Tiling triangle ABC with congruent triangles similar to ABC

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Abstract

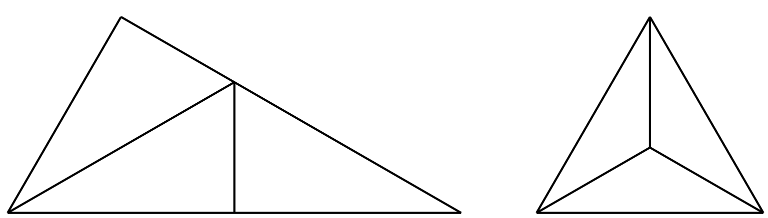
We investigate the problem of cutting a triangle ABC into N congruent triangles (the "tiles"), each of which is similar to ABC. The more general problem when the tile is not similar to ABC is not treated in this paper; see [1]. We give a complete characterization of the numbers N for which some triangle ABC can be tiled by N tiles similar to ABC, and also a complete characterization of the numbers N for which a particular triangle ABC can so tiled. It has long been known that there is always a "quadratic tiling" when N is a square. We show that unless ABC is a right triangle, N must be a square. On the other hand, if ABC is a right triangle, there are two more possibilities: N can be a sum of two squares $e^2 + f^2$ if the tangent of one of the angles is the rational number e/f, or in case ABC is a 30-60-90 triangle, N can be three times a square.

The key idea is that the similarity factor $\sqrt{N}$ is an eigenvalue of a certain matrix. The proofs we give involve only undergraduate level linear algebra.

1 Examples of Tilings

We consider the problem of cutting a triangle into N congruent triangles. Figures 1 through 4 show that, at least for certain triangles, this can be done with $N = 3$, 4, 5, 6, 9, and 16. Such a configuration is called an N-tiling.

Figure 1: Two 3-tilings



The method illustrated for $N = 4$,9, and 16 clearly generalizes to any perfect square N. While the exhibited 3-tiling, 6-tiling, and 5-tiling clearly depend on the exactly angles of the triangle, *any* triangle can be decomposed into n^2 congruent triangles by drawing $n - 1$ lines, parallel to each edge and dividing the other two edges into n equal parts. Moreover, the large (tiled) triangle is similar to the small triangle (the "tile"). We call such a tiling a *quadradtic tiling.* It follows that if we have a tiling of a triangle ABC into N congruent triangles, and m is any integer, we can tile ABC into Nm^2 triangles by subdividing the first tiling, replacing

Figure 2: A 4-tiling, a 9-tiling, and a 16-tiling

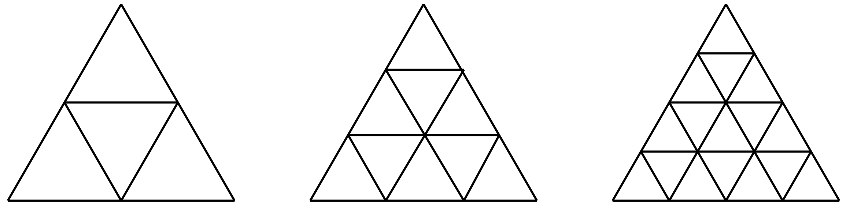
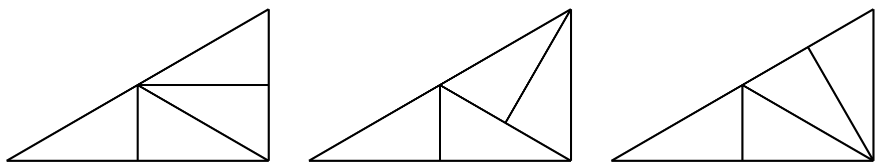


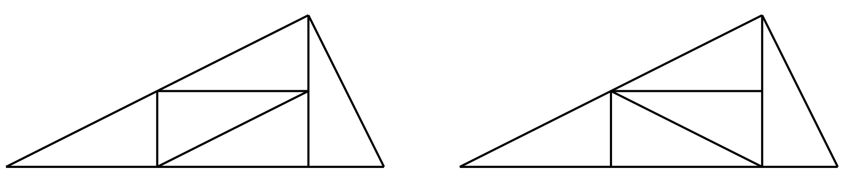
Figure 3: Three 4-tilings



each of the N triangles by m^2 smaller ones. Hence the set of N for which an N-tiling of some triangle exists is closed under multiplication by squares.

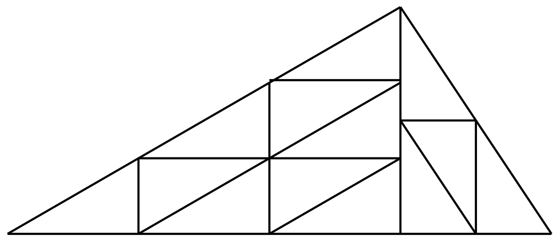
Let N be of the form $n^2 + m^2$. Let triangle T be a right triangle with perpendicular sides n and m, say with $n \geq m$. Let ABD be a right triangle with base AD of length m^2, the right angle at D and altitude mn, so side BD has length mn. Then ABD can be decomposed into m triangles congruent to T, arranged with their short sides (of length m) parallel to the base AD. Now, extend AD to point C, located n^2 past D. Triangle ADC can be tiled with n^2 copies of T, arranged with their long sides parallel to the base. The result is a tiling of triangle ABC by $n^2 + m^2$ copies of T. The first 5-tiling exhibited in Fig. 3 is the simplest example, where $n = 2$ and $m = 1$. The case $N = 13 = 3^2 + 2^2$ is illustrated in Fig. 5. We call these tilings "biquadratic." More generally, a *biquadratic tiling* of triangle ABC is one in which ABC has a right angle at C, and can be divided by an altitude from C to AB into two triangles, each similar to ABC, which can be tiled respectively by n^2 and m^2 copies of a triangle similar to ABC. The second 5-tiling shows that this can be sometimes be done more generally than by combining two quadratic tilings.

Figure 4: Two 5-tilings



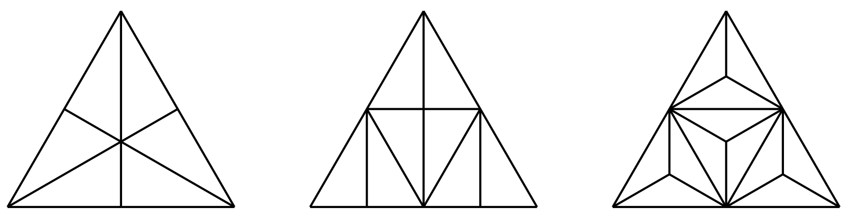
If the original triangle ABC is chosen to be isosceles, then each of the n^2 triangles can be divided in half by an altitude; hence any isosceles triangle can be decomposed into $2n^2$ congruent triangles. If the original triangle is equilateral, then it can be first decomposed into n^2 equilateral triangles, and then these triangles can be decomposed into 3 or 6 triangles each, showing that any equilateral triangle can be decomposed into $3n^2$ or $6n^2$ congruent triangles. These tilings are neither quadratic nor biquadratic. For example we can 12-tile an equilateral triangle in two different ways, starting with a 3-tiling and then subdividing each triangle into 4

Figure 5: A 13-tiling



triangles ("subdividing by 4"), or starting with a 4-tiling and then subdividing by 3.

Figure 6: A 6-tiling, an 8-tiling, and a 12-tiling



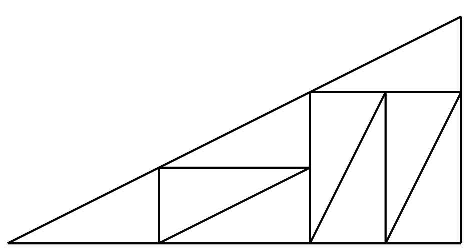
Examples like these led us to the following definitions:

A tiling E of triangle ABC (with tile T_2 is a *subtiling* of another tiling F of ABC (with tile T), if T can be tiled by the tile T_2 and the tiling E is obtained by tiling each copy of T in F with triangle T_2. It is not required that the *same* tiling be used for each copy of T. For example, we could take F to be one of the two five-tilings, and then tile each of the tiles in that tiling by one of its two five-tilings. In this way we can obtain 32 different 25-tilings, none of them quadratic.

A tiling of ABC is called *composite* if it is a subtiling of some tiling into fewer triangles. It is called *prime* if it is not composite. Note that a quadratic N^2 tiling is prime if and only if N is a prime number.

The examples above do not exhaust all possible tilings, even when N is a square. For example, Fig. 7 shows a 9-tiling that is not produced by those methods:

Figure 7: Another 9-tiling



There is another family of N-tilings, in which N is of the form $3m^2$, and both the tile and the tiled triangle are 30-60-90 triangles. The case $m = 1$ is given in Fig. 1; the case $m = 2$ makes $N = 12$. There are two ways to 12-tile a 30-60-90 triangle with 30-60-90 triangle. One is to first quadratically 4-tile it, and then subtile the four triangles with the 3-tiling of Figure 1. This produces the first 12-tiling in Fig. 8. Somewhat surprisingly, there is another way to tile the same triangle with the same 12 tiles, also shown in Fig. 8; the second tiling is prime. The next member of this family is $m = 3$, which makes $N = 27$. Two 27-tilings are shown in Fig. 9; the first obtained by subtiling a quadratic tiling, and the second one prime. Similarly, there are two 48-tilings (not shown).

Figure 8: Two 12-tilings

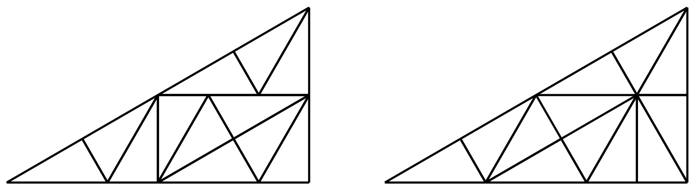
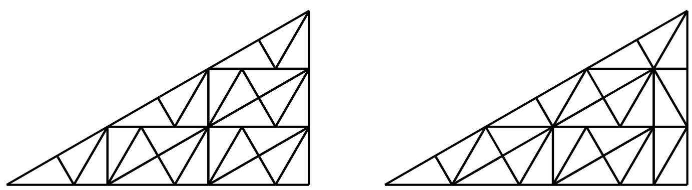


Figure 9: Two 27-tilings



Until October 12, 2008, no examples were known of more complicated tilings than those illustrated above. Then we found the beautiful 27-tiling shown in Fig. 10. This tiling is one of a family of $3k^2$ tilings (the case $k = 3$). The next case is a 48-tiling, made from six hexagons (each containing 6 tiles) bordered by 4 tiles on each of 3 sides. In general one can arrange $1 + 2 + \ldots + k$ hexagons in bowling-pin fashion, and add $k + 1$ tiles on each of three sides, for a total number of tiles of $6(1 + 2 + \ldots + k) + 3(k + 1) = 3k(k + 1) + 3(k + 1) = 3(k + 1)^2$. Figure 11 shows more members of this family.

Figure 10: A prime 27-tiling

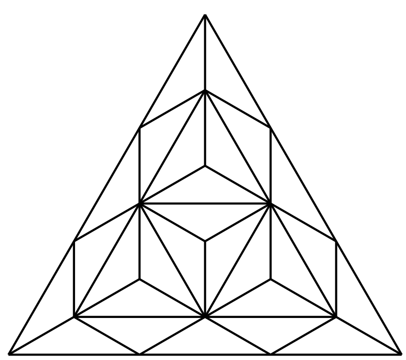
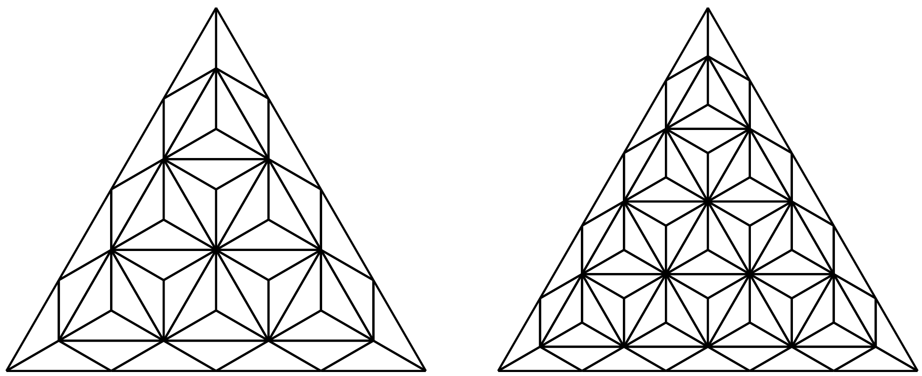


Figure 11: $3m^2$ tilings for $m = 4$ and $m = 5$



2 Previous work

The examples given in Figures 1 through 6 are well-known. They have been discussed, in particular, in connection with "rep-tiles" [7]. A "rep-tile" is a set of points X in the plane (not necessarily just a triangle) that can be dissected into N congruent sets, each of which is similar to S. An N-tiling in which the tiled triangle ABC is similar to the triangle T used as the tile is a special case of this situation. That is the case, for example, for the n^2 family and the $n^2 + m^2$ family, but not for the 3-tiling, 6-tiling, or the 12-tiling exhibited above. Thus the concepts of an N-tiling and rep-tiles overlap, but neither subsumes the other. The paper [6] also contains a diagram showing the n^2 family of tilings, but the problem considered there is different: one is allowed to cut N copies of the tile first, before assembling the pieces into a large figure, but the large figure must be similar to the original tile. The two books [2] and [3] have tantalizing titles, but deal with other problems.

Only after completing the work in this paper did I encounter Soifer's book [8], when the second edition came out, although the first edition had been out for 19 years. The book contains the observation that if the tile T is similar to the tiled triangle then $\sqrt{N}$ is an eigenvalue of a certain matrix, so that observation is, as it turns out, not new. The book, however, does not contain any examples of tilings beyond the quadratic tilings, though it gives an indication that at least the biquadratic tilings were known, since it says that the 1989 Russian Mathematical Olympiad contained the problem to show that if N is a sum of two squares then there is a triangle that can be N-tiled. Soifer states (p. 48) an open problem about triangle tiling, and says that Paul Erdös offered a \$25 prize for the first solution. He does not state where or when Erdös mentioned these problems. The problem statement is: Find all positive integers N such that at least one triangle can be cut into N triangles congruent to each other. This is Soifer's "Problem 6.7."

Soifer also states some related problems. His "Problem 6.5" is: For each triangle ABC, find all positive integers N such that T can be cut into N triangles congruent to each other, and the number of distinct partitions of T into N congruent triangles. Soifer says that his Problem 6.5 is "open and very difficult."

Soifer's "Problem 6.6" is also a \$25 Erdös problem: Find (and classify) all triangles that can only be cut into n^2 congruent triangles for any integer n.

Soifer claims, without publishing a proof, that if the sides and angles of ABC are integrally independent, then ABC admits only quadratic tilings. He proves that the perfect squares are exactly the N for which *every* triangle ABC can be N-tiled by some triangle.

Dima Fan-Der-Flaas informed me that the problem of finding an N-tiling of some triangle when $N = 1989$ was posed on the Russian Mathematical Olympiad in 1989; it was solved by a few students, who had to discover what we call here the "biquadratic tilings", and realize that 1989 is a sum of two squares and the relevance of that fact. I would like to thank Dima for his careful reading of parts of some drafts of this paper.

3 The impossibility of certain tilings

The elementary constructions just described suffice to produce N-tilings when N has one of the forms n^2, n^2+m^2, $2n^2$, $3n^2$, or $6n^2$. Of the tilings we have exhibited, many have the tile similar to the tiled triangle ABC; the others have ABC equilateral or at least isosoceles. In this paper we deal only with the case when the tile is similar to ABC; in which case it turns out that the only possible forms of N are n^2, n^2+m^2, and $3n^2$. Our main theorem implies that if N is not of one of these forms, then there is no N-tiling of any triangle ABC by a tile similar to ABC; but it also shows that the second and third form can only apply when ABC is a right triangle.

4 Definitions, notation, and some simple lemmas

We give a mathematically precise definition of "tiling" and fix some terminology and notation. Given a triangle T and a larger triangle ABC, a "tiling" of triangle ABC by triangle T is a list of triangles $T_1, \ldots, T_n$ congruent to T, whose interiors are disjoint, and the closure of whose union is triangle ABC. A "strict vertex" of the tiling is a vertex of one of the T_i that does not lie on the interior of an edge of another T_j. A "strict tiling" is one in which no T_i has a vertex lying on the interior of an edge of another T_j, i.e. every vertex is strict. For example, the biquadratic tilings (illustrated above for $N = 5$ and $N = 13$) are not strict, but all the other tilings shown above are strict. The letter "N" will always be used for the number of triangles used in the tiling. An N-tiling of ABC is a tiling that uses N copies of some triangle T.

Let a, b, and c be the sides of triangle ABC, and angles α, β, and γ be the angles opposite sides a, b, and c, i.e. the interior angles at vertices A, B, and C. An *interior vertex* in a tiling of ABC is a vertex of one of T_i that does not lie on the boundary of ABC. A *boundary vertex* is a vertex of one of the T_i that lies on the boundary of ABC.

By the law of sines we have

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

Up to similarity then we may assume

$$\begin{aligned} a &= \sin\alpha \\ b &= \sin\beta \\ c &= \sin\gamma \end{aligned}$$

Since $\gamma = \pi - (\alpha+\beta)$ we have $\sin(\gamma) = \sin(\alpha+\beta)$, so

$$p\sin\alpha + q\sin\beta + r\sin(\alpha+\beta) = 0.$$

The meanings of all these symbols will be fixed throughout the rest of the paper.

A non-strict vertex V is one that lies on an edge of T_j, with T_j on one side of the edge and (more than one) T_i having vertex V on the other side. Consider the maximal line segment S extending this edge which is contained in the union of the edges of the tiling. This is defined to be the *maximal segment* of V. Since there are triangles on each side of S, there are triangles on each side of S at every point of S (since S cannot extend beyond the boundary of ABC). Hence the length of S is a sum of lengths of sides of triangles T_i in two different ways (though the summands may possibly be the same numbers in a different order). Let us assume for the moment that the summands are not the same numbers. Then it follows that some linear relation of the form

$$pa + qb + rc = 0$$

holds, with p, q, and r integers not all zero (one of which must of course be negative), and the sum of the absolute values of p, q, and r is less than or equal to N, since there are no more than N triangles.

If S is a maximal segment containing a non-strict vertex, then there will be integers n and m such that n triangles have a side contained in S and lie on one side of S, and m triangles have a side in S and lie on the other side of S. In that case we say S is of type $m : n$. For example, Fig. 3 shows a 5-tiling with a maximal segment of type $1 : 2$. This definition does not require that the lengths of the subdivisions of the maximal segment all be the same (as they are in Fig. 3).

A *quadratic tiling* is one in which N is a perfect square, say $N = m^2$, and the tiling is produced by drawing $m - 1$ equally spaced lines parallel to each side, dividing each edge into m equal segments. In such a tiling, the tile T is similar to the large triangle ABC. An *angle relation* is an equation

$$p\alpha + q\beta + r\gamma = 2\pi$$

where p, q, and r are non-negative integers, not all equal. (Since we always have $\alpha + \beta + \gamma = \pi$, we do not count that equation or its multiples as an angle relation.)

A *split vertex* occurs when two copies of the tile in a triangle share one of the vertices of the large triangle.

The following lemma is simple and fundamental:

Lemma 1 *If, in a tiling, P is a boundary vertex (or a non-strict interior vertex) and only one interior edge emanates from P, then both angles at P are right angles and $\gamma = \pi/2$.*

Proof. If the two angles at P are different, then their sum is less than π, since the sum of all three angles is π. Therefore the two angles are the same. But $2\alpha \le \alpha + \beta < \pi$ and $2\beta \le \beta + \gamma < \pi$. Therefore both angles are γ. But then $2\gamma = \pi$, so $\gamma = \pi/2$.

The following lemma identifies those relatively few rational multiples of π that have rational tangents or whose sine and cosine satisfy a polynomial of low degree over $\mathbb{Q}$.

Lemma 2 *Let $\zeta = e^{i\theta}$ be algebraic of degree d over $\mathbb{Q}$, where θ is a rational multiple of π, say $\theta = 2m\pi/n$, where m and n have no common factor.*

Then $d = \varphi(n)$, where φ is the Euler totient function. In particular if $d = 4$, which is the case when $\tan\theta$ is rational and $\sin\theta$ is not, then n is 5, 8, 10, or 12; and if $d = 8$ then n is 15, 16, 20, 24, or 30.

Remark. For example, if $\theta = \pi/6$, we have $\sin\theta = 1/2$, which is of degree 1 over $\mathbb{Q}$. Since $\cos\theta = \sqrt{3}/2$, the number $\zeta = e^{i\theta}$ is in $\mathbb{Q}(i, \sqrt{3})$, which is of degree 4 over $\mathbb{Q}$. The number ζ is a 12-th root of unity, i.e. n in the theorem is 12 in this case; so the minimal polynomial of ζ is of degree $\varphi(12) = 4$. This example shows that the theorem is best possible.

Remark. The hypothesis that θ is a rational multiple of π cannot be dropped. For example, $x^4 - 2x^3 + x^2 - 2x + 1$ has two roots on the unit circle and two off the unit circle.

Proof. Let f be a polynomial with rational coefficients of degree d satisfied by ζ. Since $\zeta = e^{i2m\pi/n}$, ζ is an n-th root of unity, so its minimal polynomial has degree $d = \varphi(n)$, where φ is the Euler totient function. Therefore $\varphi(n) \le d$. If $\tan\theta$ is rational and $\sin\theta$ is not, then $\sin\theta$ has degree 2 over $\mathbb{Q}$, so ζ has degree 2 over $Q(i)$, so ζ has degree 4 over $\mathbb{Q}$. The stated values of n for the cases $d = 4$ and $d = 8$ follow from the well-known formula for $\varphi(n)$. That completes the proof of (ii) assuming (i).

Corollary 1 *If $\sin\theta$ or $\cos\theta$ is rational, and $\theta < \pi$ is a rational multiple of π, then θ is a multiple of $2\pi/n$ where n is 5, 4, 8, 10, or 12.*

Proof. Let $\zeta = \cos\theta + i\sin\theta = e^{i\theta}$. Under the stated hypotheses, the degree of $\mathbb{Q}(\zeta)$ over $\mathbb{Q}$ is 2 or 4. Hence, by the lemma, θ is a multiple of $2\pi/n$, where $n = 5$, 8, 10, or 12 (if the degree is 4) or $n = 3$ or 6 (if the degree is 3). But the cases 3 and 6 are superfluous, since then θ is already a multiple of $2\pi/12$.

5 Quadratic and non-quadratic tilings

In this section we give a simple sufficient condition for a tiling to be quadratic.

Lemma 3 *Suppose tile T strictly tiles triangle ABC. If the tile T is similar to the triangle ABC, and there are no angle relations, then the tiling is a quadratic tiling.*

Proof. Note that since there are no angle relations, the three angles α, β, and γ are pairwise unequal: for example, if $\alpha = \beta$, then the relation $\alpha + \beta + \gamma = \pi$ implies $2\alpha + \gamma = \pi$, which is an angle relation.

Since T is similar to triangle ABC, and angle A is the smallest angle of ABC, angle $A = \alpha$. Then consider the copy T_1 of the tile that shares vertex A. Its two sides lie on the sides of triangle ABC. We can relabel the vertices B and C if necessary so that the angle of T_1 at its vertex P_1 on side AB is β, and its angle at its vertex Q_1 on side AC is γ.

There must be exactly three copies of the tile meeting at P_1, and the three angles at P_1 are (in some order) α, β, and γ, because any other vertex behavior gives rise to an angle relation. Let the tiles meeting at P_1 be T_1, T_2, and T_3, numbered so that T_2 and T_1 share a side. That shared side is a, since it is opposite angle A in T_1. Then T_2 does not have angle α at P_1, since the α vertex of T_2 has to be opposite side P_1Q_1. T_2 does not have angle β at P_1, since T_1 has angle β there, and only one β can occur at P_1. Therefore T_2 has angle γ at P_1. Therefore T_3 has angle α at P_1. Since the tiling is strict, the angle of T_3 at its second vertex P_2 on side AB must be β; otherwise the shared sides of T_2 and T_3 will have different length, since the length of that side of T_2 is b. But now, we are in the same situation with T_2 as we originally were with T_1: the two angles along side AB are α and β (in that order). We can argue as before that the three triangles T_3, T_4, and T_5 meeting at P_2 have angles β,γ, and α at P_2, in that order. Continuing down side AB in this fashion, we eventually reach a tile T_{2m-1} that has B for a vertex; there will be m copies of the tile sharing a side with AB; there will be $m-1$ vertices $P, \ldots, P_{m-1}$ along AB, each shared by three triangles; the number of tiles used is $2m-1$. The third vertices of these triangles are points $Q_1, \ldots, Q_{m-1}$, lying on a line parallel to AB, and the last point Q_{m-1} lies on BC. The triangle Q_1CQ_{m-1} is thus tiled by the restriction of the original tiling to that triangle. This restricted tiling is still strict and has no angle relations. By induction, we can assume that this restricted tiling is quadratic. Since it has $m-1$ tiles along side Q_1Q_{m-1}, we have $(m-1)^2 = N - (2m-1)$. Then $N = (m-1)^2 + 2m - 1 = m^2$. That completes the proof.

Remark. The 5-tiling in Figure 1 has T similar to ABC, but it has an angle relation $2\alpha + 2\beta = \pi$, and it also has a non-strict vertex. It is natural to ask if the hypotheses of the lemma can be weakened by dropping one or the other of the hypotheses. Does there exist a strict non-quadratic tiling in which T is similar to ABC? (Angle relations are OK.) Does there exist a non-quadratic tiling with no angle relations in which T is similar to ABC? (Non-strict vertices are OK, but the angles meeting there would have to be exactly one each of α, β, and γ.) We do not know the answer to either of those questions. Note that if γ is a right angle we have an angle relation $2\gamma = \pi$.

6 The d-matrix, and a related eigenvalue problem

Let triangle ABC be tiled by the tile T, whose sides are a, b, and c. Let the sides of ABC be X, Y, and Z. We assume the triangle is labeled so that angles A, B, and C are listed in non-decreasing order; hence also $X \le Y \le Z$. In case triangle ABC is similar to the tile, this implies that angle $A = \alpha$, angle $B = \beta$, and angle $C = \gamma$.

Each side X, Y, and Z is a linear combination of a, b, and c, the coefficients specifying how many tiles share sides of length a, b, and c with X, Y, or Z. These nine numbers are the entries of the matrix $\mathbf{d}$, such that

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \mathbf{d} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

If the triangle ABC is similar to the tile, then we have

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \sqrt{N} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

because each side of ABC must be $\sqrt{N}$ times the corresponding side of the tile T, in order that the area of ABC can be N times the area of T. Therefore

$$\mathbf{d} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \sqrt{N} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

That is, $\sqrt{N}$ is an eigenvalue of $\mathbf{d}$, and (a, b, c) is an eigenvector for that eigenvalue. If triangle T is isosceles, then $\mathbf{d}$ is not (yet) uniquely defined. In that case we have either $a = b$ or $b = c$; our convention is to ignore b, so that when T is isosceles, the middle column of the $\mathbf{d}$ matrix is zero. We will not make use of the $\mathbf{d}$ matrix when T is equilateral, but for completeness, we define the $\mathbf{d}$ matrix in that case to have non-zero entries only in the first column. If T is not isosceles, then the coefficients in the $\mathbf{d}$ matrix are integers between 0 and $N - 1$, inclusive, assuming $N > 2$: Not all N triangles can share a side of triangle ABC, since if $N > 2$, there would be two adjacent vertices along that side at which only two triangles meet; but then by Lemma 1, the copy of the tile between those vertices would have two right angles.

For example, consider the 5-tiling shown in Figure 1. Here the shortest side of the large triangle consists of one c, so the top row of the $\mathbf{d}$ matrix is 0 0 1. The middle side of the large triangle consists of two c's, so the middle row of the $\mathbf{d}$ matrix is 0 0 2. The longest side of the large triangles consists of one a and two b's, so the bottom row is 1 2 0. Thus the $\mathbf{d}$ matrix for this example is

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$$

and the eigenvalue equation is

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \sqrt{5} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

In this example we have $\alpha = \pi/6$, $\beta = \pi/3$, and $\gamma = \pi/2$, so $a = \sin \pi/6 = 1/2$, $b = \sin \pi/3 = \sqrt{3}/2$, and $c = \sin \pi/2 = 1$. One can check the eigenvalue equation numerically with these values.

Note that the $\mathbf{d}$ matrix for a quadratic tiling is $\sqrt{N}$ times the identity. We conjecture that if N is a perfect square, say m^2, and $\mathbf{d}$ is m times the identity, then the tiling is quadratic.

7 Tilings with T similar to ABC

In this section, we assume triangle ABC is N-tiled by triangle T similar to ABC. In case N is a square, we have the quadratic tiling of ABC; in this section we assume N is not a square.

Let the sides of T be a, b, and c, in non-decreasing order; these are opposite the angles α, β, and γ of T. We start by disposing of a special case.

Lemma 4 *Suppose T and ABC are both equilateral, and there is an N-tiling of ABC by T. Then N is a square and the tiling is a quadratic tiling.*

Proof. Since all the angles of T and ABC are equal, and all the sides of T are equal, there is only one way to place tile T_1 at vertex B. Along side BC there must be a certain number m of copies of T; hence the side of ABC is mc, where c is the side of X. We prove by induction on m that such a tiling is a quadratic tiling using m^2 triangles. There are m tiles that share sides with BC. Call them $T_1, \ldots, T_m$. This sawtooth-like configuration requires the placement of $m-1$ copies of T, one between each adjacent pair of triangles $T_1, \ldots, T_m$. Now we have identified a total of $2m-1$ triangles that participate in the original tiling, and the remaining triangles tile the smaller equilateral triangle formed by deleting the tiles identified so far from ABC. The base of this triangle is smaller than the original base BC by c, the side of T. By the induction hypothesis, the tiling of this triangle is quadratic, using $(m-1)^2$ tiles. Combining this with the row of $2m-1$ triangles along BC, we have a quadratic tiling with a total of $(m-1)^2 + 2m - 1 = m^2$ tiles, completing the inductive proof.

Next we review the computation of eigenvectors by cofactors. To find an eigenvector of the $\mathbf{d}$ matrix with eigenvalue $\sqrt{N}$, consider the matrix $X := \mathbf{d} - \sqrt{N}I$. An eigenvector can be found by picking any row, and then arranging the cofactors of the elements of that row as a (column) vector. If these cofactors do not all vanish, then the result is an eigenvector. (The reader may either verify this or just check directly that the particular eigenvalues produced this way below are indeed eigenvectors.)

Now we take up the general case of a tiling T with ABC similar to T, when N is not a square.

Lemma 5 *Let triangle ABC be N-tiled by tile T similar to ABC, and suppose N is not a square. Then the diagonal entries of the* $\mathbf{d}$ *matrix are zero.*

Proof. Since the area of ABC is N times the area of T, and T is similar to ABC, the sides of ABC are $\sqrt{N}$ times a, b, and c. Then (as discussed in a previous section) we have the eigenvalue equation

$$\mathbf{d} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \sqrt{N} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

The characteristic polynomial $f(x)$ of the $\mathbf{d}$ matrix, the determinant of $\mathbf{d} - xI$, is a cubic polynomial with integer coefficients, yet has for a zero the number $\sqrt{N}$. This is only possible if it factors into a quadratic factor and a linear factor. Since N is not a square, the quadratic factor must be a multiple of $\lambda^2 - N$. The coefficient of x^3 is -1, and so for some q we have

$$f(x) = (x^2 - N)(q - x)$$

In general the coefficient of x^2 in the characteristic polynomial of any 3 by 3 matrix $\mathbf{d}$ is the trace of $\mathbf{d}$, and the constant term is the determinant of $\mathbf{d}$. Hence q is the trace of $\mathbf{d}$ and $-Nq$ is the determinant of $\mathbf{d}$. Since the entries of $\mathbf{d}$ are non-negative integers, the trace is non-negative, so $q \geq 0$.

To avoid so many subscripts, we use separate letters for the entries in the $\mathbf{d}$-matrix, writing it as

$$\mathbf{d} = \begin{pmatrix} p & d & e \\ g & m & f \\ h & \ell & r \end{pmatrix}$$

Since the similarity factor between ABC and T is $\sqrt{N}$, there cannot be more than $\sqrt{N}$ tiles with a sides along X, the short side of ABC. That is, $p \leq \sqrt{N}$. More formally, $a\sqrt{N} = X =$

$pa + db + ec \geq pa$, so $p \leq \sqrt{N}$. Similarly $m \leq \sqrt{N}$ and $r \leq \sqrt{N}$. Since N is not rational, we have strict inequalities: $p < \sqrt{N}$, $r < \sqrt{N}$, and $r < \sqrt{N}$. It follows that $pm < N$, etc.

We also note that there is just one tile sharing vertex A, where ABC has its α angle. That tile must have its b and c sides along AB and AC, or along AC and AB, we don't know which. Thus either f and ℓ are nonzero, or m and r are nonzero.

Suppose, for proof by contradiction, that q, the trace of $\mathbf{d}$, is not zero. Then $q = p+m+r > 0$. Since the three eigenvalues are distinct (because q is rational and $\sqrt{N}$ is not), the eigenspace corresponding to $\sqrt{N}$ is one-dimensional. The eigenvalue equation is

$$(\mathbf{d} - \lambda I)\begin{pmatrix} u \\ v \\ w \end{pmatrix} = 0$$

or showing the coefficients

$$\begin{pmatrix} p - \sqrt{N} & d & e \\ g & m - \sqrt{N} & f \\ h & \ell & r - \sqrt{N} \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = 0$$

We claim that there exists an eigenvector (u, v, w) whose components lie in $\mathbb{Q}(\sqrt{N})$. To prove this we will use the cofactor method described above.

The resulting eigenvector is (u, v, w), provided all three components are nonzero, where

$$\begin{aligned} u &= \begin{vmatrix} d & e \\ m - \sqrt{N} & f \end{vmatrix} \\ v &= -\begin{vmatrix} p - \sqrt{N} & e \\ g & f \end{vmatrix} \\ w &= \begin{vmatrix} p - \sqrt{N} & d \\ g & m - \sqrt{N} \end{vmatrix} \end{aligned}$$

Although we have not given a proof of the cofactor method's correctness, one can easily verify directly that the exhibited vector is indeed an eigenvector for $\sqrt{N}$; this also provides a check that no algebraic mistake has been made. The fact that all three cofactors are nonzero is really only needed to conclude directly that the eigenspace of (u, v, w) is one-dimensional; but we know that directly in our case since the eigenvalues $\sqrt{N}$, $-\sqrt{N}$, and q are distinct. It therefore suffices to check that one of the cofactors u, v, w is nonzero; then the others must automatically be nonzero because (u, v, w) is a nonzero multiple of (a, b, c). But we give direct proofs that all three cofactors are nonzero anyway, as it takes only one more paragraph.

We have $u = df - em + e\sqrt{N}$. If $u = 0$ then $e = 0$ and hence $df = 0$. If $v = 0$ then similarly $f = 0$ and $eg = 0$. Finally if $w = 0$ then $p + m = 0$ and hence $p = m = 0$, so $N = dg$.

Assume, for proof by contradiction, that $w = 0$. Then

$$\begin{aligned} (m - \sqrt{N})(p - \sqrt{N}) &= dg \\ mp + N - \sqrt{N}(p + m) &= dg \end{aligned}$$

Since $\sqrt{N}$ is irrational this means $p + m = 0$, and since p and m are nonnegative, that implies $p = 0$ and $m = 0$. Hence $dg = N$. But $d \leq (a/b)\sqrt{N}$, with equality implying that $p = d = 0$, and $g \leq (b/a)\sqrt{N}$ with equality implying $m = f = 0$. Since $dg = N$, equality must hold in both inequalities. Hence $d = (a/b)\sqrt{N}$ and $g = (b/a)\sqrt{N}$ and $p = e = m = f = 0$. But we showed above that either m and r are both nonzero, or f and ℓ are both nonzero. That is now contradicted by $m = f = 0$. This contradiction shows that $w \neq 0$.

Next we give the proof that $u \neq 0$; as remarked above, this is technically superfluous, but still it is interesting because the proof we give is not simply an abstract argument about projecting onto the one-dimensional eigenspace. Assume, for proof by contradiction, that $u = 0$. Then $df - em + e\sqrt{N} = 0$. Since $\sqrt{N}$ is not rational, $e = 0$ and $df - em = 0$. Then $df = 0$, so $d = 0$ or $f = 0$. If $d = 0$, then since both d and e are zero, side X of triangle ABC is composed of all a sides and $X = (p - \sqrt{N})a$. But since $\sqrt{N}$ is the similarity factor between T and ABC, we have $X = \sqrt{N}a$. Hence $p - \sqrt{N} = \sqrt{N}$. Hence $\sqrt{N} = p/2$, so $N^2 = p^2/4$. Hence $4N^2 = p^2$ and p is even, so N is a square, contradiction. This contradiction proves $d \neq 0$. Since $d = 0$ or $f = 0$, we have $f = 0$. Now assume, for proof by contradiction, that $g = 0$. Then since $f = 0$, side Y is composed entirely of b sides of tiles, so $Y = mb$. But $Y = b\sqrt{N}$ since $\sqrt{N}$ is the similarity factor between T and ABC. Hence $m = \sqrt{N}$, contradiction. That proves $g \neq 0$. As shown above, either f and ℓ are both nonzero or m and r are both nonzero. But $f = 0$. Hence both m and r are nonzero. Now

$$
\begin{aligned}
X &= pa + db \qquad \text{since } e = 0 \\
X &= \sqrt{N}a \\
(\sqrt{N} - p)a &= db \\
\frac{b}{a} &= \frac{\sqrt{N} - p}{d} \\
Y &= \sqrt{N}b \\
Y &= ga + mb \qquad \text{since } f = 0 \\
\frac{b}{a} &= \frac{g}{\sqrt{N} - m} \\
\frac{b}{a} &= \frac{\sqrt{N} - p}{d} = \frac{g}{\sqrt{N} - m}
\end{aligned}
$$

Cross-multiplying we have

$$
\begin{aligned}
dg &= N - (m + p)\sqrt{N} + mp \\
m + p &= 0 \qquad \text{as the coefficient of } \sqrt{N} \text{ must be zero} \\
m &= p = 0 \qquad \text{as } m \text{ and } p \text{ are nonnegative}
\end{aligned}
$$

But m was proved above to be nonzero. This contradiction completes the proof that $u \neq 0$.

Now assume $v = 0$. Then

$$pf - eg + f\sqrt{N} = 0$$

Since N is irrational we have $f = 0$ and $pf = eg$, but since $f = 0$ we have $eg = 0$. Hence either $e = 0$ or $g = 0$. Assume, for proof by contradiction, that $g = 0$. Then the middle side of ABC (corresponding to the middle row) is equal to mb but also to $b\sqrt{N}$, so $m = \sqrt{N}$, contradiction. This contradiction proves $g \neq 0$. Hence $e = 0$. Since either f and ℓ are both nonzero or m and r are both nonzero, and we have proved $f = 0$, then m and r are both nonzero. Now that we have $e = 0 = f$, and $m \neq 0$, we reach a contradiction by the same computation as in the case $u = 0$, shown in the series of displayed equations above. Hence $v \neq 0$.

Thus none of the three cofactors is zero. That completes the proof that there is an eigenvector (u, v, w) for the eigenvalue $\sqrt{N}$ with components in $\mathbb{Q}(\sqrt{N})$. Since the eigenspace is one-dimensional, this eigenvector is a (not necessarily rational) multiple of (a, b, c).

Recall that the third eigenvalue of the $\mathbf{d}$ matrix is the trace $q = p + m + r$. We can use the cofactor method to find an eigenvector for this eigenvalue as well, namely

$$
\begin{aligned}
V &= \left(\begin{vmatrix} d & e \\ m - q & f \end{vmatrix}, - \begin{vmatrix} p - q & e \\ g & f \end{vmatrix}, \begin{vmatrix} p - q & d \\ g & m - q \end{vmatrix} \right) \\
&= \begin{pmatrix} df - em + e(p + m + r) \\ -pf + eg + f(p + m + r) \\ pm - dg - (m + p)(m + p + r) + (m + p + r)^2 \end{pmatrix}
\end{aligned}
$$

$$= \begin{pmatrix} df + e(p+r) \\ eg + f(m+r) \\ pm - dg + r(m+p+r) \end{pmatrix}$$

Technically, it is not an eigenvector until we prove that the components are not zero, but we do not need that right now; it suffices that it satisfy the eigenvalue equation. The eigenvalue equation $\mathbf{d}V = (p+m+r)V$ is

$$\begin{pmatrix} p & d & e \\ g & m & f \\ h & \ell & r \end{pmatrix} \begin{pmatrix} df - e(p+r) \\ eg + f(m+r) \\ pm - dg + r(m+p+r) \end{pmatrix} = (p+m+r) \begin{pmatrix} df - e(p+r) \\ eg + f(m+r) \\ pm - dg + r(m+p+r) \end{pmatrix} \quad (1)$$

The first component of this vector equation is

$$p(df - e(p+r)) + d(eg + f(m+r)) + e(pm - dg + r(m+p+r) \quad = \quad (p+m+r)(df - e(p+r))$$

Multiplying out and cancelling like terms, and dividing by 2, we find

$$epm + er(m+p+r) = 0.$$

We argue by cases, according to whether $e = 0$ or not. We first take up the case that $e \neq 0$. Then $pm + r(m+p+r) = 0$. Since these terms are nonnegative, they are both zero. Hence $pm = 0$ and $r(m+p+r) = 0$. Hence $r = 0$ or $m+p+r = 0$. In either case $r = 0$. Writing out the third component of the eigenvalue equation, and setting $r = 0$, we have

$$\begin{aligned} h(df - ep) + \ell(eg + fm) &= (p+m)(pm - dg) \\ h(df - ep) + \ell(eg + fm) &= -(p+m)dg \qquad \text{since } pm = 0 \\ hdf - hep + \ell eg + \ell fm &= -pdg - mdg \end{aligned}$$

Now we write out the second component of the eigenvalue equation (1), setting $r = 0$:

$$\begin{aligned} g(df - ep) + m(eg + fm) + f(pm - dg) &= (p+m)(eg + fm) \\ gdf - gep + m(eg + fm) + fpm - fdg &= peg + pfm + m(eg + fm) \\ peg &= 0 \\ pg &= 0 \qquad \text{since } e \neq 0 \end{aligned}$$

Assume, for proof by contradiction, that $m \neq 0$. Then since $mp = 0$ we have $p = 0$. The equation $N(p+m) = hdg + \ell pf + \ell eg$ becomes

$$Nm = hdg + \ell eg.$$

The third component of the eigenvalue equation becomes, with $p = 0$,

$$hdf + \ell eg + \ell fm \quad = \quad -mdg$$

The left side is ≥ 0 and the right side is ≤ 0. Hence both sides are equal to zero. Since $m \neq 0$ and $e \neq 0$, we have $dg = 0$ and $hdf = 0$ and $\ell g = 0$ and $\ell f = 0$. We derived above (by observing that the b and c sides of the tile at vertex A lie on the two adjacent sides of ABC) that either f and ℓ are both nonzero or m and r are both nonzero. Since $r = 0$ we must have f and ℓ both nonzero. Hence $\ell f = 0$ is a contradiction. That contradiction completes the proof that $m = 0$.

Now assume, for proof by contradiction, that $p \neq 0$. Then since $pg = 0$ we have $g = 0$. Then the equation $N(p+m) = hdg + \ell pf + \ell eg$ becomes $Np = \ell pf$. Canceling p we have $N = \ell f$. But as proved above, $\ell f \leq N$, and equality holds if and only if AC is composed only of c sides of tiles and AB is composed only of b sides. Therefore we have $h = 0$ as well as $g = m = r = 0$. Then since $h = 0$ and $r = 0$, the long side AB of ABC is composed entirely of b sides of tiles.

If T is isosceles, then by convention the middle column of the **d**-matrix is zero. Since $\ell f = N$, we now have $\ell \neq 0$, so the middle column is not zero, and T is not isoceles.

Since side AB is composed entirely of b sides of tiles, there are equally spaced vertices $V_0 = A, V_1, \ldots, V_\ell$, spaced b apart, each one of which is one side of a tile T_i. Tile T_1, which has vertices at A and V_1, has its α angle at V_0. All these tiles have their β angles in the interior of ABC, and their α and γ angles at the V_i. If $\gamma > \pi/2$ then there is only one possible orientation for these tiles, as two γ angles will not fit at any V_i. In that case the angle of the last tile at vertex B must be γ, contradiction, since the angle there cannot exceed β, and $\beta \neq \gamma$ since then T would be isosceles. Hence $\gamma \leq \pi/2$.

In particular, the tile that shares vertex B has its b side along AB. Therefore the tile sharing vertex B and part of side AB has its α angle at B, and the angle β at vertex B splits into some number of α angles, so for some number J, we have $\beta = J\alpha$. Somewhere along AB there must occur a vertex V_k at which both the tile T_k and the tile T_{k+1} have angle γ. There is not room at V_k for a third tile, since $2\gamma + \alpha > \alpha + \beta + \gamma = \pi$. Hence there are exactly those two tiles at V_k, and we have $\gamma = \pi/2$.

Since γ is a right angle, we must have $a^2 + b^2 = c^2$. Since (u, v, w) is a multiple of (a, b, c) we also have $u^2 + v^2 = w^2$. We now compute these expressions from the formulas for (u, v, w). In view of $m = g = 0$ we have

$$\begin{aligned} u &= ef - e\sqrt{N} \\ v &= f\sqrt{N} - fp \\ w &= N - p\sqrt{N} \end{aligned}$$

Squaring these equations we have

$$\begin{aligned} u^2 &= e^2(f^2 + N - 2f\sqrt{N}) \\ v^2 &= f^2(N + p^2 - 2p\sqrt{N}) \\ w^2 &= N^2 + p^2N - 2pN\sqrt{N} \end{aligned}$$

Setting $u^2 + v^2 = w^2$ we find

$$e^2f^2 + e^2N + f^2N + f^2p^2 - 2(e^2f - f^2p)\sqrt{N} = N^2 + p^2N - 2pN\sqrt{N}$$

Equating the coefficients of $\sqrt{N}$ and equating the rational parts, we have

$$\begin{aligned} e^2f - f^2p &= pN \\ e^2f^2 + e^2N + f^2N + f^2p^2 &= N^2 + p^2N \end{aligned}$$

Since γ is a right angle, $\alpha + \beta = \pi/2$. Since $\beta = J\alpha$, we have $\alpha = \pi/(2(J+1))$, so α is a rational multiple of 2. We have $\tan\alpha = b/a = v/u$, which belongs to $\mathbb{Q}(\sqrt{N})$. We have

$$\cos\alpha = \frac{u}{u^2 + v^2}$$

which is also in $\mathbb{Q}(\sqrt{N})$. Similarly $\sin\alpha$ belongs to $\mathbb{Q}(\sqrt{N})$. Then $\zeta = d^{i\alpha}$ is of degree 4 over $\mathbb{Q}$, since $\mathbb{Q}(\zeta) = \mathbb{Q}(i, \sqrt{N})$. By Lemma 2, $4(J+1)$ is 5, 8, 10, or 12. Since 5 and 10 are not divisible by 4, we have $4(J+1) = 8$ or 10. But if $4(J+1) = 8$ then $J = 1$, while we have $J \geq 2$ since $\beta = J\alpha$. The only remaining possibility is $4(J+1) = 12$, which makes $J = 2$. Then $\alpha = \pi/6$ and $2\beta = \alpha$, so $\beta = \pi/3$. Then $a = \sin\alpha = 1/2$, $b = \sqrt{3}/2$ and $c = 1$. But now $\overline{AC} = fc = f$, and

$$\begin{aligned} \ell b &= \overline{AB} \\ &= \frac{2}{\sqrt{3}}\overline{AC} \\ &= \frac{2}{\sqrt{3}}fc \end{aligned}$$

Now we put in $c = 1$ and $b = \sqrt{3}/2$:

$$\begin{aligned} \ell\frac{\sqrt{3}}{2} &= \frac{2}{\sqrt{3}}f \\ 3\ell &= 4f \end{aligned}$$

We have $N = \ell f = (4/3)f^2$, so $3N = 4f^2$, so f is divisible by 3, say $f = 3k$; then $N = 3(2k)^2$ is three times a square.[1] It remains to show that $p = 0$; in fact we claim $p = 0$ and $e = 0$, so side AC is also composed entirely of b sides of triangles. We have

$$\begin{aligned} \overline{BC} &= \frac{1}{2}\overline{AB} \\ &= \frac{1}{2}\ell b \\ &= \frac{\ell\sqrt{3}}{4} \\ &= pa + db + ec \end{aligned}$$

Now we put in the values $a = 1/2$, $b = \sqrt{3}/2$, and $c = 1$.

$$\frac{\ell\sqrt{3}}{4} = \frac{p}{2} + d\frac{\sqrt{3}}{2} + e$$

This is an equation in $\mathbb{Q}(\sqrt{3})$. Equating the rational parts we have $0 = p/2 + e$. Since both $p/2$ and e are nonnegative, we have $p = 0$ and $e = 0$, as claimed. In particular $p = 0$ so the diagonal elements are nonzero, which is the conclusion of the theorem; or we could say, in particular $e = 0$, contradicting the assumption $e \neq 0$ and completing the analysis of that case.

Therefore we may now assume $e = 0$. Remember that $r = 0$ was derived only under the assumption $e \neq 0$, so the equation $r = 0$ is no longer in force. The third component of the eigenvalue equation (1) is (substituting $e = 0$)

$$hdf + \ell f(m + r) + r(pm - dg + r(m + p + r)) = (p + m + r)(pm + r(m + p + r))$$

Subtracting $r^2(m + p + r)$ from both sides we have

$$\begin{aligned} hdf + \ell fm + \ell fr + rpm - rdg &= (p + m + r)pm + (p + m)r(m + p + r) \\ hdf + \ell fm + \ell fr + rpm - rdg &= (p + m + r)(pm + pr + mr) \qquad (2) \end{aligned}$$

To get rid of h and ℓ, we expand the determinant of the $\mathbf{d}$ matrix by cofactors on the bottom row. That determinant is $-Nq = -N(p + m + r)$, so we have (remembering $e = 0$)

$$\begin{aligned} -N(p + m + r) &= h\begin{vmatrix} p & d \\ g & m \end{vmatrix} - \ell\begin{vmatrix} p & e \\ g & f \end{vmatrix} + r\begin{vmatrix} p & d \\ g & m \end{vmatrix} \\ &= hdf + \ell pf + rpm - rdg \end{aligned}$$

Adding and subtracting ℓpf to the left side of (2) the expression for the determinant appears, and we have

$$\begin{aligned} hdf + \ell pf + rpm - rdg + \ell fm + \ell fr - \ell pf &= (p + m + r)(pm + pr + mr) \\ -N(p + m + r) + \ell fm + \ell fr - \ell pf &= (p + m + r)(pm + pr + mr) \end{aligned}$$

Moving everything to the right side we have

$$0 = (p + m + r)(pm + pr + mr) + (N - \ell f)(m + r) + (N + \ell f)p$$

[1] Tilings of this kind actually exist, but with $e = 0$ (we are now in the case $e \neq 0$). (In the next draft I will add some figures to illustrate it.)

Since ℓ is the number of b sides of tiles on the long side $\sqrt{N}c$ of ABC, we have $\ell b \leq \sqrt{N}c$, or $\ell \leq (c/b)\sqrt{N}$. Since f is the number of c sides of tiles on AC, whose length is $\sqrt{N}b$, we have $fc \leq \sqrt{N}b$, or $f \leq (b/c)\sqrt{N}$. Hence

$$\begin{aligned} \ell f &\leq \left(\frac{c}{b}\sqrt{N}\right)\left(\frac{b}{c}\sqrt{N}\right) \\ \ell f &\leq N \end{aligned}$$

Hence all the terms on the right of the previous equation are nonnegative. Hence each of them is zero. In particular $(N + \ell f)p = 0$; but $N + \ell f > 0$, so $p = 0$. Then then equation becomes

$$(m + r)mr + (N - \ell f)(m + r) = 0$$

If $m + r = 0$ then $m = 0 = r$ and the lemma is proved. Hence we may assume $mr = 0$ and $N = \ell f$. But if $N = \ell f$ then we must have equality in the two inequalities $\ell \leq (c/b)\sqrt{N}$ and $f \leq (b/c)\sqrt{N}$. This implies that side AC is composed only of c sides of tiles and side AB is composed only of b sides of tiles, so $g = m = h = r = 0$. In particular $m = r = 0$. That completes the proof of the lemma.

We pause to observe that the **d** matrix for a biquadratic tiling, in case $N = m^2 + n^2$, has the form

$$\mathbf{d} = \begin{pmatrix} 0 & 0 & n \\ 0 & 0 & m \\ n & m & 0 \end{pmatrix}$$

which does satisfy the conditions above (as it must). The hypothesis that N is not a square is necessary, as shown by the 9-tiling in Figure 7. Its **d** matrix is

$$\begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

and as predicted, the determinant is zero, but the trace is not zero, and the characteristic polynomial is $-x(x - 3)^2$.

Continuing with the general case of N not a square, some further conclusions can be drawn about the **d** matrix. We have shown that $p = m = r = 0$. The determinant is then given by

$$\det \mathbf{d} = dfh + eg\ell$$

Since the matrix entries are nonnegative, that means that each of these two terms must contain a zero factor. In particular, at most four entries in the **d** matrix are nonzero.

The negated coefficient of λ in the characteristic equation is (since the diagonal elements are zero) the sum of paired products of off-diagonal elements:

$$N = dg + eh + fl \tag{3}$$

But at least one of these three terms will be zero, as shown above.

In view of the lemma, the **d** matrix becomes

$$\mathbf{d} = \begin{pmatrix} 0 & d & e \\ g & 0 & f \\ h & \ell & 0 \end{pmatrix} \tag{4}$$

and the matrix equation

$$\mathbf{d}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \sqrt{N}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

becomes the three equations

$$\begin{aligned} db + ec &= \sqrt{N}a \\ ga + fc &= \sqrt{N}b \\ ha + \ell b &= \sqrt{N}c \end{aligned}$$

Lemma 6 *Suppose ABC is N-tiled by tile T similar to ABC, and N is not a square. Then γ is a right angle.*

Proof. First we note that T and ABC are not equilateral, by Lemma 4. Next we will prove that T and ABC are not isosceles with $\beta = \gamma$. Assume, for proof by contradiction, that $\beta = \gamma$. Then, by our definition of the **d** matrix, the middle column of the **d** matrix is zero, i.e. b is counted as c. Then we have $d = \ell = 0$ and

$$\mathbf{d} = \begin{pmatrix} 0 & 0 & e \\ g & 0 & f \\ h & 0 & 0 \end{pmatrix} \tag{5}$$

That implies that the short side BC of triangle ABC has only c sides of tiles on it, and the long side AB has only a sides of tiles on it. At the vertex A, there can only be one tile, since the angle at A is the smallest angle α so there can be no vertex splitting. This tile has one side of length a opposite angle A and another along side AB. Hence $a = b$. Since T is not equilateral, we must have $b < c$ and $\beta < \gamma$. This contradicts the assumption that β is not less than γ, and thus completes the proof by contradiction that $\beta < \gamma$.

Since the **d** matrix has zeroes on the diagonal, no c sides of tiles occur along the longest side AB of triangle ABC; only a and b sides occur there. There are $\ell + h$ tiles along AB; for simplicity of notation, let $k := \ell + h$ and number those tiles $T_1, \ldots, T_k$ starting at vertex A. Let $V_1, \ldots, V_{k-1}$ be the vertices of those tiles on AB. Tile T_1 must have its α angle at vertex A, and since its c side is not on AB, it must have its γ angle at vertex V_1. Tile T_k cannot have a γ angle at B, since $\beta < \gamma$. Hence it has either its α or its β angle at B, and its γ angle at V_{k-1} (since its c side does not lie on AB). Continuing towards C from B with $T_2, T_3, \ldots$, and continuing towards B from C similarly, we must encounter an index j between 1 and $k-1$ such that T_j and T_{j+1} both have their γ angles at V_j. At that point we know $\gamma \le \pi/2$. Assume, for proof by contradiction, that $\gamma \neq \pi/2$. Then there is at least one additional copy T' of the tile between T_j and T_{j+1}, sharing vertex V_j, by Lemma 1. If T' has its γ angle at V_j then there are exactly those three tiles meeting at V_j (else $\gamma < \pi/3$) and we have $\gamma = \pi/3$, and hence T is equilateral, which as noted above is impossible. Hence none of the additional tiles T' meeting at V_j have a γ angle at V_j. None of the tiles T' can contribute a β angle at V_j either, since $2\gamma + \beta > \pi$. Hence there is an angle relation $2\gamma + p\alpha = \pi$, where p additional tiles contribute α each to the angle sum at V_j, and $p > 0$. But $2\gamma + p\alpha > \gamma + \beta + \alpha = \pi$, since $\gamma > \beta$. This contradiction completes the proof of the lemma.

Lemma 7 *Suppose ABC is N-tiled by tile T similar to ABC, and N is not a square. Then $f = \mathbf{d}_{12}$ is not zero.*

Proof. Suppose, for proof by contradiction, that $f = 0$. Then the middle row of the **d** matrix is $(g, 0, 0)$, which means that all the tiles along side AC of triangle ABC share their a sides with AC. At vertex A, where ABC has its smallest angle α, there is exactly one tile T_1, with angle α at a. Hence both the side of T_1 opposite that angle, and the side shared with AC, are equal to a. Thus T is isosceles. In that case, by convention we have agreed to write the **d** matrix with zeroes in the second column, so the **d** matrix has the form

$$\mathbf{d} = \begin{pmatrix} 0 & 0 & e \\ g & 0 & 0 \\ h & 0 & 0 \end{pmatrix} \tag{6}$$

Now the bottom row is $(h, 0, 0)$, which means that all the tiles along side AB share their a sides with AB. In particular the tile at vertex A has an a side along AB. But we have already seen that its other two sides are a. Hence the tile is equilateral, contradicting Lemma 4, since N is not a square. That completes the proof.

There are six letters for coefficients in the $\mathbf{d}$ matrix, but for any specific tiling, at most four of those coefficients are nonzero. We will analyze some special cases. The case corresponding to the biquadratic tilings is $d = 0$ and $g = 0$. We call that the "biquadratic case". In the biquadratic case the $\mathbf{d}$ matrix has the form

$$\mathbf{d} = \begin{pmatrix} 0 & 0 & e \\ 0 & 0 & f \\ h & \ell & 0 \end{pmatrix} \tag{7}$$

Equation (3) now becomes

$$eh + \ell f \quad = \quad N \tag{8}$$

We compute the eigenvector in the biquadratic case, using the cofactor method described above. Let

$$X = \begin{pmatrix} -\sqrt{N} & 0 & e \\ 0 & -\sqrt{N} & f \\ h & \ell & -\sqrt{N} \end{pmatrix}$$

Taking the cofactors of the bottom row (notice the minus sign in the second component, which comes from the definition of "cofactor") we find the eigenvector

$$\left(\begin{vmatrix} 0 & e \\ -\sqrt{N} & f \end{vmatrix}, - \begin{vmatrix} -\sqrt{N} & e \\ 0 & f \end{vmatrix}, \begin{vmatrix} -\sqrt{N} & 0 \\ 0 & -\sqrt{N} \end{vmatrix} \right) \quad = \quad \begin{pmatrix} e\sqrt{N} \\ f\sqrt{N} \\ N \end{pmatrix}$$

Note that $e \neq 0$ and $f \neq 0$, since the first two sides of ABC are given by ec and fc. Hence the cofactors do not vanish.

We claim that the bottom two rows of $\mathbf{d} - \sqrt{N}I$, namely $(0, -\sqrt{N}, f)$ and $(h, \ell, -\sqrt{N})$, are linearly independent. Indeed, suppose that for some constants p and q we have $p(0, -\sqrt{N}, f) + q(h, \ell, -\sqrt{N}) = 0$. From the first component we see that $qh = 0$. From the third component we see that $pf = q\sqrt{N}$. If q is not zero, then $\sqrt{N} = pf/q$, contradicting the irrationality of $\sqrt{N}$. Hence $q = 0$. Hence from the second component, $p\sqrt{N} = 0$. Hence $p = 0$. This proves that the bottom two rows of $\mathbf{d} - \sqrt{N}I$ are linearly independent. Hence $\mathbf{d} - \sqrt{N}I$ has rank 2; hence the eigenspace associated with the eigenvalue $\sqrt{N}$ is one-dimensional. It follows that the eigenvector computed above is a multiple of (a, b, c). That is, for some constant μ we have

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad = \quad \mu \begin{pmatrix} e\sqrt{N} \\ f\sqrt{N} \\ N \end{pmatrix} \tag{9}$$

The constant μ is an arbitrary scale factor; changing μ just changes the size of the tile T and the triangle ABC by the same factor. We are therefore free to choose μ to suit our convenience. We choose to take $\mu = \sqrt{N}$; then we have

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad = \quad \begin{pmatrix} e \\ f \\ \sqrt{N} \end{pmatrix} \tag{10}$$

Lemma 8 *Let triangle ABC be N-tiled by T, and suppose N is not a square and T is similar to ABC, and $d = g = 0$ (the biquadratic case). Then N is a sum of squares, specifically $N = e^2 + f^2$ where e and f are as above, and $\tan\alpha = e/f$. In particular $\tan\alpha$ is rational.*

Proof. By Lemma 6, we have $\gamma = \pi/2$. By the Pythagorean theorem and (10) we see that $\gamma = \pi/2$ if and only if $e^2 + f^2 = N$. Since $\gamma = \pi/2$, we have $\tan\alpha = a/b$, and by (9), $\tan\alpha = e/f$. That completes the proof of the lemma.

Lemma 9 *Suppose ABC is N-tiled by tile T similar to ABC, and N is not a square, and $d = g = 0$ (the biquadratic case). Then the right angle of ABC is split by the tiling, and the tangents of the other angles of ABC are rational.*

Proof. We suppose, as always, that the γ angle of ABC is at C, the β angle at B, and the α angle at A. Since the $\mathbf{d}$ matrix has the form given in (7), all the tiles along side BC share their c sides with BC (there are e of them) and all the tiles along side AC share their c sides with AC (there are f of them). Suppose, for proof by contradiction, that the vertex at C is not split. Then a single tile shares vertex C, so the tile has two c sides, and hence is isosceles with $b = c$. But by (10), we have $c/b = \sqrt{N}/e$. Hence if $b = c$ we have $N = e^2$, contradicting the hypothesis that N is not a square. Hence the vertex C is split as claimed. The tangents of the other two angles are e/f and f/e, which are rational. This completes the proof of the lemma.

We now turn to another important case, when $e = 0$. We call this the "triple-square case", because it will turn out that in this case N must be three times a square. The following lemma and its proof give a complete analysis of this case.

Lemma 10 *Suppose ABC is not equilateral and is N-tiled by tile T similar to ABC, and N is not a square, and $e = 0$ (the triple-square case). Then $\alpha = \pi/6$, $\beta = \pi/3$, and $N = 3d^2$ is three times a square.*

Remark. There do exist tilings for each N of the form $3d^2$ that fall under the triple square case, as we showed in Figures 9 and 10.

Proof. Under the hypotheses of the lemma we have

$$\mathbf{d} = \begin{pmatrix} 0 & d & 0 \\ g & 0 & f \\ h & \ell & 0 \end{pmatrix}.$$

In this matrix, d and $h + \ell$ are not zero, since they represent the number of tiles along BC and AB, respectively. By Lemma 7 we have $f \neq 0$. We have

$$X = \mathbf{d} - \sqrt{N}I = \begin{pmatrix} -\sqrt{N} & d & 0 \\ g & -\sqrt{N} & f \\ h & \ell & -\sqrt{N} \end{pmatrix}$$

We will prove that the bottom two rows of X are linearly independent. If they are linearly dependent, then for some p and q, we have

$$\begin{aligned} 0 &= p(g, -\sqrt{N}, f) + q(h, \ell, \sqrt{N}) \\ &= pg + pf + q\ell + qh + \sqrt{N}(q - p) \end{aligned}$$

Since N is not a square, the coefficient of $\sqrt{N}$ is zero, so $q = p$, and

$$\begin{aligned} 0 &= pg + pf + q\ell + qh \\ &= pg + pf + p\ell + ph \\ &= p(g + f + \ell + h) \end{aligned}$$

Since the entries of the $\mathbf{d}$ matrix are non-negative, and $h + \ell$ is strictly positive, we conclude $p = q = 0$. That proves that the bottom two rows of X are linearly independent, so X has rank 2 and the eigenspace of $\sqrt{N}$ is one-dimensional. We then compute the eigenvector by the cofactor method. Taking the cofactors of the bottom row, we find the eigenvector

$$\left(\begin{vmatrix} 0 & e \\ -\sqrt{N} & f \end{vmatrix}, -\begin{vmatrix} -\sqrt{N} & 0 \\ g & f \end{vmatrix}, \begin{vmatrix} -\sqrt{N} & d \\ g & -\sqrt{N} \end{vmatrix} \right) = \begin{pmatrix} df \\ f\sqrt{N} \\ N - dg \end{pmatrix}$$

Since (a, b, c) is an eigenvector and the eigenspace is one dimensional, (a, b, c) is a multiple of this computed eigenvector. By scaling the triangle appropriately we can assume (a, b, c) is actually equal to the computed eigenvector:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} df \\ f\sqrt{N} \\ N - dg \end{pmatrix}$$

It follows that $\sin\alpha = a/c = fd/(N - dg)$ is rational and $\tan\alpha = a/b = d/\sqrt{N}$ is of degree 2 over $\mathbb{Q}$.

According to the first row of the $\mathbf{d}$ matrix, the tiles along BC have only b sides on BC. Assume, for proof by contradiction, that vertex B does not split. Then there is a single tile T_1 at vertex B, which therefore shares one side with AB and one side with BC. Triangle T is not isosceles, since then by definition the $\mathbf{d}$ matrix would have zeroes in the middle column. Hence the unique b side of T_1 must be opposite angle B; but it must also lie on BC, which is a contradiction. Hence vertex B does split. Therefore for some integer P we have $\beta = P\alpha$. Since by Lemma 6, $\gamma = \pi/2$,we have

$$\begin{aligned} \frac{\pi}{2} &= \alpha + \beta \\ &= \alpha + P\alpha \\ &= (P + 1)\alpha \end{aligned}$$

Therefore

$$\alpha = \frac{\pi}{2(P+1)} = \frac{2\pi}{4(P+1)}.$$

By Lemma 2 we conclude that $4(P + 1)$ is one of the numbers $n = 3, 4, 5, 8, 10$, or 12 for which $\phi(n) = 4$. Of these numbers, only 4, 1, and 12 are divisible by 4, which implies $P = 2$, since the values $P = 0$ and $P = 1$ do not correspond to vertex splitting. Hence $P = 2$ and we have $\beta = 2\alpha$, so $\alpha + \beta = \pi/2 = 3\alpha$, and $\alpha = \pi/6$. Hence $\tan\alpha = df/(f\sqrt{N}) = d/\sqrt{N} = 1\sqrt{3}$. Hence $N = 3d^2$. That completes the proof of the lemma.

Now we have dealt with the biquadratic case (when $d = g = 0$) and the triple-square case (when $e = 0$). It remains to show that these are the only two possible cases, when N is not a square and T is similar to ABC. Recall that $dfh = 0$ and $eg\ell = 0$ since $\det \mathbf{d} = 0$; that leaves only a few possibilities to consider. We begin by showing that if $d = 0$ then we are already in the biquadratic case.

Lemma 11 *Assume triangle ABC is N-tiled by T, that N is not a square, that T is similar to ABC, and that d and g are two entries in the* $\mathbf{d}$ *matrix of the tiling, in the notation used above (the ones that are zero in the biquadratic case). Then $d = 0$ implies $g = 0$, i.e. we are in the biquadratic case as soon as $d = 0$.*

Proof. We have

$$X = \mathbf{d} - \sqrt{N}I = \begin{pmatrix} -\sqrt{N} & d & e \\ g & -\sqrt{N} & f \\ h & l & -\sqrt{N} \end{pmatrix}$$

By the cofactor method described above we compute the eigenvector

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} df + e\sqrt{N} \\ f\sqrt{N} + eg \\ N - dg \end{pmatrix}$$

Suppose, for proof by contradiction, that $d = 0$ and $g \neq 0$. Then

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} e\sqrt{N} \\ f\sqrt{N} + eg \\ N \end{pmatrix}$$

By Lemma 6, γ is a right angle, so by the Pythagorean theorem, we have $a^2 + b^2 = c^2$. That is,

$$\begin{aligned} e^2N + (f\sqrt{N} + eg)^2 &= N^2 \\ e^2N + f^2N + 2egf\sqrt{N} + e^2g^2 &= N^2 \end{aligned}$$

Since N is not a square, the coefficient of $\sqrt{N}$ is zero; that is, $egf = 0$. By hypothesis, $g \neq 0$, so $ef = 0$. The first row of the $\mathbf{d}$ matrix is $(0, d, e) = (0, 0, e)$, so $e \neq 0$ because there must be some triangles on the first side of ABC. Therefore $f = 0$. Then the $\mathbf{d}$ matrix is

$$\begin{pmatrix} 0 & 0 & e \\ g & 0 & 0 \\ h & \ell & 0 \end{pmatrix}$$

Hence all the tiles on the middle side AC of ABC have their a side on AC, and all the tiles on the hypotenuse AB do not have their c side on AB. Consider the tile T_1 sharing vertex A (there is only one, since ABC has angle α there). It has its a side on AC and does not have its c side on AB. Hence its b side is on AB and its c side opposite angle A, which is α. Hence $a = c$ and triangles T and ABC are equilateral, which is a contradiction since $\gamma = \pi/2$. This contradiction shows that the assumption $d = 0$ and $g \neq 0$ is untenable, which completes the proof of the lemma.

Lemma 12 *Assume triangle ABC is N-tiled by T, that N is not a square, that T is similar to ABC, and that $d \neq 0$. Then $e = 0$, i.e. we are in the triple-square case as soon as $d \neq 0$.*

Proof. We have as in the proof of the previous lemma

$$\mathbf{d} = \begin{pmatrix} 0 & d & e \\ g & 0 & f \\ h & \ell & 0 \end{pmatrix}$$

$$X = \mathbf{d} - \sqrt{N}I = \begin{pmatrix} -\sqrt{N} & d & e \\ g & -\sqrt{N} & f \\ h & \ell & -\sqrt{N} \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} df + e\sqrt{N} \\ eg + f\sqrt{N} \\ N - dg \end{pmatrix}$$

By Lemma 6 and the Pythagorean theorem we have

$$\begin{aligned} 0 &= c^2 - a^2 - b^2 \\ &= (N - dg)^2 - (df + e\sqrt{N})^2 - (eg + f\sqrt{N})^2 \\ &= -2(def + egf)\sqrt{N} + \text{ rational} \end{aligned}$$

Since N is not a square and the entries of the $\mathbf{d}$ matrix are nonnegative integers, we have $def = 0$ and $egf = 0$. Since $f \neq 0$ by Lemma 7, and $d \neq 0$ by hypothesis, we have $e = 0$. That completes the proof of the lemma.

8 The main theorem

The following theorem completely answers the question, "for which N does there exist an N-tiling in which the tile is similar to the tiled triangle?" Namely, N is a square, or a sum of two squares, or three times a square.

Theorem 1 *Suppose ABC is N-tiled by tile T similar to ABC. Then either N is a square, or T is a right triangle and exactly one of the following holds:*

(i) T is a right triangle, and N is a sum of two squares, specifically $N = e^2 + f^2$, where $\tan \alpha = e/f$, or

(ii) N is three times a square and T is a 30-60-90 right triangle, i.e. $\alpha = \pi/6$.

Proof. Suppose N is not a square. Then by Lemma 6, γ is a right angle. Now consider the **d** matrix. By Lemma 5 the diagonal entries are zero, so as stated in (4) the **d** matrix has the form

$$\mathbf{d} = \begin{pmatrix} 0 & d & e \\ g & 0 & f \\ h & \ell & 0 \end{pmatrix}$$

By Lemma 11, if $d = 0$ then also $g = 0$, i.e. we are in the "biquadratic case". Then by Lemma 8, $N = e^2 + f^2$ f and $\tan \alpha = e/f$. If $e = 0$ then by Lemma 10, N is three times a square and T is a 30-60-90 triangle. Finally, Lemma 12 shows that the cases $d = 0$ and $e = 0$ are exhaustive. That completes the proof that at least one of the two given alternatives holds.

We now will prove that N cannot be both a sum of squares and three times a square, since the equation $x^2 + y^2 = 3z^2$ has no integer solutions. To see that, we can assume without loss of generality that x, y, and z are not all even. Note that squares are always congruent to 0 or 1 mod 4, so the left side is 0, 1, or 2 mod 4. Then z^2 must be congruent to 0 mod 4, since if not, the right side is congruent to 3 mod 4. Hence z is even. But x and y must also be even to make the left side congruent to 0 mod 4, contradiction. Hence the equation has no solutions. Thus the alternatives in the theorem are mutually exclusive, as claimed. That completes the proof of the theorem.

9 Conjectures

While the theorem characterizes the possible N for which triangle ABC can be N-tiled by tiles similar to ABC, it does not completely characterize the possible tilings themselves. Note that the 9-tiling in Figure 7 shows that not every m^2-tiling is a quadratic tiling, so we have not classified all the m^2-tilings. Briefly we conjectured that a tiling in which the **d**-matrix is m times the identity should be a quadratic tiling, but even that is not true. One can extend the 9-tiling in Figure 7 by adding more triangles to the right and below, producing a 25-tiling in which the **d**-matrix is 5 times the identity. We did show, in Lemma 9, that if N is not a square, then the right angle of ABC is split by the tiling.

We conjecture that if ABC is not a right triangle, then the only tilings of ABC by a tile similar to ABC are the quadratic tilings. In case ABC is a right triangle, the possibility arises that two tiles form a rectangle with diagonal. One can then erase that diagonal and draw the other diagonal, creating another tiling. Similarly, as in the 9-tiling in Figure 7, several tiles can form a square, and the square can be rotated. Call two tilings "immediately equivalent" if one is obtained from the other by re-drawing a diagonal or rotating a square. Let the relation of "equivalence" between tilings be the transitive closure of immediate equivalence. In other words, two tilings are equivalent if one can be obtained from the other by a finite number of immediate equivalences. Then we conjecture that, for any triangle ABC, every tiling of ABC by tiles congruent to ABC is equivalent to a quadratic tiling.

References

[1] Beeson, M., Tiling a triangle with congruent triangles, in progress.

[2] Boltyanskii, V. G., and Gohberg, I. T. *The decomposition of figures into smaller parts*, translated and adapted from the Russian edition by Henry Christoffers and Thomas P. Branson.

[3] Boltyanskii, V. G. *Equivalent and Equidecomposable Figures*, Translated and adapted from the 1st Russian ed. (1956) by Alfred K. Henn and Charles E. Watts. D. C. Heath (1963).

[4] Borevich, Z. I., and Shafarevich, I. R., *Number Theory*, Academic Press, New York and London (1966).

[5] Cohen, Henri, *Number Theory, Volume 1: Tools and Diophantine Equations*, Springer (2007).

[6] Goldberg, M., and Stewart, B. M., A dissection problem for sets of polygons, *Amer. Math. Monthly* **71** (1964) 1077–1095.

[7] Golomb, Solomon W., Replicating figures in the plane, *The Mathematical Gazette***48** No. 366. (Dec., 1964), pp. 403-412.

[8] Soifer, Alexander, *How Does One Cut a Triangle?*, second edition, Springer (2009).