TILING AN EQUILATERAL TRIANGLE

MICHAEL BEESON

Abstract. Let ABC be an equilateral triangle. For certain triangles T (the "tile") and certain N, it is possible to cut ABC into N copies of T. It is known that only certain shapes of T are possible, but until now very little was known about the possible values of N. Here we prove that for $N > 3$, N cannot be prime.

1. Introduction

The subject of this paper is N-tilings of the equilateral triangle. More generally, triangle ABC is said to be N-tiled by a triangle (the "tile") with angles (α, β, γ), if ABC can be cut into N smaller triangles congruent to the tile. In this paper, we restrict attention to the case of ABC equilateral. A few pictures of N-tilings of equilateral triangles are given in the figures.

Figure 1. A 3-tiling, a 6-tiling, and a 16-tiling

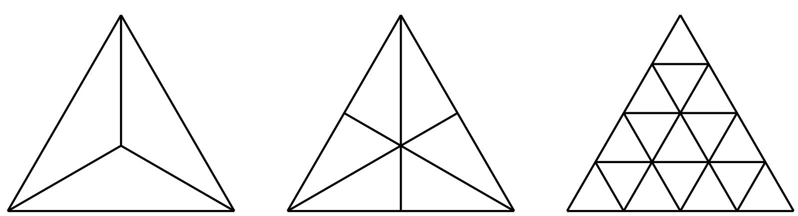


Figure 2. A 27-tiling due to Major MacMahon 1921, rediscovered 2011

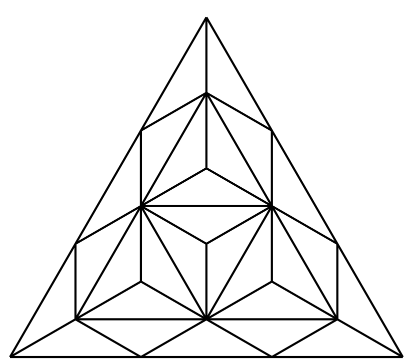


FIGURE 3. $3m^2$ (hexagonal) tilings for $m = 4$ and $m = 5$

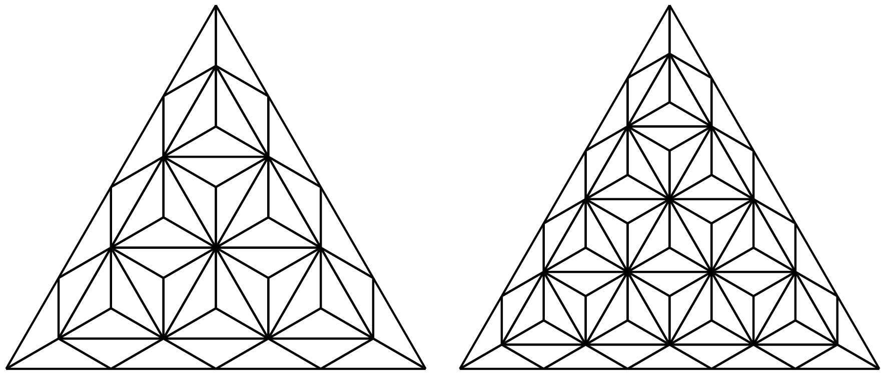
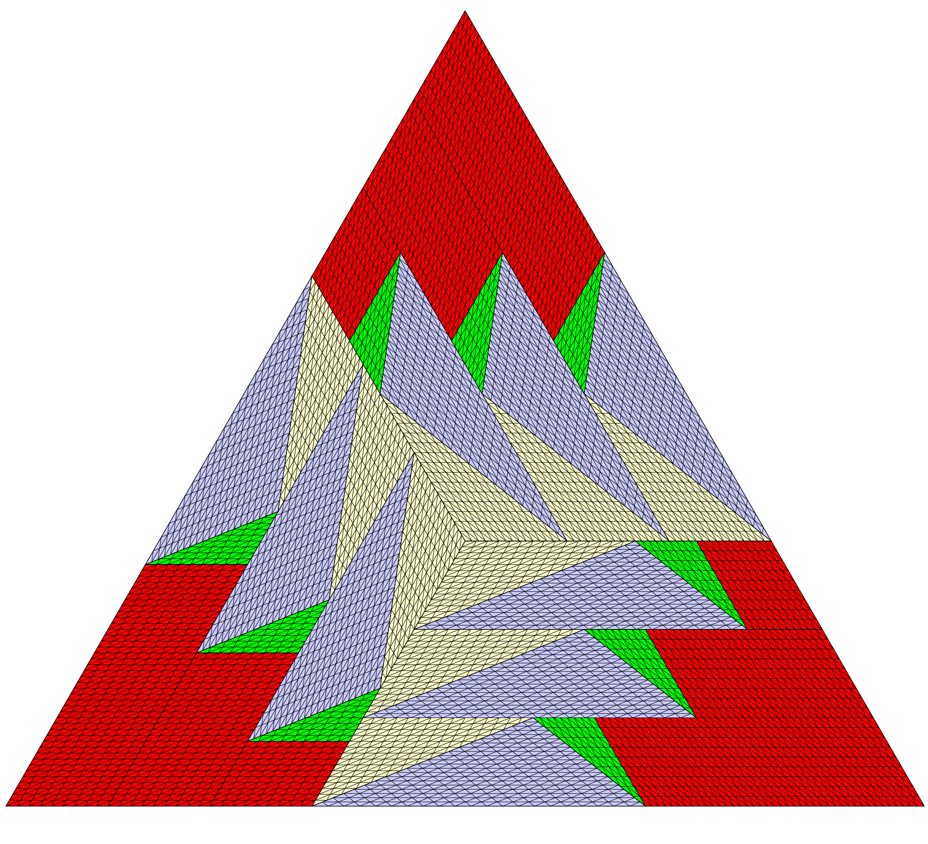


FIGURE 4. $N = 10935$. The tile is $(3, 5, 7)$.



Date: February 24, 2019.

Miklós Laczkovich wrote many papers on the subject of tilings, considering not only triangles but convex polygons, and considering not only tilings by congruent tiles, but by similar tiles as well. In [2] and [3] (especially Theorem 3.3 of [3]) he narrowed the list of possible tiles that can be used to tile some equilateral triangle. He proved that the tile (α, β, γ) (in some order) must be one of the following

(i) $(\pi/3, \pi/3, \pi/3)$ (equilateral)
(ii) $(\pi/6, \pi/6, 2\pi/3)$
(iii) $(\pi/6, \pi/2, \pi/3)$
(iv) $(\alpha, \beta, \pi/3)$ with α not a rational multiple of π
(v) $(\alpha, \beta, 2\pi/3)$ with α not a rational multiple of π

The possible N in the first three cases are known. In case (i), N must be a square, and any square corresponds to a tiling. (That is true for any triangle ABC and a tile similar to ABC.) For a proof see [4] or [5]. In cases (ii) and (iii), N must have the form $3n^2$ or $6n^2$ respectively [1]. None of these can be prime except for case (ii), when $N = 3$ is possible. That leaves the last two cases as the focus of this paper.

It may come as a surprise to the reader that tilings falling under cases (iv) and (v) do exist. In 1995, Laczkovich already gave a method for constructing a tiling of *some* equilateral triangle from any tile satisfying (iv) or (v), but the N involved can be quite large. The smallest one I have been able to construct has $\gamma = 2\pi/3$ and $N = 10935$; see Fig. 4. In [1], it is shown that N must be at least 12. There is a big gap between 12 and 10935. For tilings when the tile has a $2\pi/3$ angle, the smallest one I have been able to calculate has more than five million tiles, so it cannot be drawn; already one million tiles is too many to make a nice picture, unless you have the side of a large building available to display it.

In this paper, we prove that if an equilateral triangle is N-tiled according to cases (iv) or (v) above, then N is not a prime. Laczkovich proved in Theorem 3.3 of [3] that in those two cases, the tile is rational; that is, the ratios of the sides of the tile are rational. This gives us a good starting point. On that basis, we proceed in case (iv) by elementary, if somewhat intricate, algebraic computations, arriving at some equations that must be satisfied by N. While we could not reduce these equations to an elegant number-theoretical criterion, at least we can show that N cannot be prime. The algebraic calculations have been performed or checked using the computer algebra system SageMath [6].

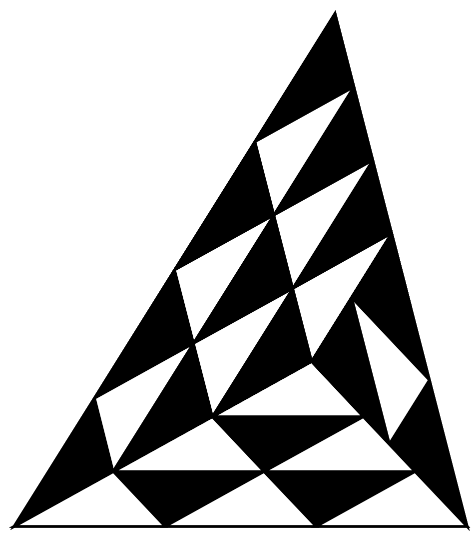
To deal with case (v), when the tile has a $2\pi/3$ angle, we make use of a beautiful method introduced by Laczkovich in his 2012 paper [3]. In this paper, we give the required definitions and state the two lemmas we need, but for the proofs of those lemmas, we refer to the cited paper.

The result of this paper, that equilateral triangles cannot be N-tiled when N is a prime larger than 3, answers a question posed in [5]. Namely, until now it was not known whether there are arbitrarily large N such that no equilateral triangle can be N-tiled. Now, in view of Euclid's theorem that there are infinitely many primes, we know the answer.

2. The coloring equation

In this section we introduce a tool that is useful for some, but not all, tiling problems. Suppose that triangle ABC is tiled by a tile with angles (α, β, γ) and sides (a, b, c), and suppose there is just one tile at vertex A. We color that tile black, and then we color each tile black or white, changing colors as we cross tile boundaries. Under certain conditions this coloring can be defined unambiguously, and then, we define the "coloring number" to be the number of black tiles minus the number of white tiles. An example of such a coloring is given in Fig. 5.

FIGURE 5. A tiling colored so that touching tiles have different colors.



The following theorem spells out the conditions under which this can be done. In the theorem, "boundary vertex" refers to a vertex that lies on the boundary of ABC or on an edge of another tile, so that the sum of the angles of tiles at that vertex is π. At an "interior vertex" the sum of the angles is 2π.

Theorem 1. *Suppose that triangle ABC is tiled by the tile (a, b, c) in such a way that*

(i) There is just one tile at A.

(ii) At every boundary vertex an odd number of tiles meet.

(iii) At every interior vertex an even number of tiles meet.

(iv) The numbers of tiles at B and C are both even, or both odd.

Then every tile can be assigned a color (black or white) in such a way that colors change across tile boundaries, and the tile at A is black. Let M be the number of black tiles minus the number of white tiles. Then the coloring equation

$$X \pm Y + Z = M(a + b + c)$$

holds, where Y is the side of ABC opposite A, and X and Z are the other two sides. The sign is $+$ or $-$ according as the number of tiles at B and C is odd or even.

Proof. Each tile is colored black or white according as the number of tile boundaries crossed in reaching it from A without passing through a vertex is even or odd. The hypotheses of the theorem guarantee that color so defined is independent of the path chosen to reach the tile from A. The total length of black edges, minus the total length of white edges, is $M(a+b+c)$, since $a+b+c$ is the perimeter of each tile. Each interior edge makes a contribution of zero to this sum, since it is black on one side and white on the other. Therefore only the edges on the boundary of ABC contribute. Now sides X and Y contain only edges of black tiles, by hypotheses (i) and (ii). Side Y is also black if the number of tiles at B and C is odd, and white if it is even. Hence the difference in the total length of black and white tiles is $X \pm Y + Z$, with the sign determined as described. That completes the proof.

3. A tile with an angle $\gamma = \pi/3$ and α/π irrational

In this case we have $\alpha + \beta = 2\pi/3$. Then the possible ways to write π and 2π as an integer linear combination of (α, β, γ) are these:

$$\begin{aligned} \pi &= \alpha + \beta + \gamma \\ 2\pi &= 2\alpha + 2\beta + 2\gamma \\ 2\pi &= 6\gamma \\ 2\pi &= \alpha + \beta + 4\gamma \\ 2\pi &= 2\alpha + 2\beta + 2\gamma \end{aligned}$$

Hence every vertex with total angle π has an odd number of tiles sharing that vertex and every vertex with total angle 2π has an even number of tiles sharing that vertex. Moreover at each vertex of ABC there is just one tile, with its γ angle at the vertex. Thus the coloring theorem, Theorem 1, applies. It tells that, when the tiles are colored black and white with the vertex at A black, if M is the number of black tiles minus the number of white tiles,

$$M(a+b+c) = 3X \tag{1}$$

where X is the length of each side of ABC.

Our second tool is the area equation, obtained by equating the area of ABC to N times the area of the tile. Twice the area of ABC is $X^2 \sin\gamma$, and twice the area of the tile is $ab \sin\gamma$, so the area equation is

$$X^2 = Nab \tag{2}$$

These tools enable us to formulate a necessary condition for such a tiling to exist, and characterize the tile, i.e., compute the tile from N and the coloring number of the tiling.

Theorem 2. *Let triangle ABC be equilateral. Suppose it is N-tiled using a tile with angles $(\alpha, \beta, \frac{\pi}{3})$, where β is not a rational multiple of π. Let M be the coloring number of the tiling. Then*

(i) $\zeta = e^{i\alpha}$ satisfies a quadratic equation with coefficients (involving N and M) in $\mathbb{Q}(i\sqrt{3})$.

(ii) N and M determines the shape of the tile uniquely; that is, there are algebraic formulas for (b/a) and (c/a) in terms of N and M. Specifically, a/c and b/c are given by

$$\frac{1}{2}\left(\frac{3N+M^2}{3N-M^2} \pm \frac{\sqrt{(9N-M^2)(N-M^2)}}{3N-M^2}\right)$$

(iii) $M^2 < N$

FIGURE 6. SageMath code to derive (3)

```
var('p,q,r,N,M,x')
c = sqrt(3)/2
a = (x-x^(-1))/(2*i)
b = (sqrt(3)/2)* (x+x^(-1))/2 + (1/2)*(x-x^(-1))/(2*i)
X = (M/3)*(a+b+c)
f = 24*(X^2-N*a*b*x^2
print(f.full_simplify())
```

Proof. Suppose that equilateral triangle ABC is N-tiled using a tile with angles $(\alpha, \beta, \frac{\pi}{3})$. Define

$$\zeta := e^{i\alpha}.$$

Then

$$\begin{aligned} c &= \sin\gamma = \frac{\sqrt{3}}{2} \\ a &= \sin\alpha = \frac{\zeta - \zeta^{-1}}{2i} \\ b &= \sin\beta \\ &= \sin(2\pi/3 - \alpha) \\ &= \sin(2\pi/3)\cos\alpha - \cos(2\pi/3)\sin\alpha \\ &= \frac{\sqrt{3}}{2}\frac{\zeta+\zeta^{-1}}{2} + \frac{1}{2}\frac{\zeta-\zeta^{-1}}{2i} \end{aligned}$$

Substituting the value for X from the coloring equation (1) into the area equation (2), we find (by hand or using the SageMath code in Fig. 6)

$$\begin{aligned} 0 = \; & \Big(M^2\Big(-i\sqrt{3}-1\Big) + N\Big(3i\sqrt{3}+3\Big)\Big)\zeta^4 \\ & +2M^2\Big(-i\sqrt{3}+1\Big)\zeta^3 \\ & +6\left(M^2-N\right)\zeta^2 \\ & +2M^2\Big(i\sqrt{3}+1\Big)\zeta \\ & +M^2\Big(i\sqrt{3}-1\Big) + N\Big(-3i\sqrt{3}+3\Big) \end{aligned} \tag{3}$$

Observe first $\zeta = 1$ is not a solution, since $f(1) = 8M^2$, and by the coloring equation, $M > 0$. (The command `f.substitute(x=1).simplify()` will save you the trouble.) Next observe that $\zeta = -1$ is a solution. Hence the non-real solutions, which are the ones of interest, satisfy a cubic equation. That equation is $f(\zeta)/(\zeta + 1) = 0$. The equation can be calculated by long division, or by the SageMath command[1]

```
g = (f.maxima_methods().divide(x+1)[0]).full_simplify()
```

[1] Here we see explicitly that SageMath calls on Maxima to do polynomial division. The `divide` method produces a list of length 2, with the quotient and remainder.

The equation resulting is

$$\begin{aligned} 0 &= \Big(M^2\Big(-i\sqrt{3}-1\Big)+N\Big(3i\sqrt{3}+3\Big)\Big)\zeta^3 \\ &\quad +\Big(M^2\Big(-i\sqrt{3}+3\Big)+N\Big(-3i\sqrt{3}-3\Big)\Big)\zeta^2 \\ &\quad +\Big(M^2\Big(i\sqrt{3}+3\Big)+N\Big(3i\sqrt{3}-3\Big)\Big)\zeta \\ &\quad +M^2\Big(i\sqrt{3}-1\Big)+N\Big(-3i\sqrt{3}+3\Big) \end{aligned}$$

Eventually I realized that this equation has another explicit solution, namely $\zeta = e^{-i\pi/3}$, as one can verify by hand, or by asking SageMath for the value of `g(x=exp(-i*pi/3)`. Dividing by $z - e^{-\pi/3}$ we find a quadratic equation for ζ:

$$\begin{aligned} 0 &= \Big(M^2\Big(-i\sqrt{3}-1\Big)+N\Big(3i\sqrt{3}+3\Big)\Big)\zeta^2 \\ &\quad +\Big(M^2\Big(-i\sqrt{3}+1\Big)+N\Big(-3i\sqrt{3}+3\Big)\Big)\zeta \\ &\quad +2\,M^2-6\,N \end{aligned}$$

That is the quadratic equation mentioned in (i) of the theorem.

Solving that equation we find

$$\begin{aligned} \zeta &= \frac{M^2\big(i\sqrt{3}-1\big)+N\big(3i\sqrt{3}-3\big)}{M^2\big(-2i\sqrt{3}-2\big)+N\big(6i\sqrt{3}+6\big)} \\ &\quad \pm\frac{\sqrt{M^4\big(6i\sqrt{3}+6\big)+M^2N\big(-60i\sqrt{3}-60\big)+N^2\big(54i\sqrt{3}+54\big)}}{M^2\big(-2i\sqrt{3}-2\big)+N\big(6i\sqrt{3}+6\big)} \\ &= \frac{1}{2}\frac{3N+M^2}{3N-M^2}\left(\frac{i\sqrt{3}-1}{i\sqrt{3}+1}\right) \\ &\quad \pm\frac{\sqrt{M^4\big(6i\sqrt{3}+6\big)+M^2N\big(-60i\sqrt{3}-60\big)+N^2\big(54i\sqrt{3}+54\big)}}{2(3N-M^2)(i\sqrt{3}+1)} \\ &= \frac{1}{2}\frac{3N+M^2}{3N-M^2}\left(\frac{i\sqrt{3}-1}{i\sqrt{3}+1}\right)\pm\frac{\sqrt{6}\sqrt{i\sqrt{3}+1}\sqrt{(M^2-9N)(M^2-N)}}{2(3N-M^2)(i\sqrt{3}+1)} \\ &= \frac{1}{4}\frac{3N+M^2}{3N-M^2}(1+i\sqrt{3})\pm\frac{\sqrt{3}e^{i\pi/6}}{2e^{i\pi/3}}\frac{\sqrt{(M^2-9N)(M^2-N)}}{3N-M^2} \\ &= \frac{1}{4}\frac{3N+M^2}{3N-M^2}(1+i\sqrt{3})\pm\frac{\sqrt{3}e^{-i\pi/6}}{2}\frac{\sqrt{(9N-M^2)(N-M^2)}}{3N-M^2} \\ &= \frac{1}{4}\frac{3N+M^2}{3N-M^2}(1+i\sqrt{3})\pm\frac{3-i\sqrt{3}}{4}\frac{\sqrt{(9N-M^2)(N-M^2)}}{3N-M^2} \end{aligned}$$

The tile edges a and b are the imaginary parts of the two values of ζ. We calculate them explicitly:

$$\begin{aligned}
a &= \frac{\sqrt{3}}{4}\left(\frac{3N+M^2}{3N-M^2} - \frac{\sqrt{(9N-M^2)(N-M^2)}}{3N-M^2}\right)\\
b &= \frac{\sqrt{3}}{4}\left(\frac{3N+M^2}{3N-M^2} + \frac{\sqrt{(9N-M^2)(N-M^2)}}{3N-M^2}\right)\\
c &= \frac{\sqrt{3}}{2}
\end{aligned}$$

These immediately imply the formula in part (ii) of the theorem.

We next prove that $M^2 < N$, which is part (iii) of the theorem. From the area and coloring equations we have

$$\begin{aligned}
X^2 &= Nab\\
\left(\frac{M}{3}(a+b+c)\right)^2 &= Nbc\\
M^2 &= \frac{9Nbc}{(a+b+c)^2}
\end{aligned}$$

From the formulas for (a,b,c) above, we have

$$\begin{aligned}
a+b+c &= \left(\frac{\sqrt{3}}{2}\right)\left(1+\frac{3N+M^2}{3N-M^2}\right)\\
&= \sqrt{3}\left(\frac{3N}{3N-M^2}\right)
\end{aligned}$$

This already implies $M^2 < 3N$, since if not, the right side is negative, but the left side is positive. Now notice that the equation in part (ii) contains $\sqrt{(9N-M^2)(N-M^2)}$. Since $M^2 < 3N$, the first factor is positive. Hence the second factor $N - M^2$ must be nonnegative, or the square root will not be real, but the ratio a/b is real. Therefore $M^2 \le N$. We cannot have $M^2 = N$ since that will make $a = b$, which would make $\alpha = \beta$, contrary to hypothesis. Hence $M^2 < N$. That proves part (iii) of the theorem, and completes the proof.

Lemma 1. *Let triangle ABC be equilateral. Suppose it is N-tiled using a tile with angles $(\alpha, \beta, \frac{\pi}{3})$, where α is not a rational multiple of π. Let (a,b,c) be the sides of the tile. Then*

$$\frac{ab}{c^2} = \frac{4M^2N}{(3N-M^2)^2}$$

Proof. We start with the formulas for a and b from Theorem 2, namely that a/c and b/c are given by choosing the $+$ and $-$ signs respectively in

$$\frac{1}{2}\left(\frac{3N+M^2}{3N-M^2} \pm \frac{\sqrt{(9N-M^2)(N-M^2)}}{3N-M^2}\right)$$

Using the identity $(u-v)(u+v) = u^2 - v^2$ we have

$$\frac{ab}{c^2} = \frac{1}{4}\frac{(3N+M^2)^2 - (9N-M^2)(N-M^2)}{(3N-M^2)^2}$$

Simplifying the numerator we have

$$\frac{ab}{c^2} = \frac{1}{4}\frac{16M^2N}{(3N-M^2)^2}$$

Canceling 4 we have the formula mentioned in the lemma. That completes the proof.

Theorem 3. *Let triangle ABC be equilateral. Suppose it is N-tiled using a tile with angles $(\alpha, \beta, \frac{\pi}{3})$, where α is not a rational multiple of π. Then for some integer $M < \sqrt{N}$ (the coloring number of the tiling),*

(i) $(9N - M^2)(N - M^2)$ is a square, and

(ii) N is not prime.

Remarks. (1) In particular N cannot be 7, 11, or 19. In [1], we proved that there is no 7 or 11 tiling, and the case studied here was handled purely computationally, by a method that does not work for $N = 19$.

(2) Laczkovich has shown that each rational tile with an angle $\gamma = \pi/3$ in which α is not a rational multiple of π can be used to tile *some* equilateral triangle. But the N required might be very large. Indeed, we tried to construct a tiling following Laczkovich's instructions in [2], but N came out to be over a million, so we could not draw the tiling, and thus cannot present even one picture.

Proof. Suppose equilateral ABC is N-tiled as in the statement of the theorem. Let the sides of the tile be (a, b, c). According to Theorem 2, $M < \sqrt{N}$ as mentioned in the theorem, and the ratios a/c and b/c are given by

$$\frac{1}{2}\left(\frac{3N+M^2}{3N-M^2} \pm \frac{\sqrt{(9N-M^2)(N-M^2)}}{(3N-M^2)}\right)$$

According to Theorem 3.3 of [3], the tile (a, b, c) is rational, so a/c and b/c are rational. Therefore the expression under the square root is an integer square. That proves part (i) of the theorem.

Recall the area equation $X^2 = Nab$. Since the tile is rational, we may re-scale it so that (a, b, c) are integers with no common factor. (That changes the size of ABC and makes X an integer.) Let s be the square-free part of ab. Then s divides X^2, so (being square-free) it divides X. Hence s^2 divides Nab. Hence s divides N. Now assume, for proof by contradiction, that N is prime. Then s is either 1 or N. If s is 1, then ab is a square, so $N = X^2/ab$ is a rational square, and since N is an integer, it is an integer square. But that contradicts the assumption that N is prime. Hence the other case must hold: $s = N$. Since N is presumed prime, s is also prime, and thus one of (a, b) is N times a square and the other is a square. Say it is a that is not square; then $a = Nd^2$ and $b = e^2$. Then $ab = Nd^2e^2$ is N times a square. Well, that does not contradict Lemma 1, provided $(3N - M^2)$ divides $2M$. Let the quotient be q; then $(3N - M^2)q = 2M$, so $qM^2 + 2M = 2N = (qM + 2)M$. Since N is prime we must have $M = 2$ and $qM + 2 = N$. But with $M = 2$, we then have $qM + 2 = 2q + 2 = 2(q + 1) = N$, contradicting the assumption that N is prime unless $q = 0$, but in that case $N = 2$, which is impossible, as at least three tiles are required, one with its $\pi/3$ angle at each vertex of ABC. That completes the proof of the theorem.

3.1. An algorithm to decide if there is an N-tiling.

Theorem 4. *Given N and an equilateral triangle ABC, there is a finite set S of not more than $\sqrt{3N}$ tiles such that, if any tile with angles $(\alpha, \beta, \pi/3)$ and α not a multiple of π can N-tile ABC, then one of the tiles in S can do so. Whether such a tiling exists is computable in a finite (though perhaps large) number of steps.*

Remark. We are not claiming an *efficient* algorithm.

Proof. By Theorem 2, the coloring number M of any N-tiling of ABC is at most $\sqrt{3N}$, and (N, M) together determine the sides (a, b, c) of the tile. This provides the finite set S of tiles, and Theorem 2 says that any N-tiling uses one of those tiles. All of the tiles in S satisfy the area equation that the area of ABC is N times the area of the tile. Given such a tile (a, b, c), it is solvable by well-known graph-search algorithms (for example depth-first search) whether (a, b, c) can tile ABC. This may seem obvious, but we consider the details briefly.

We can consider connected partial tilings as nodes in a graph, where there is an edge between two nodes p and q if partial tiling q is obtained from p by adding one more tile within the boundaries of ABC, the new tile sharing at least one vertex and at least part of at least one edge with the tiling p, and not overlapping any tile of p. Each partial tiling p has finitely many neighbors, which can be algorithmically generated by enumerating the possible ways to extend a given partial tiling. In so doing we need to test whether a triangle overlaps another triangle; the precision issue involved is discussed below. No path has length more than N, since the area of a partial tiling cannot exceed N times the area of the tile. Finally, we can test algorithmically whether a given partial tiling is actually a tiling of ABC; again a precision issue arises. The point of these precision issues is that we need to determine in a finite number of steps of computation whether two points coincide or not, whether two tile edges coincide or not, and whether a point lies on a given line segment.

We represent a partial tiling as a list of triangles, where a triangle is three points, and a point is given by a pair of coordinates in a suitable field $\mathbb{K}$, or perhaps just by finite-precision complex numbers. It is well-known that algebraic number fields have "decidable equality", so if we use algebraic numbers as coordinates of points, the precision issues will be not arise, i.e. the computations will be exact. That completes the proof.

Remark. We could write the program described in the proof in Python, calling on SageMath for arithmetic in algebraic number fields. We have not done that. Instead we wrote the program in C++, using fixed-precision real numbers. Theoretically finite-precision real numbers would always work, although the precision might theoretically have to be large if the tile has a very small angle. Since the angles of the tiles at each vertex are made of α, β, and γ, two tile edges either coincide, or they miss by a lot. In practice we did not compute with triangles containing tiny angles, so the usual fixed-precision real numbers caused no problems. This program played no role in our proofs, and did not succeed in finding any new tilings. It did, however, enable us to rule out values of $N < 105$, as described below.

3.2. Comparison with Laczkovich's results.

Given N, we have shown how to determine a finite set of "possible tiles" that include every tile (with a $\pi/3$ angle and α not a rational multiple of π) that can be used in an N-tiling of an equilateral triangle. In other words, given N, the tile (and hence ABC) are determined; or more accurately, all other possibilities for (N, ABC) are eliminated. We don't know if a tiling really exists, except by trial and error.

Laczkovich proved that any such tile can be used to N-tile an equilateral triangle ABC, if we choose N large enough. In other words, given the tile, at least one pair (N, ABC) is determined such that a tiling exists.

Comparing those results, the question naturally arises, whether the tile actually determines N. That is, can the same tile be used for tiling two equilateral triangles of different sizes? Well, given one tiling, we can always replace each tile by a quadratic tiling of m^2 smaller tiles, thus producing an m^2N tiling. So the best we can hope for is that the squarefree part of N might be determined by the tile. Since we have explicit formulas for a/c and b/c, this question can be answered.

Theorem 5. *Suppose T is a triangle with angles $(\alpha, \beta, \pi/3)$ with β not a rational multiple of $\pi/3$. Suppose T can be used to N-tile an equilateral triangle, and also to K-tile a (different) equilateral triangle. Then N and K have the same squarefree part.*

Proof. Let M be the coloring number of the N-tiling and J the coloring number of the K-tiling. By Lemma 1 we have

$$\frac{3N+M^2}{3N-M^2} \pm \frac{\sqrt{(9N-M^2)(N-M^2)}}{3N-M^2} \quad = \quad \frac{3K+J^2}{3K-J^2} \pm \frac{\sqrt{(9K-J^2)(K-J^2)}}{3K-J^2}$$

(where the same sign is taken for both $\pm$ signs). Adding the two equations (obtained by taking different signs for $\pm$) we have

$$\frac{3N+M^2}{3N-M^2} \quad = \quad \frac{3K+J^2}{3K-J^2} \tag{4}$$

Define

$$f(x) \quad := \quad \frac{3x+1}{3x-1}$$

We are interested only in the domain of rational $x > 1$. Then f' is negative, so f is decreasing and hence one-to-one. By (4),

$$f\left(\frac{N}{M^2}\right) \quad = \quad f\left(\frac{K}{J^2}\right)$$

Since f is one-to-one, $N/M^2 = K/J^2$. Hence $J^2N = M^2N$. Hence M and N have the same square-free part. That completes the proof of the theorem.

3.3. **Numerical results.** The main question of interest in this paper is, for which N do there exist N-tilings of an equilateral triangle? In this section we restrict attention to tiles with angles $(\pi/3, \beta, \gamma)$ where β is not a rational multiple of π. We proved above that there is no N-tiling if N is prime, but we were not able to formulate a more general number-theoretic condition on N. However, we have reduced the problem to two computational steps:

- Determine if the equations of Lemma 1 have rational solutions for the ratios a/c and b/c. If they do not, there is no N-tiling. If they do, let (a, b, c) be integers with no common divisor whose ratios solve the equations.
- Determine by trial and error (e.g., depth-first search) whether a tiling actually exists.

Here we investigate the limits of computation in this matter.

Theorem 6. *For $N < 105$, there is no N-tiling of any equilateral triangle by a tile with angles $(\alpha, \beta, \pi/3)$ where β is not a rational multiple of π.*

Proof. If there is an entry for (N, M), then any N-tiling with coloring number M must use the tile mentioned in Table 3.3. The SageMath code for computing the table up to $N = 200$ is given in Fig. 7; it checks for a solution of the equations of Lemma 1. This computation is instantaneous, and we could compute as many pages of this table as we want to read.

Table 1. Tilings not ruled about by the area and coloring equations

N	M	the tile
54	6	(3,8,7)
66	4	(11,96,91)
70	5	(7,40,37)
85	6	(17,80,73)
96	8	(3,8,7)
105	7	(5,21,19)
105	9	(7,15,13)
130	9	(40,117,103)
150	10	(3,8,7)
153	5	(17,225,217)
156	9	(13,48,43)
198	10	(72,275,247)

Figure 7. SageMath code to produce the table

```
def nov19(J):
   var('N,M')
   for N in range(3,J):
      for M in range(1,sqrt(N)):
         x = (9*N-M^2)*(N-M^2)
         if not is_square(x):
            continue
         den = 3*N-M^2
         num = 3*N+M^2
         C = 2*den;
         A = num - sqrt(x)
         B = num + sqrt(x)
         assert(C^2 == A^2 + B^2 - A*B)
         g = gcd(A,gcd(B,C))
         (a,b,c) = (A/g,B/g,C/g)
         print(N,M,(a,b,c))
```

We then used a special-purpose C++ program to carry out depth-first search for a tiling, using the values from Table 3.3 as inputs, one row at a time. This program hits the limits of practical search at $N = 105$. It produced a negative answer in a few minutes (but not seconds) for each row up to $N = 96$, but for $N = 96$ it took overnight, generating 13450826 partial tilings in the search; and for the first $N = 105$ entry, it took five hours and generated 3004129 partial tilings. The second $N = 105$ entry ran several days, searching more than 300 million partial tilings before a power failure terminated the search.

The question naturally arises whether we should trust a complicated program whose output is a simple statement that a search had no result. Setting a `verbose` option in the program allowed it to draw pictures of the partial tilings it was considering. In this way we produced TEX documents with several thousand pages, each page containing a picture of a partial tiling. Somewhat surprisingly, TEXand the associated software could correctly process these documents! The program appeared to be searching as intended. That is as close as we are going to come to a proof of this theorem, at least by this method. We could probably bring more computing power and patience to bear and rule out a few more values of N, but the search is fundamentally exponential, so this is not a promising line of attack.

4. A tile $(\alpha, \beta, 2\pi/3)$ with α/π irrational

Every vertex of the tiling with total angle π either is composed of $\alpha + \beta + \gamma$ or of $3\alpha + 3\beta$. Since the latter form has six tiles meeting at the vertex, there is no coloring equation, since that would require an odd number of tiles at each such vertex. Even in a tiling without such vertices, there still could not be a coloring equation, because there will have to be a "center" somewhere in the tiling, with three tiles each having angle $2\pi/3$. The existence of a "center" follows from the observation that at each vertex of ABC, there will have to be two tiles with angles α and β; we do not give details since our only purpose here is to explain why we cannot use the coloring equation for these tilings.

Laczkovich has proved [2] that, given a rational tile of the shape we are now considering, there is an N-tiling of *some* sufficiently large equilateral ABC; but N might have to be large. In fact, the smallest N for which we have been able to find such a tiling is $N = 10935$. See Fig. 4.

In [3], Laczkovich made a significant advance: a tile with a $2\pi/3$ angle that tiles an equilateral triangle must either have both its other angles $\pi/6$, or else both the following conditions hold:

(i) the tile is rational (that is, the ratios of the sides are rational), and

(ii) the other two angles of the tile are not rational multiples of π.

These statements are proved in Theorem 3.3 and Lemma 3.2 of [3], respectively. It is the rationality of the tile that is the significant advance of 2012 [3], as (ii) was already proved in 1995 [2].

We need some terminology. Given a tiling of (in our case) a triangle ABC, an *internal segment* is a line segment connecting two vertices of the tiling that is contained in the union of the boundaries of the tiles, and lies in the interior of ABC except possibly for its endpoints. A *maximal segment* is an internal segment that is not part of a longer internal segment. A *left-maximal segment* is an internal segment XY that is not contained in a longer segment UXY, i.e., a segment UY with X between U and Y. A tile is *supported by* XY if one edge of the tile lies on XY. The internal segment XY is said to have "all c's on the left" if the endpoints X and Y are vertices of tiles supported by XY and lying on the left side of XY, and all tiles supported by XY lying on the left of XY have there c edges on XY. Similarly for "all c's on the right."

An internal segment XY is said to *witness the relation* $jc = \ell a + mb$ if XY has all c's on one side, and exactly j of them (that is, the length of XY is jc), and on the other side XY supports ℓ tiles with their a edges on XY and m tiles with their b edges on XY (in any order) and no other tiles, and the endpoints X and Y are vertices of tiles on both sides of XY. This implies that c is not a linear combination of a and b with nonnegative rational coefficients, but it is stronger than that statement, in some way limiting the size of the (numerators and denominators of the) coefficients. Similarly we use the terminology "XY witnesses a relation $jc = \ell a + mc$", which implies "c is a rational multiple of a", but is stronger.

Laczkovich defined a tiling to be *regular* if there are two angles (say α and β) of the tile such that at each vertex V of the tiling, the number of tiles having angle α at V is the same as the number of tiles having angle β at V. According to Lemma 3.2 of [3], in an irregular tiling, (α, β, γ) are linear combinations with rational coefficients of the angles of the tiled polygon. In this paper the tiled polygon is the equilateral triangle, so an irregular tiling has angles that are rational multiples of π. Therefore, the case of interest in this section, when α and β are not rational multiples of π and $\gamma = 2\pi/3$, only can occur in regular tilings.

In the work below, we shall make use of Lemmas 4.5 and 4.6 of [3]. These lemmas make use of the directed graph Γ_c defined on page 346. We review that definition now.

Definition 1 (The directed graph Γ_c)**.** *Given a tiling of some triangle, the nodes of the graph Γ_c are certain vertices of the tiling. An edge of Γ_c connects vertices X and Y if the segment XY is a left-maximal internal segment having all c's on one side of XY, and there is another tile supported by XY on that side of XY past Y that does not have its c edge on that side.*

Example. In Fig. 4, look at the longest side of one of the light blue components of the tiling. That segment is composed of 21 c edges. At one end it cannot be extended: that is X. At the other end, it does extend beyond the blue triangles, but there it has a edges on both sides. Hence, there is an edge of Γ_c from X to Y. We need only Γ_c in this paper, but Γ_a and Γ_b are defined similarly, and the reader may wish to see what the graphs look like in Fig. 4.

We now state the versions of Laczkovich's lemmas that we need. These lemmas presuppose a regular tiling of a convex polygon, in our application an equilateral triangle.

Lemma 2 (Laczkovich's Lemma 4.5)**.** *Suppose the tiling does not witness any relation $jc = \ell a + mb$. Let XY be a segment of the tiling (internal or on the boundary) and let V be a vertex of the tiling lying on the interior of XY, lying either on the boundary or on an internal point of an edge of some tile. Suppose that of the two tiles supported by XY with a vertex at V, one has edge c on XY and the other has edge a or b on XY. Then there is an edge of Γ_c starting from V.*

Remark. In Fig. 4, the edge of Γ_c mentioned as an example does witness a relation $21c = 24a + 15b$.

Proof. Laczkovich's statement replaces the first sentence of the lemma by "c is not a linear combination of a and b with nonnegative rational coefficients." But the proof actually proves our version.

Lemma 3 (Laczkovich's Lemma 4.6)**.** *If a and b are commensurable and the tiling does not witness any relation of the forms $jc = \ell a + mc$ or $ja = \ell c + ma$, with j and ℓ positive, or $jc = \ell a + mb$ with j positive, then the graph Γ_c is empty: it has no edge.*

Proof. Again, our statement differs from Laczkovich's in that his hypothesis is that c is not a rational multiple of a, but his proof actually proves our version.

Before coming to the main theorem, we prove a lemma. It may seem obvious, but it does actually need a proof.

Lemma 4. *In an N-tiling of any triangle ABC with $N > 3$, no segment of the tiling can support all N tiles.*

Proof. Suppose segment XY supports all N tiles. If there is no vertex of the tiling lying on the interior of XY then there is only one tile on each side of XY. Then $N = 1$ if XY lies on the boundary of ABC, and otherwise $N = 2$, contradiction. Therefore there is a vertex V between X and Y. If at least three tiles meet at V on the same side of XY, then the middle one is not supported by XY. Hence exactly two tiles on the same side of XY meet at V. Then both tiles must have their γ angle at V (assuming here that $\alpha < \beta < \gamma$), since otherwise the sum of the angles cannot be π. Then $\gamma = \pi/2$. Let T be one of those two tiles and let W be its other vertex on XY. Then T does not have its γ angle at W. If W is in the interior of XY then the other two tiles on the same side of XY as T and with a vertex at W must together make more than a $\pi/2$ angle at W, so there must be at least two such tiles. Then the one adjoining T is not supported by XY, contradiction. Therefore W is not in the interior of XY, but must be X or Y. Hence only two tiles on that side of XY are supported by XY, and their other sides (the ones not shared between the two tiles or lying on XY) lie on the boundary of ABC, and the angles of those tiles at X and Y are acute. Now if XY lies on the boundary of ABC, then $N = 2$, contradiction. Hence XY is an interior segment. Then the same argument applies to the other side of XY, so there are exactly two tiles supported on that side of XY as well, whose other edges lie on the boundary of ABC and have acute angles at X and Y. But then ABC is a quadrilateral, with diagonal XY. (Note that some quadrilaterals can indeed be tiled in such a way that the diagonal supports all four tiles.) That contradicts the hypothesis that ABC is a triangle. That completes the proof.

Theorem 7. *Let equilateral triangle ABC be N-tiled by a tile with angles (α, β, γ), with $\gamma = 2\pi/3$ and α not a rational multiple of π. Then N is not a prime number.*

Proof. By Theorem 3.3 of [3], (a, b, c) are pairwise commensurable. Without loss of generality we can assume that (a, b, c) are integers with no common factor. As explained above, Lemma 3.2 of [3] implies the tiling is regular. Assume, for proof by contradiction, that N is a prime number. Let X be the length of the sides of equilateral ABC. Then the area equation is

$$X^2 \sin(\pi/3) \quad = \quad Nab\sin(2\pi/3)$$

Since $\sin(\pi/3) = \sin(2\pi/3)$ we have

$$X^2 \quad = \quad Nab \qquad \text{area equation}$$

Since each side of ABC is the disjoint union of a set of tile edges, we have for some non-negative integers (p, q, r),

$$X = pa + qb + rc.$$

Then X is an integer. Since $X^2 = Nab$, N divides X^2, which we write as usual $N|X^2$. Then we have

$$\begin{array}{ll} N|X^2 & \\ N|X & \text{since } N \text{ is prime} \\ N^2|X^2 & \\ N^2|Nab & \text{since } X^2 = Nab \\ N|ab & \end{array}$$

Since N is prime, N divides a or N divides b. Since so far, nothing distinguishes α from β except the name, we may assume without loss of generality that N divides a. Then there is an integer $e \geq 0$ such that $a = Ne$. Then $X^2 = N^2eb$. Then eb is a rational square, and hence an integer square.

By the law of cosines, we have

$$\begin{array}{rcll} c^2 & = & a^2 + b^2 - \cos(2\pi/3)\, ab & \\ c^2 & = & a^2 + b^2 + ab & \text{since } \cos(2\pi/3) = -1/2 \\ c^2 & = & N^2e^2 + b^2 + Neb & \end{array} \tag{5}$$

Therefore c is congruent to $\pm b$ mod N and also N does not divide b, since if $N|b$ then also $N|c^2$ and hence $N|c$, contradiction, since then N would divide all three of (a, b, c), but (a, b, c) have no common divisor. Since X is a sum of tile edges, there are nonnegative integers (p, q, r) such that

$$X \quad = \quad pa + qb + rc$$

Moreover, we may assume not both q and r are zero, since at each vertex of ABC, one of the tiles there has its α angle at that vertex, and hence does not have its a edge on the boundary of ABC. We choose such a side of ABC to pick the decomposition of X. Then not both q and r are zero. Substitute for c from (5). Then

$$X \quad = \quad pNe + qb + r\sqrt{N^2e^2 + b^2 + Neb}$$

Since $N|X$, looking at the equation mod N we have

$$0 \quad = \quad (q \pm r)b \bmod N \tag{6}$$

But N does not divide b. Therefore either $N|(q + r)$ or $N|(q - r)$, according as c is congruent to b or $-b$ mod N. We have $q + r < N$, since at least one tile does not have an edge on the side of ABC that decomposes into $pa+qb+rc$. (We can choose one at a vertex of ABC, for example.) Since not both q and r are zero, we have $0 < (q + r) < N$, so N cannot divide $q + r$. Therefore $N|q - r$. Hence $q = r$ and $c \equiv -b$ mod N. Hence $b + c$ is divisible by N. Then the equation $X = pa + qb + rc$ becomes

$$X \quad = \quad pa + q(b + c)$$

Suppose $jc = \ell a + mb$ is witnessed on some internal segment of the tiling. Then

$$jc + jb \quad = \quad \ell a + (m + j)b$$

and mod N we have $j(b+c) \equiv 0$, and since $N|a$ and N does not divide b, and N is prime, we have $N|(m+j)$. But since the relation is witnessed in the tiling, (m, ℓ, j) are each less than N, hence $m + j = N$. Then every tile touches that internal line, which is impossible, by Lemma 4. (This is the only place we use that lemma, but it does seem to be needed here if $\ell = 0$.) Hence no internal segment of the tiling witnesses a relation $jc = \ell a + mb$.

I say that the tiling also does not witness any relation of the forms $jc = \ell a + mc$ with j and ℓ positive. For suppose it does. Then $\ell a = c(j - m)$. Since there are altogether N tiles, we have $j + \ell + m \le N$, and since $\ell > 0$ we have $j < N$ and $m < N$. Since $N|a$ and N is prime, and N does not divide c, $N|(j - m)$. Since $0 < j < N$ and $0 \le m < N$, we have $|j - m| < N$. Then $N|(j - m)$ implies $j = m$. Then $jc = \ell a + mc = jc$ implies $\ell = 0$, contradicting the hypothesis that $\ell > 0$.

Similarly, the tiling does not witness any relation of the form $ja = \ell c + ma$ with j and ℓ positive. For suppose it does. Then $\ell c = (j - m)a$. Since $N|a$, N divides the left side ℓc. Since N is prime and N does not divide c, N divides $j - m$. But since the tiling witnesses the relation, $j+\ell+m < N$. Hence, as above, $|j-m| < N$. Hence $j = m$. But then $ja = \ell c + ma$ implies $\ell = 0$, contradiction.

Hence, by Lemma 3, the graph Γ_c is empty. Therefore, by Lemma 2, every maximal segment XY in the tiling that supports a tile with a c edge with a vertex at X has *only* c edges on that side of XY.

In particular, each side of ABC consists only of c edges if it has any c edges at all. But we also proved that it has equal numbers of b and c edges. Hence the number of b and c edges on the boundary is zero. In that case, however, each tile on side AB of ABC would have a γ angle on AB, and since $\gamma > \pi/2$, there is at most one γ angle at each vertex, and no γ angle at the endpoints A and B, since $\gamma = 2\pi/3$ is greater than the angles of an equilateral triangle. Then by the pigeonhole principle, one vertex on AB must have two tiles with their γ angles at that vertex. But that is a contradiction, since $2\gamma > \pi$. That completes the proof.

Remark. The proof above works also for the case of an equilateral triangle tiled by $(\alpha, \beta, \pi/3)$. Then the law of cosines gives us $c^2 = a^2 + b^2 - ab$ instead of $c^2 = a^2 + b^2 + ab$, but the argument still goes through. Since we already gave one proof that N cannot be prime in that case in Theorem 3, we do not spell out the details.

References

[1] Michael Beeson. *No triangle can be cut into seven congruent triangles.* 2018. Available on ArXiv and the author's website.
[2] M. Laczkovich. Tilings of triangles. *Discrete Mathematics*, 140:79–94, 1995.
[3] Miklós Laczkovich. Tilings of convex polygons with congruent triangles. *Discrete and Computational Geometry*, 38:330–372, 2012.
[4] Stephen L. Snover, Charles Waiveris, and John K. Williams. Rep-tiling for triangles. *Discrete Mathematics*, 91:193–200, 1991.
[5] Alexander Soifer. *How Does One Cut a Triangle?* Springer, 2009.
[6] W. A. Stein et al. *Sage Mathematics Software (Version 8.0).* The Sage Development Team, 2017. http://www.sagemath.org.