (in an interval). The geometric line segment must be "the sum of its parts"–it is composed of points, so there must be "enough" points to fill up the line, without leaving any "gaps". Is it really possible that all those gaps are filled up with computable numbers?

All the notions in these principles (computation, algorithm, real number, geometric line segment) are notions about which we have strong intuitions, but which required Herculean labors of mathematics to bring to their modern precise formulations. Even before these concepts were made precise, people such as Weyl and Brouwer were uncomfortable, but with the advent of Turing machines the matter reached a sharper formulation. From the classical point of view, there are only countably many computable real numbers, so no, not all the gaps can be filled with computable numbers. Of course, the standard constructivist reply is that the computable numbers are not constructively enumerable, so you cannot point out an unfilled gap. But is this a fully satisfactory answer? One might compare that answer to saying, after sweeping the dirt under the carpet, that you cannot point it out.

In this paper, we will examine this question more closely.

The continuum in the history of logic

I will begin, not with the well-known contributions of Dedekind and Cantor, but with another historical stream, from the development of geometry.

Geometry was in attendance at the birth of logic in Euclid's *Elements*, and the nature of the continuum was already giving philosophers difficulty before that (Zeno's paradox). In the middle of the nineteenth century, Staudt (*Geometrie der Lage 1847*) took steps towards a modern deductive geometry; but Felix Klein observed in 1873 the difficulties about continuity in Staudt's treatment, complaining of the necessity "to conceive points, also if these are defined by means of an infinite process, as already existing." The second half of the nineteenth century saw the development of an increasingly rigorous axiomatic approach to geometry, for example Pasch's *Vorlesung über neuere Geometrie* appeared in 1882; but as Freudenthal observes in his fascinating history ([8], pp. 106–107), Pasch had a number of Italian contemporaries: Veronese, Enrique, Pieri, Padoa. These developments set the stage for the appearance in 1899 of Hilbert's *Grundlagen der Geometrie* [9]. [1] As Freudenthal says, the opinion is widespread that it was Hilbert who first gave a completely deductive logical system for Euclidean geometry, in which nothing was left to intuition. But in view of the achievements of his predecessors just mentioned, *what was there left for Hilbert to do?*[2]

Freudenthal answers this rhetorical question as follows: From antiquity an axiom was an evident truth, that could not be proved, but also needed no proof:

[1] Quite possibly they also influenced the work of Peano, who introduced modern logical notation a few years later.

[2] Was blieb für Hilbert eigentlich noch zu tun? [8], p. 110.

> Whether one believed with Kant that axioms arose out of pure contemplation, or with Helmholtz that they were idealizations of experience, or with Riemann that they were hypothetical judgements about reality, in any event nobody doubted that axioms expressed truths about the properties of actual space and were to be used for the investigation of properties of actual space.

Hilbert's book, on the other hand, begins with "Wir denken uns drie verschiedene Systeme von Dingen: die Dinge des ersten Systems nennen wir *Punkte*; die dingen des zweiten Systems nennen wir *Gerade*...". Freudenthal says *"With this the umbilical cord between reality and geometry is severed."*

But a few years later, Brouwer was again attempting to characterize and elucidate the properties of the *actual* continuum. He wrestled for the first time with the problem that will concern us in this paper: How can we reconcile the computability of individual real numbers with our notion that the continuum itself is "full of points", so full as to make a geometric line? Apparently he did not think that one can fill the gaps with computable numbers: he invented his theory of choice sequences. He said, for example, there is a real number 0.334434333444344..., where we are free to choose at any stage a 3 or a 4 arbitrarily, according to our free will. He did not require that we specify at any finite stage an algorithm for making all the rest of the choices.[3] The necessity for functions defined on [0,1] to be defined on all choice sequences led to his continuity principle, that all such functions are continuous. This principle flatly contradicted classical mathematics, and was responsible in no small part for the well-known public relations problems of Brouwer's intuitionistic mathematics.

Hermann Weyl was also concerned with the problem of "filling the gaps". Weyl (in the preface to the 1917 edition of *Das Kontinuum*) wrote:

> At the center of my reflections stands the conceptual problem posed by the continuum–a problem which ought to bear the name of *Pythagoras* and which we currently attempt to solve by means of the arithmetical theory of irrational numbers.

In the twenties and thirties, Hilbert applied his axiomatic viewpoint to other mathematical theories than geometry, and formulated his program to secure mathematics from the dangers of the paradoxes by proving the consistency of axiom systems. In his view, consistent axioms had to be "about something"–consistency guaranteed existence. In some sense, that is what was proved in the completeness theorem of Gödel, but in the twenties, let alone the 1910's, the concepts in the completeness theorem were not yet clear; not until Hilbert-Ackerman 1929 does one find the question clearly formulated, and it was answered two years later by Gödel.

In 1932, Weyl's book was reprinted, and he remarked in the preface,

[3]Technically the notion of choice sequence allows us to restrict our future choices; we could say, for example, that from now on we will choose every other digit to be a 3, and use our free will only on the remaining digits.

> ...in the period since its appearance, my work has been superseded by two trends identified by the catchwords Intuitionism and Formalism. Still, this...has not led to an even moderately satisfying or defensible conclusion...it seems not to be out of the question that the limitations prescribed in the present treatise–i.e., unrestricted application of the concepts "existence" and "universality" to the natural numbers, but not to sequences of natural numbers–can once again be of fundamental significance.

Recursion theory and the continuum

The recursive continuum (if this is not a contradiction in terms) has been of interest from the dawn of computability theory: Turing's original paper on Turing machines had the phrase "computable numbers" in the title. A recursive real number is given by a recursive sequence of rational approximations converging at a pre-specified rate, for example

$$|x_k - x_j| \leq 1/2^k + 1/2^j.$$

(One cannot simply say that the decimal expansion is recursively computable because of technicalities about 0.49999. . ..)

In the nineteen-forties, Stephen Kleene developed recursive realizability, giving for the first time a concrete and classically comprehensible interpretation of the notion of "constructive existence." The main idea of recursive realizability is that the quantifier combination $\forall x \exists y$ is replaced by a recursive function that produces y from x. For details see [10], pp. 501-516, or [14].

Kleene also observed the following fundamental fact:

Theorem 1 (Kleene's singular tree) *König's lemma is false in the recursive continuum. More precisely, there is an infinite binary tree with no infinite recursive path.*

Notation. We use the following standard notations from recursion theory: $T(e, x, k)$ means that k is a computation by the e-th computable partial function at input x, and the result (output) of the computation is $U(k)$. We also write $\{e\}(x)$ for this value $U(k)$. Since these are partial functions, $\{e\}(x)$ may be undefined for some e and x; we write $\{e\}(x) \cong y$ to mean that $\{e\}(x)$ is defined and is equal to y, i.e., $\exists k(T(e, x, k) \wedge U(k) = y)$.

Proof. One starts with two r.e., recursively inseparable sets, for example $A = \{n : \{n\}(n) \cong 0\}$ and $B\{n : \{n\}(n) \cong 1\}$. The tree will be constructed so that any path f will separate A and B: if $n \in A$ then $f(n) = 0$ and if $n \in B$ then $f(n) = 1$. The definition of the tree K is this: the finite binary sequence t of length n belongs to K if for each $k \leq n$, n steps of computation of $\{k\}(k)$ do not reveal that $(t)_k \neq \{k\}(k)$; that is,

$$K = \{t : \neg \exists j \leq lh(t)(T(k, k, j) \wedge U(j) = (t)_k\}).$$

We give 2^N the product topology and measure, induced by the norm $|x| = 2^{-k}$ where k is the least integer such that $x(k) \neq 0$. Kleene's construction also shows that the Heine-Borel theorem fails in recursive 2^N, since the finite sequences t which are not in K, but whose initial segments are all in K, form a covering of recursive 2^N without a finite subcover.

Closely related to Kleene's singular tree is the following construction of a "singular cover."

Theorem 2 (Lacombe's singular cover) *The set of recursive members of* 2^N *has measure zero.*

Proof. Let $\epsilon > 0$ be given and let k be a fixed integer with $1/2^k < \epsilon$. Let $y_1, \ldots, y_n$ be an enumeration of all indices of partial recursive functions y whose initial segments of length $k + y$ are defined. Let A_n be the set of all extensions of the initial segment t of $\{y_n\}$ of length $y_n + k$. The measure of A_n is $1/2^{y_n+k}$. Every recursive member of 2^N belongs to one of the A_n, but their total (classically defined) measure is bounded by

$$\sum_{n=1}^{\infty} \frac{1}{2^{y_n+k}} \leq \frac{1}{2^k} < \epsilon.$$

Remarks. Credit for this theorem is shared by Zaslavskiĭ and Čeitin [16], who made the argument constructive instead of classical, and adapted the construction to the unit interval, paying attention to making the intervals overlap only at endpoints, but Lacombe's publication was first. The neighborhoods produced in Lacombe's construction are not disjoint; in general A_n will meet A_m when $\{y_n\}$ and $\{y_m\}$ have a common initial segment longer than $\max(y_n, y_m) + k$.

Lacombe's construction is a "double shocker": the recursive members of [0,1] or 2^N have measure less than 1, and what's more, they have measure zero! Contrary to what one might initially suspect, neither of these shocks is implicit in Kleene's construction. This matter is worth investigating. We can get a cover of the recursive elements of 2^N from Kleene's tree by taking the collection of neighborhoods determined by finite sequences t that do not belong to the tree, but all their initial segments do belong. The elements of this cover are pairwise disjoint. What is their total measure? One, or less than one? The following lemma answers this question.

Lemma 1 *The cover of the set of recursive members of* 2^N *determined by Kleene's singular tree has total measure 1 (classically); constructively, the partial sums of the lengths of the cover are not bounded by* $1 - \epsilon$ *for any* $\epsilon > 0$.

Proof. Let U_0 be empty and $K_0 = 2^N$. Let U_{m+1} be the set of members of 2^N whose initial segment of length $m + 1$ does not belong to Kleene's tree K, but whose initial segment of length m does belong to K. Then the U_m are pairwise disjoint and their union covers the recursive elements of 2^N. Let K_{m+1} be the set of members of 2^N whose initial segments of length $m+1$ belong to K. Thus $U_{m+1} \cup K_{m+1} = K_m$. What sequences t of length m are initial segments of members of U_m? Those such that we have $\exists j < m(T(j,j,m) \wedge U(m) = (t)_j)$. Given m, there is at most one j such that $T(j,j,m)$; if there is no such j or if $j \geq m$, then $U_m = U_{m-1}$. Otherwise (if there is such a j), then half the sequences in K_{m-1} will drop out in U_m. In other words: for each m, either U_{m+1} is empty and $K_{m+1} = K_m$, or U_{m+1} consists of exactly half of K_m. The total measure of the U_m is thus (classically) the sum of the series $1/2 + 1/4 + \ldots 1/2^j + \ldots = 1$. We have not proved the constructive convergence of this series, since we do not know how large we must take m to get within a specified distance of 1. But if the cover had the partial sums of its measures bounded below 1, then there would be a maximum on the values of m such that U_{m+1} is nonempty, and the tree would be finite, which it is not. So we have proved constructively that for every ϵ, it is not the case that the partial sums are bounded by $1-\epsilon$.

In addition to these failures in topology and measure theory, the final public-relations disaster for recursive analysis is the failure in the recursive reals of the theorem that bounded monotone sequences have limits. Specker [13] discovered the existence of (what are now called) Specker sequences: recursive, strictly increasing sequences of rationals belonging to [0,1] but not converging to any recursive real number. Such a sequence can be given as follows: We construct the decimal expansion of the numbers to contain only 3 and 4 (so as to avoid problems with tails of nines). The k-th digit of x_n will be 3 if n steps of computation of $\{k\}(k)$ do not yield a value, or 4 if they do yield a value. This sequence is monotone since digits only change from 3 to 4; and the limit number, if it existed, would enable us to solve the halting problem, since $\{k\}(k)$ halts if and only if the k-th digit of the limit number is 4.

The Kreisel basis theorem ([12], p. 187) says that a recursive binary tree always has a Δ^0_2 path. The fact that Δ^0_2 is best-possible is illustrated by the Kleene's singular tree. König's lemma also fails in the class of functions recursive in α, for a fixed α. To avoid this phenomenon, we must go to the collection of partial Π^1_1 functions. König's lemma holds in this class: The total functions in this class are hyperarithmetic, and every hyperarithmetic infinite binary tree has a hyperarithmetic path. Thus from a purely recursion-theoretic point of view, there appears a connection between the fullness of the continuum and our ability to quantify over the integers. In a collection of functionals of finite type, we need the numerical quantifier E (considered as a type 2 functional) to guarantee the geometric fullness of the type 1 functions. (The hyperarithmetic functions are exactly those functions of type 1 recursive in E.) Feferman's analyses of predicativity [5, 6] show that the hyperarithmetic reals form a model of theories of predicativity, so Weyl might consider his viewpoint, and the doubts quoted above, to be partially justified by these results.

Bishop's constructive mathematics

These three failures of important classical results of analysis might seem to be the death knell of Church's thesis, since they appear to flatly contradict the principle of geometric completeness. Brouwer died in 1966, so he lived to see these results, but in his usual style, he never commented on them in print. They may have made him glad that he had developed the theory of choice sequences. Nevertheless, the Russian constructivists under the leader of Markov pursued the development of constructive mathematics assuming Church's thesis for some decades. At the time of Brouwer's death it appeared that your choices were:

(1) *accept Brouwer's theories, give up most of mathematics and give up talking to most mathematicians; or*

(2) *accept Church's thesis, give up analysis and give up talking to most mathematicians; or*

(3) *reject constructive mathematics entirely.*

This was not a difficult choice for most mathematicians; but Errett Bishop refused the prongs of this dilemma and published a book [2] in 1967 (the year after Brouwer's death) in which he developed constructive mathematics without using either Church's thesis or choice sequences. Since he didn't assume every real is recursive, the recursive counterexamples do not apply directly. Since he didn't assume there are some non-recursive reals (e.g. choice sequences), the classical theorems are not directly contradicted. His idea was to show that by suitable choices of definitions, the constructive content of classical mathematics could be brought to the fore, and was *substantial.*

Logicians labored in the subsequent decade to analyze what Bishop had done, by constructing suitable formal theories and studying their formal interpretations. This work is summarized in [1]. These studies verified (for various formal theories) that Bishop's work is indeed consistent with Church's thesis as well as with classical mathematics, and is constructive in the sense that "when a person proves an integer to exist, he or she can produce that integer". This is reflected in the "numerical existence property" of a formal theory T: if T proves $\exists x A(x)$ then for some numeral $\bar{n}$, T proves $A(\bar{n})$.

Bishop's measure theory

One of Bishop's achievements was the development of a constructive version of measure theory, according to which the unit interval has measure one. This measure theory was revised in [3]. A similar revised version appears in the second edition of Bishop's book, which added Douglas Bridges as co-author, and whose final version was completed by Bridges after Bishop's death.[4] In this section we extract from Bishop's theory the definition of "set of measure zero", and the statement and proof of the fundamental lemma that permits Bishop to prove

[4] See also the discussion of the formalization of Bishop's measure theory in [7].

that sets of positive measure are nonempty.[5] At first this seems surprising, since merely knowing that a set has positive measure does not seem to provide enough information to actually compute a member of the set. The following simple piece of constructive order theory will be needed: if $u + v < 1$ then either $u < 1/2$ or $v < 1/2$. One proves this as follows: for some $\epsilon > 0$ we have $u+v \le 1-\epsilon$. Then it is contradictory that both $u \ge 1/2+\epsilon/2$ and $v \ge 1/2+\epsilon/2$. Hence $u < 1/2-\epsilon/4$ or $v < 1/2-\epsilon/4$. At the last step we used the usual constructive replacement for trichotomy: if $a < b$ then for all x we have $x < b$ or $a < x$.

It will turn out to be sufficient to understand the concept "set of measure zero" in Bishop's measure theory. Bishop's measure theory applies to "complemented sets"; for our purposes we can define

$$-A = \{x : \forall y \in A(y \neq x)\}.$$

Here all variables range over [0,1], and $x \neq y$ means what constructivists usually call "apartness"; that is, it means that for some rational $\epsilon > 0$, $|x-y| > \epsilon$. What we usually call the measure of A is then the measure of the complemented set $(-A, A)$.

Bishop's definition on page 159 of [2] (with f identically zero) implies that A has measure zero if and only if for each $\epsilon > 0$ there exists a sequence of nonnegative functions f_j of bounded variation such that[6]

(a) $\sum_{j=1}^{\infty} \int f_j(x)\,dx$ exists and is less than ϵ.

(b) $x \in -A$ whenever $\exists\delta > 0 \forall N(\sum_{j=1}^{N} f_j(x) \le 1-\delta)$.

Bishop gives as an example the case when A is the set of rational numbers in [0,1]. Enumerate A by a sequence q_n. For each $\epsilon > 0$ there is a function f_n which is 1 at q_n and decreases linearly to zero on either side of q_n so that its integral is at most $\epsilon/2^n$. Thus condition (a) is satisfied. For condition (b): suppose the condition on the right of (b) holds for x, and let q_n be a given rational; then since f_n decreases linearly its slope is known, and we can bound x away from q_n in terms of n and δ, so $x \neq q_n$, so $x \in -A$.

The fundamental lemma in Bishop's measure theory (p. 160 of [2], with g identically 1; compare p. 219 of [4]) connects a measure-theoretic statement with an existence statement about a point. Essentially, it says that if a set X has measure less than the whole space, then we can find a point x in $-X$. We will give a more precise statement. In the statement, "test function" means continuous function with compact support.

Lemma 2 (Basic lemma of constructive measure theory) *Let X be a locally compact space. Let g be a nonnegative test function and let f_j be nonnegative*

[5] Bishop's measure theory is quite complicated: it has been presented in three different forms in the literature and the final form in [4] occupies more than eighty pages. It is therefore worthwhile to extract here the information relevant to the questions at hand.

[6] Bishop's definition has a condition (c) which falls away in the case of measure zero, when the f in Bishop's definition is taken to be identically zero.

test functions such that $\sum_{j=1}^{\infty} \int f_j\,dx$ *converges and is less than* $\int g(x)\,dx$*. Then we can find an* x *in* X *and* $\epsilon > 0$ *such that for all positive integers* m

$$\sum_{j=1}^{m} f_j(x) \leq g(x) - \epsilon.$$

The basic idea of the proof of this lemma is considerably easier to grasp in a "totally disconnected" space such as 2^N, where the space can be divided into, for example, two disjoint subspaces each of measure half that of the whole space. The plan of the proof is "divide and conquer". Consider the illustrative case of $X = 2^N$ and g identically 1. We divide the space into two pieces, $A = \{f : f(0) = 0\}$ and $B = \{f : f(0) = 1\}$. Given a set X with measure less than 1, we argue that either $X \cap A$ or $X \cap B$ has measure less than 1/2. We can make this choice constructively, by the piece of inequality reasoning given above: if $u + v < 1$ then either $u < 1/2$ or $v < 1/2$. Then we can continue in the fashion of the usual proof of Bolzano-Weierstrass, determining at the n-th stage a neighborhood U_n of radius $1/2^n$ such that the measure of $X \cap U_n$ is less than half the measure of U_n, and at the $n + 1$-st stage dividing the neighborhood U_n in half, selecting one of the halves as U_{n+1}. Let g_i be characteristic function of the neighborhood determined at the i-th stage, and x_i its center. Then the x_i form a Cauchy sequence whose limit will be the desired x. Of course, we still have to argue that x belongs to $-X$, but this is easy: assume that $\sum_{j=1}^{m} f_j(x) > 1 - \epsilon$ for some positive integer m. Then (by the continuity of f_j and the characteristic functions g_i), for sufficiently large n we have

$$\sum_{j=1}^{m} f_j(x) g_n(x) \geq (1 - \epsilon) g_n(x).$$

Integrating, we have

$$\int (1 - \epsilon) g_n\,dx \leq \sum_{j=1}^{\infty} f_j(x) g_n(x)\,dx$$

contradicting the hypothesis.

If the space is not disconnected (for example [0,1]) then one has to use a *partition of unity*. A partition of unity is a finite collection of functions $g_1, \ldots, g_m$ whose sum is identically 1, but each of which is nonzero only on a set of small diameter, say $1/n^2$. In a totally disconnected space, partitions of unity are trivial to construct; for example in 2^N, let N_t be the set of functions with finite initial segment t, and pick any collection of neighborhoods N_t that cover the whole space, and let the g_i be the characteristic functions of these neighborhoods. The use of partitions of unity is implicit in [2], p. 160-161, and more explicit in [4], p. 219, although in neither case is the phrase "partition of unity" actually mentioned.

Remark. What I want to call attention to is the absolutely crucial role played by the hypothesis that $\sum_{j=1}^{\infty} \int f_j\,dx$ *converges.* We needed that number to be constructively well-defined in order to use the $u + v$ argument to to decide whether to "go left or right" at the n-th stage in computing x. If all we knew was that the partial sums were bounded, we wouldn't be able to make that decision.

Singular covers and constructive measure theory

The main point of this section is to explain, without requiring a full exposition of constructive measure theory, how it is possible that Bishop's measure theory could be consistent with Church's thesis, in spite of the singular covers described above. This question is already indirectly approached in Exercise 2, page 70 of [1], which addresses the following technicality: the sum of the lengths of the intervals of the singular cover is not a recursive real. A more precise statement is that the cover consists of a recursive sequence of intervals, and the total length of any finite number of them is less than ϵ. But the series whose terms are the lengths is not recursively convergent. Putting the matter another way, if the singular cover consists of intervals I_n whose lengths are $L(I_n)$, then the partial sums s_n of the series $\sum L(I_n)$ form a Specker sequence. The total length of the covering intervals cannot be computed to a predetermined accuracy. In Bishop's terminology it might be a *fugitive sequence*–it can always jump by an unknown amount, no matter how long you have already been computing. Exercise 2 asks the reader to prove this fact, but for a hint it suggest that otherwise, Bishop-Cheng measure theory would be contradicted if Church's thesis is assumed, while we know from metamathematical studies that Bishop-Cheng measure theory can be formalized in theories that are consistent with Church's thesis. This may technically count as a solution, but from the point of view of understanding the situation, it is circular. What follows is a technically useless attempt to prove what we know to be impossible, but it is helpful for understanding.

Let us try to use the singular covers A_n defined above to prove that the set A of all recursive members of 2^N has measure zero, imitating the proof that the rationals have measure zero. In 2^N, characteristic functions of neighborhoods are continuous, we can take f_n to be the characteristic function of A_n, which depends on a given ϵ even though the notation A_n does not show this dependence. Condition (b) works: if the right-hand side of (b) holds, then x is not in any A_n, and hence is not a total recursive member of 2^N. Turning to (a), the integral $\int f_j(x)\,dx$ is bounded above by $1/2^{y_n+k}$, where $1/2^k < \epsilon$, but as we remarked above, the neighborhoods in Lacombe's cover do overlap, so the actual (classical) value of the sum on the left may be less than the bound, and we do not have a constructive proof that it converges. The fact that we can't estimate the rate of convergence of this sum prevents us from proving that the set of recursive reals has measure zero.

Let us ignore that difficulty for a moment, and try to use the fundamental lemma of constructive measure theory to construct a non-recursive real. To determine the first value $x(0)$ we consider $U = \{x : x(0) = 0\}$ and $V = \{x : x(0) = 1\}$.

Let S be the union of the singular cover, so S has measure < 1 and we need to know whether it is $S \cap U$ or $S \cap V$ that has measure less than $1/2$. Again we encounter the same problem: the measure of S is not a number that we can compute to any desired accuracy.

To cap this discussion, we will show directly that the measure of the union of the singular cover A_n is not a recursive number; in other words, the measures of the union of the first N terms form a Specker sequence. Recall the definition of the singular cover; it suffices to take $k = 1$ so the measure comes out less than $1/2$. Then we enumerate as $y_1, \ldots, y_n$ those y such that $\overline{\{y\}}(y+1)$ is defined, and we define A_n to be the set of elements f of 2^N such that $f(x) = \{y_n\}(x)$ for $x < y_n + 1$. Thus the measure of A_n is $2^{-y_n - 1}$. Any two of the A_n are either disjoint, or one contains the other. Suppose the measure of the union were a computable number. Then, given a rational $\epsilon > 0$, we could compute an integer $K = \kappa(\epsilon)$ such that for $j > K$, either A_j is contained in one of the first K sets A_n, or the measure of A_j is less than ϵ. Let $t_j = \overline{\{y_j\}}(y_j + 1)$. Then either $\{y\}$ extends t_n for some $n \leq K$, or $2^{-y_j - 1} < \epsilon$.

The condition $2^{-y_j - 1} < \epsilon$ is equivalent to $y_j > 1 + \lg(1/\epsilon)$, so it says that we won't have short programs y coming out at a late stage of the enumeration y_n. We will show, however, that this possibility cannot be prevented. The enumeration y_n is constructed in the first place by enumerating all computations, and putting y in the sequence y_n when we have successfully computed all values $\{y\}(x)$ for $x < y$. Some of those computations might take a long time, so a short program y might come out arbitrarily late in the sequence y_n. That is the intuitive reason why the measure of the union of the singular cover is not computable. We can use the recursion theorem to make this intuition into a proof, as shown in the next paragraph. The recursion theorem permits us to use the number y in the definition of the partial recursive function $\{y\}$.

By the recursion theorem define a recursive function y as follows: to compute $\{y\}(x)$, first compute $\epsilon = 2^{-y-1}$ and then $K = \kappa(\epsilon)$. Then compute (i.e., search for) a sequence number t that does not extend t_n for any $n \leq K$, but t is at least as long as all the t_n with $n \leq K$. The search for such a t will succeed, since the measure of the union of the first K of the A_n is less than $1/2$. Then the value to return as $\{y\}(x)$ is $(t)_x$ if $x < lh(t)$, and 0 otherwise. But before returning this value, we (artificially) enter a long loop, so that the computation of $\{y\}(x)$ will take a long time, specifically at least $K+1$ steps. Then the index J of y in the sequence y_n will be at least $K+1$, so by hypothesis we have either y extends t_n for some $n \leq K$, or $2^{-y-1} < \epsilon$. The first alternative does not hold since $\{y\}$ extends t and t is at least as long as t_n but does not extend t_n. Hence $2^{-y-1} < \epsilon$; but by definition of ϵ, $2^{-y-1} = \epsilon$. This contradiction completes the proof. This proof gives a direct solution of Exercise 2, p. 70 of [1], without reference to Bishop's measure theory or any metamathematical argument, and thereby demonstrates why the singular cover does not contradict Bishop's measure theory: the latter has the hypothesis that the sum of the measures of the cover should be a constructively-defined real number, but that hypothesis, which is crucial for the fundamental theorem that sets of positive measure contain some element, fails for the singular cover in recursive mathematics.

Geometric completeness and constructive logic

Bishop's beautiful construction of a measure theory that is consistent both with classical mathematics and with Church's thesis is dazzling, but somehow the singular cover constructions still leave one with the feeling that Church's thesis is at odds with the geometric completeness principle. In this section we try to capture that feeling in formulas.

Definition 1 *For subsets X of a locally compact metric space, we define **X has measure at most t** to mean that X is covered by a union of (a sequence of) neighborhoods such that the sum of the measures of any finite number of those neighborhoods is less than or equal to t.*

Thus the set of recursive members of [0,1] has measure at most ϵ, for every $\epsilon > 0$. This concept drops the condition imposed by Bishop that the measure of the cover itself must be computable.

With the aid of this definition we can consider the following principle, expressing the "fullness" of the continuum:

Fullness Principle (FP): *If [0,1] has measure at most t then $t \geq 1$.*

Although FP is meant to express an intuition about [0,1], its equivalent expression as a statement about 2^N is also interesting. There is a natural connection between binary trees and covers of 2^N. For t a finite binary sequence and $f \in 2^N$, let $t \subseteq f$ mean that f has t for an initial segment. Each finite binary sequence t determines a neighborhood $U_t = \{f : t \subseteq f\}$. The cover associated with a tree T consists of all the U_t such that $t \notin T$ but all initial segments of t are in T. Distinct members of this cover are distinct, since if U_t and U_s are in this cover then neither t nor s has the other for an initial segment. A tree has *size at most t* if the sum of the measures of any finite union of the U_t is at most t. A tree is *well-founded* if every path eventually runs out of the tree. FP is then closely related to this statement:

Tree Fullness Principle (TFP) *If a well-founded binary tree has size at most t then $t \geq 1$.*

At this point we review the standard notation for sequences coded as integers: we assume that every integer is a sequence number; $lh(t)$ is the length of the sequence encoded by t and its members are $(t)_0, \ldots, (t)_{n-1}$, where $n = lh(t)$. We write $t \subseteq q$ or $q \supseteq t$ to mean "q extends t", that is, $lh(t) \leq lh(q)) \wedge \forall j < lh(t)((t)_j = (q)_j)$. For β a function (of type 1) we write $t \subset f$ to mean $\forall j < lh(t)((t)_j = f(j))$. We write $\forall \beta \in 2^N \ldots$ to abbreviate $\forall \beta(\forall k \beta(k) < 2 \rightarrow \ldots$.

FP is formulated with a function variable for the sequence of neighborhoods; TFP admits several formalizations: it could be a second-order principle (with a set variable for the tree), or a first-order schema, with a formula for the complement of the tree. For definiteness, we take a version that is similar to FP, in that a function variable α is used to enumerate a sequence of neighborhoods.

$$\forall \beta \in 2^N \exists k \exists n (\overline{\beta(k)} \supseteq \alpha(n)) \wedge \forall m \big(\sum_{j=0}^{m} 2^{-lh(\alpha(j))} \leq t \big) \rightarrow t \geq 1 \qquad \text{TFP}$$

By Church's thesis CT, we mean, as usual, the assertion that every sequence of integers is given by some recursive function. In view of the singular cover, FP contradicts Church's thesis CT. Hence FP is not derivable in Bishop's constructive mathematics (BCM) or in formal systems that are consistent with CT.

Not only that, FP contradicts "there exists a real α such that every other real is recursive in α." This can be proved by relativizing the singular-cover construction to functions recursive in α.

FP is intended to express a formal version of the geometric completeness principle, that there are enough points to fill up a geometric line. Since it refutes CT, it proves not-not there exists a non-recursive member α of N^N, so it proves not-not there exists a non-recursive subset of $N \times N$, namely the graph of α. There is therefore some metamathematical work to do to check that this principle is not completely non-constructive.

FP is carefully formulated to avoid hidden assertions about the computability of numbers; it is intended to express that a lot of points exist (or technically, do not fail to exist) to fill up a line, without asserting that we can find any specific ones among those points. Contrast FP with the following:

Strong Fullness Principle (SFP) *If a subset X of [0,1] has measure at most $1 - \epsilon$, and $\epsilon > 0$, then we can find a member of $-X$.*

This principle strengthens the fundamental lemma of constructive measure theory by dropping the requirement that the measure of X must constructively exist.

We also will consider several constructive versions of König's lemma, or more precisely, of "weak König's lemma". Weak König's lemma is "weak" in that the tree is binary (or equivalently of bounded branching), rather than just of finitary (but possibly unbounded) branching. We formalize this in theories without set variables as a schema, using a formula to represent the tree. The usual formulation of weak König's lemma (considered in the proof theory of subsystems of classical analysis) says that every infinite binary tree has an infinite path.

To formalize WKL and related principles, we must decide how to represent a binary tree: a set variable, or a formula (resulting in a schema), or a function variable. To choose the version most closely related to FP, we suppose the complement of the tree is given by a sequence of neighborhoods. A function α will be thought of as enumerating sequence numbers $\alpha(n)$, and sequence number t belongs to the complement of the tree if $\alpha(n) \subseteq t$ for some n. Thus the tree itself consists of those sequence numbers t such that $\forall n \neg(\alpha(n) \subseteq t)$. In this formulation, the closure of the tree under subsequence is automatic: if $s \subseteq t$ and t does not extend any $\alpha(n)$, then s does not extend any $\alpha(n)$ either, since if it did, t would extend that same $\alpha(n)$. Weak König's lemma becomes:

$$\forall m \exists t (lh(t) \geq m \wedge \forall k \neg(\alpha(k) \subseteq t)) \rightarrow$$
$$\exists \beta \in 2^N \forall j \forall k \neg(\alpha(k) \subset \bar{\beta}(j)) \quad \text{(WKL)}$$

Constructively, we consider the version "every infinite binary tree cannot fail to have an infinite path," which we call "intuitionistic weak König's lemma":

$$\forall m \exists t (lh(t) \geq m \wedge \forall k \neg(\alpha(k) \subseteq t)) \rightarrow$$
$$\neg\neg\exists \beta \in 2^N \forall j \forall k \neg(\alpha(k) \subset \bar{\beta}(j)) \quad \text{(IWKL)}$$

We call a tree "well-founded" if every path eventually runs out of the tree. Thus IWKL says "every infinite tree is not well-founded". The following are equivalent expressions of IWKL: "every well-founded binary tree is not infinite," and "there are no well-founded infinite binary trees."

That Lacombe's result (for some $\epsilon < 1$) already implies Kleene's (without assuming that all members of 2^N are recursive) is the content of the following:

Theorem 3 *With constructive reasoning, IWKL implies TFP and FP.*

Proof. Suppose IWKL; we will prove TFP. Suppose that 2^N has measure at most t. That is, there is a cover of 2^N, the partial sums of whose lengths are bounded by t. Since 2^N is totally disconnected (open neighborhoods are also closed) we can define a new cover whose elements do not overlap, by removing from each U_n the part covered by the union of the U_k for $k < n$. Let the elements of this disjoint cover be denoted by V_n. Since $V_n \subseteq U_n$, the partial sums of the lengths of V_n are bounded by t. Each V_n is the set of all $f \in 2^N$ which extend some binary sequence t_n. The proper initial segments of these t_n form a tree T. Since V_n is a cover of 2^N, this tree has no infinite path.

Hence by IWKL, T is not an infinite tree. That is, not not there exists an integer m such that every member of T has length less than m. If there were such an m, then since T has no infinite path, the sum of the measures of the V_n would be exactly 1. Hence, not not $t \geq 1$. But $\neg\neg t \geq 1$ implies $t \geq 1$, completing the proof.

Remark. The tree T is not necessarily decidable, since the enumeration of the U_n might at any time spit out some relatively large neighborhoods, corresponding to short members of T. Intuitively, T would be decidable only if the partial sums of the measures of the U_n do not form a "fugitive sequence".

The converse question amounts, intuitively, to whether the existence of Kleene's singular tree already implies the existence of Lacombe's singular cover (dropping the assumption that all members of 2^N are recursive). Technically, it amounts to the question whether $\neg$WKL implies not not there exists a number $t < 1$ and a singular cover of measure at most t.

Open Question 1 *Does FP imply IWKL? Or even the restriction of IWKL to decidable trees?*

Remark. We show here how one line of attack fails. Suppose given an infinite binary tree T with no infinite path. We must derive a contradiction. Form the cover of neighborhoods $N_t = \{f : t \text{ is an initial segment of } f\}$, where t is not in T but all its initial segments are in T. (This step makes the extra assumption that T is decidable, but even with this assumption the argument won't work.) By TFP the partial sums of the measures of this cover are not bounded by any number less than 1. This situation, however, is not contradictory, as shown in the discussion of Kleene's singular tree. In general, then, the cover constructed in this way from a singular tree need *not* be a singular cover. But whether there is some *other* way to construct a singular cover from a singular tree, I do not know.

Brouwer's intuition about the continuum led him to formulate the *fan theorem*, which implies as a special case that a well-founded binary tree is finite; this is essentially Heine-Borel's theorem for 2^N:

$$\forall \beta \in 2^N \exists k A(\bar{\beta}(k)) \rightarrow \exists m \forall \beta \in 2^N \exists k \leq m A(\bar{\beta}(k))$$

To follow the pattern we have used for FP and IWKL, we consider only the special case in which the predicate A is given by a sequence of neighborhoods:

$$\forall \beta \in 2^N \exists k \exists n(\bar{\beta}(k) \subseteq \alpha(n)) \rightarrow \forall \beta \in 2^N \exists k \leq m \exists n(\bar{\beta}(k) \subseteq \alpha(n)) \qquad \text{HB}$$

In words: "every well-founded binary tree is finite." The converse of HB is then just IWKL, so our investigations connect in this way to the century-old investigations of Brouwer; but it seems to me that an intuition other than the geometric completeness principle is needed to justify HB; and the fan theorem, which allows any predicate to define the complement of the tree, rather than just a sequence of neighborhoods, is (presumably) even stronger. From the philosophical viewpoint we are arguing for the acceptance of FP based on geometric completeness, which is (on the face of it) weaker than HB and possibly weaker than IWKL.

Numerical Existence Property of FP

In this section we will use a well known form of realizability to show that FP has the disjunction property and the numerical existence property when added to the usual formal theories for constructive mathematics. TFP and WKL can be expressed in any of the formal theories discussed in [1]. For simplicity we consider here intuitionistic arithmetic of finite types HA^ω, which has variables for integers, and for functions of finite type, and a scheme for defininig function(al)s by primitive recursion. WKL is formalized by using a type-1 variable for the characteristic function of the tree. To formalize TFP, a cover can be considered as determined by a sequence of neighborhoods, where a neighborhood N_t is given by the sequence number t. Thus a cover is a function from integers to integers. The formula $f \in t$ abbreviates $t \subset f$ which in term abbreviates $\forall k < lh(t) f(k) = (t)_k$, where $(t)_k$ is the $k-th$ element of the sequence coded by the integer t. Thus a cover of 2^N is a function α such that

$$\forall \beta \in 2^N \exists k (\beta \in \alpha(k)).$$

The measure of the neighborhood t is $2^{-lh(t)}$, so to say that the partial sums of cover α are bounded by rational number r is to say that

$$\forall k \sum_{i=0}^{k} 2^{-lh(\alpha(i))} \leq r.$$

It is well-known how to formalize the arithmetic of rational numbers in the arithmetic of integers; the indexed-sum functional can be defined by the recursion scheme of HA^ω.

Theorem 4 *(i) $HA^\omega + FP$ has the disjunction property and the numerical existence property.*

(ii) If $HA^\omega + FP$ proves $\exists \alpha A(\alpha)$, then for some term t, it proves $A(t)$.

(iii) $HA^\omega + FP$ is closed under Church's rule. Explicitly: if $HA^\omega + FP$ proves $\forall n \exists m A(n, m)$, then for some numeral $\bar{e}$, it proves $\forall n \exists k (T(\bar{e}, n, k) \wedge A(n, U(k))$.

Remark. Since the terms of HA^ω denote primitive recursive functionals, (ii) implies that $HA^\omega + FP$ is closed under Church's rule: if it proves a function with property A exists, then it proves there is a recursive function with property A. In particular, $HA^\omega + FP$ cannot prove the existence of a non-recursive function, in spite of the fact that it does prove not all functions are recursive.
Proof. We use modified **q**-realizability, written e **mq** A. See [14], especially p. 259, Theorem 3.72. In view of that theorem it suffices to show that $HA^\omega + FP$ proves that FP is **mq**-realized. Let us review the syntactic form of FP:

$$\forall \alpha [\forall \beta \exists k (\beta \in \alpha(k)) \wedge \forall m (\sum_{i=0}^{m} 2^{-lh(\alpha(i))} \leq t) \rightarrow t \geq 1].$$

This has the form $\forall \alpha [Q(\alpha, t) \rightarrow t \geq 1]$. To prove this is **mq**-realized: let α be given and suppose j **mq** $Q(\alpha, t) \wedge Q(\alpha, t)$. In particular $Q(\alpha, t)$, so by FP we have $t \geq 1$. The formula $t \geq 1$ is a Π^1_0 formula and will be realized by the identically zero function of the correct type, if true. Hence $t \geq 1$ is realized by the identically zero function of the type to realize $t \geq 1$. Hence $\lambda \alpha z$ **mq** FP. That completes the proof.

Conservativity over HA

In this section we use intuitionistic forcing to prove the conservativity over HA of some of the principles considered in this paper. We begin with some preliminaries.

Next, regarding the exact formulation of HA^ω: we can either formulate HA^ω with λ-terms, in which case we take $\mathbf{s}$ to be $\lambda xyz.xz(yz)$, or we can formulate it using combinatory logic, in which case $\mathbf{s}$ is primitive and $\lambda x.t$ denotes a term built up using $\mathbf{k}$ and $\mathbf{s}$. We follow [14] and use the combinatory-logic formulation. We assume that the variables of type σ are v_n^σ for $n = 0, 1, 2, \ldots$; letters x,y, and so on are meta-variables ranging over these actual object variables.

Regarding notation for substitution: We sometimes write $A[x := t]$ to mean the result of substituting t for x in A; more often we write $A(x)$ and $A(t)$, the former indicating that x may occur free in A and the latter indicating $A[x := t]$.

Let T be HA^ω, or some other suitable constructive formal theory. Augment T by a constant α and an axiom that says that α defines a sequence of neighborhoods giving the complement of an infinite binary tree.

$$\forall k \exists t (lh(t) \geq k \wedge \forall j \neg(\alpha(j) \subseteq t))$$

That is, there are arbitrarily long sequence numbers that are not contained in any neighborhood $\alpha(j)$. As remarked before, the condition that the set of sequence numbers not contained in any $\alpha(j)$ is closed under subsequence is automatic. This augmented theory we call $T\alpha$.

Lemma 3 *$T\alpha$ is conservative over T.*

Proof. Let $Q(\beta)$ be the axiom for α, with the constant α replaced by a variable β. Let $A(\alpha := \beta)$ be the result of replacing α by β in formula A. Define, for each formula A of $T\alpha$, the formula A^* of T as follows:

$$A^* \textit{ is } \forall \beta (Q(\beta) \to A(\alpha := \beta),$$

where β is a variable that does not occur in A. By induction on the length of proofs in $T\alpha$ we have: If $T\alpha$ proves A then T proves A^*. When A does not contain α then A^* is provably equivalent to A; that completes the proof.

In $T\alpha$ we can formulate WKL this way:

$$\exists \beta \forall k, j \neg(\alpha(j) \subseteq \bar{\beta}(k)) \qquad \text{(WKL}\alpha\text{)}$$

In other words, any proof in $T\alpha$ plus this axiom can be syntactically transformed into a proof in T from WKL.

We will use forcing to prove our conservative extension result. Let $T\mathbf{b}$ be the theory $T\alpha$ augmented by another constant $\mathbf{b}$ for a function from integers to integers. We will define forcing for $T\alpha$ in $T\mathbf{b}$. The definition presupposes that we possess a formula of $T\alpha$, say $C(p)$, that defines the forcing conditions p to be used. Specifically we define $p \Vdash A$ for each formula A of $T\mathbf{b}$; the resulting formula $p \Vdash A$ is a formula of $T\alpha$. The atomic clauses in the definition of forcing are arranged so that

$p \Vdash \mathbf{b}(n) = m$	*is*	$C(p) \wedge n < lh(p) \wedge (p)_n = m$
$p \Vdash A$	*is*	A for A atomic not involving B

but since there can be atomic clauses involving higher-type terms mentioning **b**, this does not suffice as a definition. To give the correct definition, we will assign a term $\hat{t}$ to each term t of HA^ω, of type $(1,\sigma)$ where σ is the type of t, and then define

$$p \Vdash t = s \ := \ \forall\beta(p \subset \beta \to \hat{t}\beta = \hat{s}\beta).$$

The definition of the terms $\hat{t}$ is as follows. In this definition, variables p,q,r are implicitly relativized to the formula C defining the forcing conditions.

$$\begin{aligned} \hat{\mathbf{b}} &:= \lambda\beta.\beta \\ \hat{t} &:= \lambda\beta.t \ \text{ for } t \text{ a constant} \\ \hat{v}_k^\sigma &:= v_k^{\sigma,1} \\ \hat{tq} &:= \mathbf{s}\hat{t}\hat{q} \end{aligned}$$

so that $\hat{tq}(\beta) = (\hat{t}\beta)(\hat{q}\beta)$. The remaining clauses in the definition of forcing, as given in [1], Chapter XV, are as follows:

$$\begin{array}{rcl} p \Vdash \forall xA & is & \forall\hat{x}\forall q \supseteq p\exists r \supseteq q(r \Vdash A(x)) \\ p \Vdash (A \to B) & is & \forall q \supseteq p(q \Vdash A \to \exists r \supseteq q(r \Vdash B)) \\ p \Vdash \exists xA & is & \exists\hat{x}(p \Vdash A) \\ p \Vdash A \wedge B & is & (p \Vdash A) \wedge (p \Vdash B) \\ p \Vdash A \vee B & is & (p \Vdash A) \vee (p \Vdash B) \\ p \Vdash \bot & is & \bot \end{array}$$

The free variables of the formula $p \Vdash A$ are p, together with the $\hat{x}$ such that x is free in A; hence the use of $\hat{x}$ instead of x in the clauses above for $\exists$ and $\forall$. A clause for negation is not needed since we treat $\neg A$ as $A \to \bot$.

The following lemma is what makes forcing useful for conservative extension results:

Lemma 4 *$T\alpha$ proves $(p \Vdash A) \leftrightarrow A$ for arithmetic formulae A.*

Proof. A straightforward induction on the complexity of A, using the lemma that, when A has free variables, $(p \Vdash A)[\hat{x} := \hat{t}] \leftrightarrow p \Vdash A[x := t]$.

If ϕ is a formula of $T\alpha$, then "ϕ is generically valid" is the formula $\forall p\exists q \supseteq pq \Vdash \phi$. The soundness theorem for forcing ([1], Ch. XV, p. 348), says that if $T\alpha$ proves ϕ, and all the axioms of $T\alpha$ are provably generically valid in T, then all theorems of $T\alpha$ are provably generically valid in T. It follows from Lemma 4 that if some formula or schema S is provably generically valid in T then S is conservative over T for arithmetic theorems.

As a warmup exercise, we reprove a known result of Simpson:

Theorem 5 *WKL is conservative over Peano arithmetic PA.*

Proof. The tree defined by α will be used to define a set of forcing conditions, which we will use to add a generic path $\mathbf{b}$ through the tree. The set C of forcing conditions is the set of sequence numbers p that do not have only finitely many extensions in the tree determined by α. Formally, write $T(p)$ for $\neg\exists n(\alpha(n) \subseteq p)$. Then T is the tree whose complement consists of the neighborhoods enumerated by α, and we define

$$C(p) \;:= T(p) \wedge \neg\exists n \forall q(q \supseteq p \wedge T(q) \rightarrow lh(q) \leq n).$$

Since we are working with classical logic, this is the same as the set of p with infinitely many extensions in T. With classical logic then, we can prove that for each n there is a p of length n with $C(p)$, from which it follows that

$$p \Vdash \forall n(T(\bar{\mathbf{b}}(n)) = 0),$$

i.e., $\bar{\mathbf{b}}$ is a path through T. Formalizing this argument we see that $PA^{\omega}\mathbf{b}$ proves that WKL is generically valid; and as remarked before the theorem, that is sufficient for the conservativity of WKL over PA for arithmetic theorems.

To prove the conservativity of IWKL over HA, we will not be able to imitate the above proof directly; indeed WKL is not conservative over HA, so we cannot just add a generic path through the tree T determined by α. Our proof is more complicated; forcing will be combined with the model of HA^{ω} in Kleene's "countable functionals". This notion is introduced in [11], and described in [1], p. 135. We review the relevant features of the definition to establish notation. Kleene defines the concept of a type 1 function being an *associate* of a type σ function. This definition can be given in HA^{ω} by means of formulas $Ass^{\sigma}(f^{\sigma}, \gamma^{1})$. A notion of application is defined on type 1 functions by

$$\alpha|\beta = \lambda n.(\alpha(\mu k.\alpha(\bar{\beta}(k)) > 0) - 1)),$$

where $-$ is cutoff subtraction. Terms of type 1 are defined to interpret the constants $\mathbf{k}$ and $\mathbf{s}$ of type σ as well as the recursion constants. The functions of type σ will all be interpreted in the model as functions of type 1.[7] The functions that will interpret type σ are defined by a formula $T^{\sigma}(\gamma)$, given by

$$T^{(\sigma,\tau)}(\beta) := \forall x(T^{\sigma}(x) \rightarrow T^{\tau}(\beta|x).$$

Here x is a type 1 variable. Of course this starts with $T^{0}(x) := x = x$ and $Ass^{0}(\alpha, n) := \alpha(0) = n + 1$.

The "model", expressed as a syntactical interpretation, assigns to each term t of HA^{ω} a corresponding term t^{*} and to each formula A of HA^{ω} a corresponding formula A^{*}, expressing that A holds in the model, in such a way that

[7] It is not necessary that the interpretations of distinct types be disjoint, although this is not difficult to arrange, say by using the first value of each function as a type label, and modifying $\alpha|\beta$ to ignore (or type-check) those first values.

(i) $(tq)^* = t^*|q^*$
(ii) $(v_k^\sigma)^* = v^1_{(k,\sigma)}$ where v_k^σ is the k-th variable of type σ
(iii) $(\forall x^\sigma A)^*$ is $\forall\beta(T^\sigma(x^*) \rightarrow A^*)$
(iv) $(\exists x^\sigma A)^*$ is $\exists\beta(T^\sigma(x^*) \rightarrow A^*)$
(v) $\bot^*$ is $\bot$

and the map * commutes with the propositional connectives. Then the soundness of the model is expressed by: if A is a closed theorem of HA^ω then A^* is a theorem of HA^ω. To prove this we have to prove a more general theorem applicable to formulas A with free variables x of types σ_j in which the conclusion has the variables x^* relativized to T_j^σ.

$$Ass^{\sigma,\tau)}(t,\theta) \wedge Ass^\sigma(q,\gamma) \rightarrow Ass^\tau(tq,\theta|\gamma),$$

which is provable in HA^ω for each fixed pair of types (σ,τ).

In order to use forcing, we need a variation of IWKL that asserts the existence of something, so that we can use forcing to add a generic "something". To construct such a variation, we turn to the Gödel Dialectica interpretation of $IWKL$ for inspiration, although the Dialectica interpretation is not used in the proof. Namely, let us define $PathEnder(e,\alpha)$ to express that e of type (1,0) is a witness to the fact that the sequence α of neighborhoods defines the complement of a well-founded tree:

$$\forall\gamma(\alpha(e(\gamma)_0) \subseteq \bar{\gamma}(e(\gamma)_1)).$$

That is, $e(\gamma)$ is a pair (n,k) such that the initial segment $\bar{\gamma}(k)$ of γ witnesses that γ belongs to the neighborhood (consisting of all all extensions of) $\alpha(n)$. Now, the formula we need is the following formula "No Path Ender" (NPE) of $T\alpha$:

$$\exists F\forall e\neg(\alpha(e(F(e))_0) \subseteq \overline{F(e)}(e(F(e))_1)) \qquad (NPE)$$

In words: $F(e)$ is a γ which serves as a counterexample to $PathEnder(e,\alpha)$.

Lemma 5 *$HA^\omega + AC_{1,0}$ proves NPE implies IWKL.*

Proof. Suppose NPE; in order to derive IWKL, suppose α is a sequence of neighborhoods defining the complement of a well-founded infinite binary tree. We must derive a contradiction. Since the tree is well-founded we have

$$\forall\beta \in 2^N \exists n,k(\bar{\beta}(k) \supseteq \alpha(n)).$$

Applying $AC_{1,0}$ we have some e such that $e(\beta)$ is the pair (n,k):

$$\forall\beta \in 2^N(\bar{\beta}(e(\beta)_1) \supseteq \alpha(e(\beta)_0).$$

That is, $PathEnder(e,\alpha)$. But using NPE, we can construct a member $\gamma = F(e)$ of 2^N such that γ is a counterexample to $PathEnder(e,\alpha)$. This contradiction completes the proof of the lemma.

Theorem 6 *NPE is conservative over HA.*

Proof. We will use forcing to add a generic function (of type 1) $\mathbf{f}$ which will serve as an associate of a function F as in NPE. The forcing conditions will be sequence numbers which might be initial segments of such an associate. An associate of such a function would operate on initial segments of an associate of a pathender e for the tree given by α. An initial segment of an associate of some function e of type (1,0) can be visualized as a finite tree, the leaves of which are labeled with values $1 + z$ where $z = e(\beta)$ for every β extending that leaf. Therefore, an initial segment of a pathender e for a tree given by a sequence of neighborhoods α is essentially a finite tree whose leaves are all covered by the neighborhoods in the range of α. Initial segments of $\mathbf{f}$ must assign values to such finite labeled trees; the values must be zero if the information in the finite tree is not enough to determine $F(e)$, or $1 + F(e)$ if it is enough.

A forcing condition will be a sequence number that could serve as an initial segment of an associate $\mathbf{f}$ of F. It thus codes a finite set X of pairs (T, v) where T is a finite labeled tree and v the associated value, and where if v is nonzero, then $v - 1$ is a pair (n, k) showing that no function γ could be an associate of a pathender e with an initial segment of γ corresponding to the finite labeled tree T. Specifically:

Given a sequence number q specifying a finite set X of pairs (T, v) where T is a finite labeled tree, consider the labels on the tree leaves as pairs (n, k) (every number is a pair), and take the maximum value K of these k over all the trees T with (T, v) in X; and take the maximum value N of these n. The first N of the neighborhoods $\alpha(n)$ determine a finite cover; since α determines the complement of an infinite tree, there is a binary sequence t of length $\max(N, K+1)$ that does not extend any $\alpha(n)$ with $n \le N$. Any such sequence t will be said to be "ok with respect to X". The set C of forcing conditions is defined as the set of p such that, for $q < lh(p)$, if q specifies a set X as above, then $p_q = 1 + t$ where t is ok with respect to X; and if q does not specify such a set X, then $p_q = 0$. Forcing is defined as usual, so that $p \Vdash \mathbf{f}(q) = (p)_q$ for all $q < lh(p)$.

As just shown, in $T\alpha$ we can prove there are arbitrarily long forcing conditions. We now claim that NPE^* (the intepretation of NPE in the Kleene countable-functional model) is generically valid. Suppose $p \Vdash \mathit{PathEnder}^*(e, \alpha)$ and p forces α determines an infinite tree. We must show that some extension of p forces a contradiction. Since $p \Vdash \mathit{PathEnder}^*(e, \alpha)$, if β is any type-1 function then every condition extending p has a further extension forcing, for some m, n, and k, $e * (\bar{\beta}(m)) = 1 + (n, k)$ and $\bar{\beta}(k) \supseteq \alpha(n)$. Replacing m by $\max(m, k)$ we still have $e * (\bar{\beta}(m)) = 1 + (n, k)$, and now we have $p \Vdash \bar{\beta}(m) \supseteq \alpha(n)$.

Let M be the length of p, and consider those $q < M$ such that q specifies a finite set X of pairs (T, v), where T is a finite labeled tree. Then for each such q, $p_q = 1 + t$ where t is ok with respect to X. Consider, as above, the maximum values N and K of the n and k such that (n, k) is a label on one of the trees T such that (T, v) belongs to X. As above, we can choose a sequence t of length at least $\max(N, K+1)$ that does not extend any $\alpha(n)$ with $n \le N$. Let β be any type 1 function with initial segment t. Choose m and n as above so that $p \Vdash \bar{\beta}(m) \supseteq \alpha(n)$. Then in fact $\beta(m) \supseteq \alpha(n)$; in particular β extends $\alpha(n)$. Since t does not extend any $\alpha(n)$ with $n \le N$, but t is longer than all $\alpha(n)$ with $n \le N$, β also not extend any $\alpha(n)$ with $n \le N$. This is a contradiction, and completes the proof that NPE is generically valid. Since for arithmetic formulae ϕ, we have $p \Vdash \phi^*$ equivalent to ϕ, the conservative extension result for NPE follows from the soundness of forcing.

IWKL (and hence FP), when added to $HA^\omega + AC$, are conservative over HA.

Since IWKL implies NPE using $AC_{1,0}$, it would suffice to show that if ψ is an instance of $AC_{1,0}$ then ψ^* is generically valid. That would incidentally give yet another proof of Goodman's theorem on the conservativity of AC over HA.

The interpretation of $AC_{1,0}$ in Kleene's countable functionals is equivalent to the following principle of "continuous choice":

$$\forall\beta\exists n\phi(\beta,n) \to \exists g[\forall\beta\exists k g(\bar{\beta}(k)) > 0 \land \forall t(g(t) > 0 \to \forall\gamma \supset t(\phi(\gamma, g(t)-1)))] \quad \text{CC}$$

CC in turn can be decomposed: it is equivalent to the conjunction of $AC_{1,0}$ and "Brouwer's principle":

$$\forall\beta\exists m\phi(\beta,m) \to \forall\beta\exists m,k\forall\gamma(\bar{\gamma}(k) = \bar{\beta}(k) \to \phi(m)) \quad BP_0$$

Theorem 7 *$HA^\omega + CC$ is conservative over HA.*

Proof. We use the technique of [1], Chapter XV, namely, the composition of realizability and forcing. Let $T\mathbf{b}$ be HA^ω with a constant $\mathbf{b}$ for a type-1 function. The realizability interpretation $e\ \mathbf{r}\ A$ of A goes from HA^ω to $T\mathbf{b}$. It can be taken as modified realizability in the countable functionals recursive in $\mathbf{b}$. It is straightforward to show that CC is realized. Let ϕ be an arithmetic sentence; then in the cited chapter (p. 356)[8] it is shown how to construct a notion of forcing such that

$$\forall p \exists q \supseteq p(q \Vdash ((e\ \mathbf{r}\ \phi) \to \phi)).$$

Suppose $HA^\omega + CC$ proves ϕ. Then by the soundness of realizability, for some e we have HA^ω proves $e\ \mathbf{r}\ \phi$. By the soundness of forcing, HA^ω proves that $e\ \mathbf{r}\ \phi$ is generically valid. By the property of this particular notion of forcing, $e\ \mathbf{r}\ \phi \to \phi$ is provable; hence HA^ω proves ϕ. But HA^ω is conservative over HA, so HA proves ϕ.

Theorem 8 *$HA^\omega + AC_{1,0} + IWKL + FP$ is conservative for arithmetic theorems over HA.*

[8] In the cited reference $\supset$ is used where we now are using $\supseteq$, and on the cited page there are two typographical errors in which $\subset$ is used for $\supset$.

Proof. Since IWKL implies FP in HA^ω, and IWKL implies NPE in $HA^\omega + AC_{1,0}$, it suffices to show that NPE is conservative for arithmetic theorems over $HA^\omega + CC$. Suppose NPE proves an arithmetic statement ϕ in $HA^\omega + AC_{1,0}$. Then, using the notion of forcing in the previous proof, $HA^\omega + CC$ proves $\forall p \exists q \supseteq p (q \Vdash \phi)$. Since ϕ is arithmetic, $q \Vdash \phi$ is provably equivalent to ϕ, by Lemma 4, so $HA^\omega + CC$ proves ϕ. Then by the previous theorem, HA^ω proves ϕ.

Physics and the Continuum

The theme of this paper is to explore our geometrical intuitions about the continuum. In this section we show that the source of those intuitions is definitely *not* the nature of physical space. Around 1880 the idea that our geometric intuitions were about physical space was widely accepted. Recall the quote from Freudenthal given earlier about geometric axioms:

> Whether one believed with Kant that axioms arose out of pure contemplation, or with Helmholtz that they were idealizations of experience, or with Riemann that they were hypothetical judgements about reality, in any event nobody doubted that axioms expressed truths about the properties of actual space and were to be used for the investigation of properties of actual space.

The developments of non-Euclidean and Riemannian geometry, and their subsequent application to the general theory of relativity by Einstein, dealt a death blow to this idea.[9] This took place in the early twentieth century, and showed that on very large scales Euclidean geometry breaks down. Later it was also shown that Euclidean geometry must break down on *small* scales; how far physics has progressed towards the utter destruction of Kantian ideas about space deserves to be more widely appreciated by mathematicians and logicians. What follows is an explanation of the "Planck length" and its implications for the nature of space.

Apparently Planck was the first to note that $\sqrt{G\hbar/c^3}$ has the dimensions of length, but he offered no explanation. What follows is a simple calculation showing that distances smaller than this length cannot exist in the usual sense; i.e., spacetime cannot be considered to be smooth at that scale. The calculation uses two fundamental equations: The uncertainty principle from quantum mechanics, and the Schwarzschild radius for the formation of a black hole, from general relativity. It is often stated that "general relativity and quantum mechanics are not consistent", but seems not to be so well known to non-physicists that the inconsistency can be derived in one paragraph. (No claim of originality is made here; the argument is well-known to physicists and was shown to me by my friend Bob Piccioni.) These two equations will be combined to show that there

[9] Supposedly Gauss already attempted much earlier to verify empirically that the angle sum of a physical triangle formed by mountaintops is the Euclidean 180°. This shows that he didn't think physical space had to be Euclidean.

is a minimum radius given by the Planck formula just mentioned, below which spacetime cannot be regarded as smooth. The smoothness of spacetime (possibly except at isolated singularities) is a fundamental starting point for general relativity, so this calculation shows the inconsistency of general relativity and quantum mechanics.[10]

The uncertainty principle is $\Delta E \Delta t \geq \hbar$, where ΔE is the uncertainty in energy and Δt the uncertainty in time. Quantum mechanics allows the spontaneous creation of a particle-antiparticle pair of total mass M, which can travel a distance r and back to annihilate each other, provided that the uncertainty principle is respected when we take the uncertainty ΔE to be the whole energy E of the particles and Δt to be their lifetimes. Using Einstein's equation $E = mc^2$, and taking for the lifetime Δt the time it takes light to travel the distance r and back, namely $2r/c$, we have

$$[Mc^2][2r/c] \geq \hbar$$

or

$$2Mrc \geq \hbar \tag{1}$$

Now the Schwarzschild solution of Einstein's equations, expressed in units where $c = G = 1$, is

$$ds^2 = -(1 - 2M/r)dt^2 + \frac{dr^2}{1 - 2M/r} + r^2\psi(\theta, \phi)$$

for some function ψ of the angular coordinates θ and ϕ. This is valid in the exterior of a spherical body of mass M. The value $r = 2M$ gives a zero denominator; what this means is that whenever a mass is compressed within its "Schwarzschild radius" $r = 2M$, the mass will collapse into a black hole.[11] Putting the factors of c and G into the equation $r = 2M$, we get $rc^2 = 2MG$. Solving for M we get$M = rc^2/(2G)$. Putting that into equation (1) we have

$$r^2c^3/G \geq \hbar$$

Solving for the smallest permissible value of r, the Planck length comes out:

$$r \geq \sqrt{\hbar G/c^3}$$

Evaluating this numerically we have

$$r \geq 1.616 \times 10^{-33}\text{cm}$$

[10]There are books about "relativistic quantum mechanics", but they are about *special* relativity and quantum mechanics, e.g. the Dirac equation.

[11]The Schwarszchild radius is, curiously, what you get if you use the classical Newtonian equation for the escape velocity $v^2 = 2GM/r$ and set the escape velocity equal to the speed of light c. However, for establishing its connection to black holes, we need the Schwarszchild solution of Einstein's equations.

Now we'll go over the argument again without equations. The uncertainty principle allows spontaneous creation of sufficient energy to momentarily (for Planck times on on the order of 10^{-43} seconds) collapse spacetime, i.e. creating a small temporary black hole. Thus the topology of spacetime itself may be uncertain at these dimensions; it may be multiply connected or have more than three spatial dimensions. Wheeler called this situation Quantum Foam. It is worth pointing out that since the argument derives a contradiction, we haven't really proved the existence of quantum foam or of tiny black holes at the Planck scale. All we have proved is that something happens at that scale that we cannot understand with our present collection of equations of physics.

The contradiction between quantum mechanics and relativity implies that our intuition of the continuum does not correspond to physical reality at lengths smaller than the Planck length. The contradiction depends on general relativity, and a fundamental assumption of general relativity is that space can be assigned coordinates; or in other words, numbers can be assigned to points on a line in such a way that to every point there corresponds a number and vice-versa. In other words, the geometric completeness principle is assumed by general relativity. But as we just derived, this cannot continue to be the case for distances smaller than the Planck length.[12]

The Source of Intuition about the Continuum

Philosophers have argued over whether our intuition of the continuum is derived from physical reality, from our experience of physical reality, from the nature of our minds, or from some "mathematical reality" that we experience with our minds. It has now been clearly shown that it is *not* derived from physical reality.

In the past people thought that physical space was Euclidean. Since Einstein we have known that our intuition about lines does not correspond to the physical "lines" determined by light paths in a vacuum. This we can call the "failure of (Euclidean) geometry in the large." The Planck-length argument we can call the "failure of geometry in the small." It might be argued that these failures imply that our intuition of the continuum has a non-physical source, which must therefore be either our minds, or a mathematical but non-physical reality that can be apprehended by the mind. For example, Gödel suggested that our minds can

[12] The Planck length can be "discovered" in various ways, the simplest of which is to ask for an expression in G and c that has dimension length. Another way is to use the equation $E = h\nu = hc\lambda$ for the energy of a photon of frequency ν and wavelength λ. This photon would distort space in the same way as a mass given by $mc^2 = E = hc/\lambda$, so it would have a Schwarzschild radius of $2mG/c^2 = 2hG/(c^3\lambda)$. Now observe that for λ small enough, λ will be less than the Schwarzschild radius, so the Schwarzschild solution should apply in the exterior of the photon, and the photon would be sealed into the black hole of its own creation, and hence unobservable by us. The critical wavelength is obtained by setting λ equal to the Schwarzschild radius $2hG/c^3\lambda$. Solving, we find $\lambda = \sqrt{2hG/c^3}$, approximately the Planck length. This argument, though interesting and strange, is not apparently contradictory, unlike the argument given in the text.

serve as sense organs to apprehend the non-physical reality of the continuum. We argue against this implication, instead supporting Helmholtz's view (as quoted by Freudenthal) that intuition is derived from our *experience* of physical reality. Specifically, part of our intuition about the continuum is derived as an abstraction from a simple physical *process*. The process is familiar from computer graphics: *zooming in* and *zooming out*. While viewing a finite interval that is part of a longer line, we can double the length of the viewed interval (zooming out) or halve the length of the interval (zooming in). Then we can adjust the size of the display so that the new selected interval appears congruent to the previous one. The abstraction involved here is abstracting from the finite thickness of the line and the small but possibly nonzero deviation from being perfectly straight. After the zoom, the thickness and straightness of the (view of) the line will be adjusted to be as before.

We can imagine this process, say zooming in, going on indefinitely. Specifically, as many times as there are positive integers. We can define I_0 to be $[0, 1]$ and I_{n+1} to be the middle half of I_n. The Planck length argument shows that this is not physically the case–after about 100 zooms, space itself is no longer coordinatizable, and the zooming-in process breaks down. The exact manner of its failure is unknown! But nevertheless, the zooming-in process itself is easily intuited, and we can distinguish two parts of this intuition:

- Intuition of zooming in and zooming out (once)
- Intuition of iterating a process any number of times

The intuition of iterating a process is reducible to the concept of natural number, once the process to be iterated is understood.

This argument in support of Helmholtz's view (and against Gödel's) is not definitive, since the zooming processes do not account entirely for our intuition of the continuum. Here are some aspects not accounted for: *linearity*, *composition*, *continuity*, and *fullness*. By linearity, we mean the quality Euclid had in mind when he wrote that a line is that which has length but not breadth. We can zoom in on a plane or even on a self-similar fractal set. By *composition* we mean the question whether the continuum is composed of (infinitely many) *points*, each of which has zero length, but whose aggregation can make intervals of nonzero length. The alternative conception is that somehow these points need to be actively created "at run time", as a computer scientist might say; perhaps by Brouwer's "free choices" or by some sort of quantum-mechanical device. The zooming processes also do not address the *continuity* of the continuum, the property that Dedekind addressed with his definition of completeness (every cut determines a real) and Cauchy with his definition of completeness (Cauchy sequences converge). By specifying that there are no visible gaps, we rule out the possibility that we are zooming in on some kind of fractal set rather than the true continuum. But, there are also invisible gaps to worry about: while zooming, how can we tell whether we are seeing the whole continuum or only, say, the rational numbers? No matter how many times we zoom in on $\sqrt{2}$, the gap in the rationals never becomes visible. We can call attention to it by placing a right isosceles triangle with its hypotenus on the number line. We can similarly call attention to any recursive real number; but how about the gap in the recursive reals at the limit of a Specker sequence? Can you visualize that gap? As the predicates used to define a Dedekind cut increase more and more in logical complexity, the existence of a point filling that cut seems less and less closely related to a fundamental geometric intuition. Finally, the property of *fullness* of the continuum, as axiomatized in this paper using the fullness principle FP, seems similar to continuity, but distinct, since the recursive reals satisfy continuity (in the sense that recursively Cauchy sequences of recursive reals converge to recursive reals), but they do not satisfy FP. Fullness does not require reference to specific "gaps", since it is defined by coverings. Perhaps the formulation of the fullness property and the recognition that it is not the same as continuity may help in future efforts to elicidate our intuitions about the continuum.

Conclusions

Our intuitions point to two principles:

(Church's thesis) *Every real number can be computed to any desired approximation by an algorithm.*

(Geometric Completeness) *The points on a geometric line segment correspond to real numbers in an interval.*

These seem to be contradictory in view of Kleene's singular tree and Lacombe's singular cover. We have formalized this feeling by exhibiting the principle FP, which is justified by geometric completeness, and contradicts CT. On the other hand FP is otherwise constructive, since

- FP has the numerical existence and disjunction properties.
- FP satisfies Church's rule; in particular it does not prove the existence of a non-recursive function
- FP is conservative for arithmetic theorems when added to HA^{ω} or other constructive theories.

The geometric continuum is "filled" with non-recursive members, even though we cannot prove their individual existence. Perhaps we should say, the continuum is not-not filled with non-recursive members. These unspecifiable points correspond, perhaps, to "generic" reals; or perhaps, to Brouwer's choice sequences; or perhaps, some of them can be generated by quantum-mechanical processes; or perhaps, they are figments of our mathematical imagination. This conclusion, however, does not necessarily destroy the basic premises of constructive mathematics, nor does it even necessitate accepting classically false axioms as Brouwer did. The principle FP, for example, is an example of a system expressing some of our intuition about the non-recursive "gap-fillers" in the continuum, and still possessing the usual properties of constructive systems. There may be other, stronger axiom systems that capture yet more of our intuition about the continuum.

In searching for such additional principles, it may be fruitful to examine the source of our intuitions about the continuum. At any rate, our intuition about the continuum is *not* related to the physical space we inhabit, but only to our mental conceptions about a possible idealization of that space, since modern physics tells us that physical space cannot be coordinatizable and indefinitely divisible.

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