Triangle Tiling V: Tilings by a tile with integer sides

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June 4, 2012

Abstract

An N-tiling of triangle ABC by triangle T is a way of writing ABC as a union of N triangles congruent to T, overlapping only at their boundaries. The triangle T is the "tile". The tile may or may not be similar to ABC. We wish to understand possible tilings by completely characterizing the triples (ABC, T, N) such that ABC can be N-tiled by T. In particular, this understanding should enable us to specify for which N there exists a tile T and a triangle ABC that is N-tiled by T; or given N, to determine which tiles and triangles can be used for N-tilings; or given ABC, to determine which tiles and N can be used to N-tile ABC. This is the fifth paper in a series of papers on this subject. The previous papers have reduced the problem to the case when T has a 120° angle and integer side lengths. That is the problem we take up in this paper. We are still not able to completely solve the problem, but we prove that if there are any N-tilings by such tiles, then $N \geq 96$. Combining this results with our earlier work, we can remove the exception for a 120° tile, obtaining definitive non-existence results. For example, there is no 7-tiling, no 11-tiling, no 14-tiling, no 19-tiling, no 31-tiling, no 41-tiling, etc.

Regarding the number $N = 96$: There are several possible shapes of ABC, and for each shape, we exhibit the smallest N for which it is presently unknown whether there is an N-tiling. For example, for equilateral ABC, the simplest unsolved case as of May, 2012 is $N = 135$. For each of these minimal-N examples, the tile would have to have sides $(3, 5, 7)$.

1 Introduction

For a general introduction to the problem of triangle tiling, see [1]. This is our fifth paper on the subject; in [2] we prove some nonexistence theorems; in [3] we found a new family of tilings, and proved they are the only ones possible when $3\alpha + 2\beta = \pi$ (where the angles of the tile are α, β, and γ). In [4], we took up the remaining case, when the tile has a 120° angle; in that paper, we reduced the problem to the case when the sides of the tile are integers, and in which the tiling has a total of six tiles at the vertices of ABC. It is that case that we take up here.

Although there are some quite interesting ways of fitting together tiles of this shape, one never seems to be able to make a triangle. After the efforts presented here, we have still not ruled out the existence of such tilings. The main theorem of this paper is that, if there is such a tiling, then $N \geq 96$.

Along the way we prove some other interesting things. For example, if tile T with integer side lengths (a, b, c) tiles triangle ABC, whether or not the tile has a 120° angle, then each side of ABC is composed of edges of the tile in a special way: there must be at least one c edge, and there cannot be both a and b edges. It came as a surprise that this is true for any shape of tile.

In [1], we introduced the $\mathbf{d}$ matrix and the $\mathbf{d}$ matrix equation,

$$\mathbf{d} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

where a, b, and c are the sides of the tile, and X, Y, and Z are the lengths of the sides of ABC, in order of size. The angles of ABC are, in order of size, A, B, and C, so $X = \overline{BC}$, $Y = \overline{AC}$, and $Z = \overline{AB}$. We keep this convention even if some the angles are equal. The $\mathbf{d}$ matrix has nonnegative integer entries, describing how the sides of ABC are composed of edges of tiles. Sometimes we assume $a < b$, and sometimes not; but always $a < c$ and $b < c$.

The d matrix is used in almost all our proofs. To avoid having every page filled with cumbersome subscript notation $\mathbf{d}_{ij}$ for the entries of the matrix, we introduce letters for the entries. While this eliminates subscripts, it does require the reader to remember which element is denoted by which letter. Here, for reference, we define

$$\mathbf{d} = \begin{pmatrix} p & d & e \\ g & m & f \\ h & \ell & r \end{pmatrix}$$

2 The equation $c^2 = a^2 + b^2 + ab$

Suppose the triangle with sides (a, b, c) has a 120° angle. The law of cosines gives us the equation in the section title (as will be proved below); in this section, we study this equation from the point of view of number theory.

Lemma 1 *Let a, b, and c be the sides of a triangle with a 120° angle opposite side c. Then*

$$c^2 = a^2 + b^2 + ab. \tag{1}$$

Proof. By the law of cosines, we have

$$c^2 = a^2 + b^2 - 2ab\cos(2\pi/3).$$

But $\cos(2\pi/3) = -1/2$. That completes the proof of the lemma.

If (a, b, c) have a common factor, we can divide by it, which amounts to rescaling the tile. Hence we can assume without loss of generality that they have no common factor. The following lemma shows that even more is true.

Lemma 2 *Suppose that a, b, and c are integers with no common factor, forming the sides of a triangle with a 120° angle. Then a, b, and c are pairwise relatively prime.*

Proof. The law of cosines tells us $c^2 = a^2 + b^2 - 2ab\cos\gamma = a^2 + ab + b^2$, since $\cos\gamma = -1/2$. Hence any common factor of a and b is also a factor of c. Hence a and b are relatively prime. Similarly, $a^2 = c^2 - b(a + b)$ shows that any common factor of b and c is also a factor of a; hence b and c are relatively prime. The law of cosines equation can also be written in the form $b(a+b) = c^2 - a^2$, so any common prime factor of c and a also divides b or $a+b$, but if it divides $a + b$ then it also divides b; hence c and a are also relatively prime. That completes the proof of the lemma.

Here are seven small solutions of $c^2 = a^2 + b^2 + ab$:

```
(a,b,c)

(3, 5, 7)
(5, 16, 19)
(7, 8, 13)
(7, 33, 37)
(9, 56, 61)
(11, 24, 31)
(11, 85, 91)
```

Merely because we can compute these solutions (by enumerating small values of a and b) does not prove that one cannot find, for example, a solution with $a = 2$ and b extremely large. That requires a proof:

Lemma 3 *Let a, b, and c be integers such that $c^2 = a^2 + b^2 + ab$ and $a < b$. Then $a \geq 3$ and $b \geq 5$, and $(3, 5, 7)$ is the only solution with $a = 3$. In general there are (zero or) finitely many solutions for each fixed value of a, because we have the bound*

$$b < c \leq \frac{3a^2 + 1}{2}.$$

Proof. Regard $b^2 + ab + a^2 - c^2 = 0$ as a quadratic equation for b. For the solution to be an integer, the discriminant must be a square. The discriminant is $D = a^2 - 4(a^2 - c^2) = c^2 - 3a^2$. The largest square less than c^2 is $(c - 1)^2$, which is less than c^2 by $2c - 1$. So when $2c - 1$ exceeds $3a^2$ there can be no more solutions found for that a by increasing c. When $a = 1$, there are no solutions with $2c - 1 > 3$; that is $c > 2$. Hence there are no solutions for $a = 1$. When $a = 2$ there are no solutions with $2c - 1 > 12$, which means $c > 6$. One can check by hand that $2^2 + b^2 + 2b = c^2$ has no solutions for $c \leq 6$, so there are no solutions with $a = 2$. When $a = 3$ there are no solutions with $2c - 1 > 27$, which means $c > 14$. We do have the solution $(3, 5, 7)$, but no other solution with $a = 3$. If $a \geq 4$ then $b \geq 5$ because $a < b$. That completes the proof of the lemma.

Whether anything further of interest for the geometry of triangle tiling can be extracted from the number theory of $c^2 = a^2 + ab + b^2$, we do not know. We have explored some possibilities, but they were not fruitful in the end, and we mention only one of them here, namely, the parametrization of the solutions. This is a fundamental result about the equation, and an easy application of known number theory, so we include it here, even though we are not able to derive anything useful from it.

Lemma 4 (Parametrization of the solutions) *Let a, b, and c be a solution of $c^2 = a^2 + b^2 + ab$ with a and b coprime and $a < b$. If both a and b are odd, then there are two coprime positive integers s and t (exactly one of which is even) such that*

$$\begin{aligned} a &= 2st + s^2 - 3t^2 \\ b &= 2st - s^2 + 3t^2 \\ c &= s^2 + 3t^2 \end{aligned}$$

where $t\sqrt{3} < s < 3t$, or with a and b switched and $t < s < t\sqrt{3}$.

If one of a or b is even, then instead we have

$$\begin{aligned} a &= 2st - t^2 \\ b &= s^2 - 2st \\ c &= s^2 + t^2 - st \end{aligned}$$

with $0 < t < s$ and $2st < (s - t)^2$, or with a and b switched and $0 < t < s$ and $2st > (s - t)^2$.

Examples. With $t = 1$ and $s = 2$ we have $t < s < t\sqrt{3}$, and we find $a = 3$, $b = 5$, $c = 7$. But the solution $(5, 16, 19)$ is not given by the first parametrization, since b is even. In the second parametrization, if we take $t = 2$ and $s = 5$, then we have $2st = 20 > (s - t)^2 = 9$, so we have $a = s^2 - 2st = 5$ and $b = 2st - t^2 = 16$, so the solution $(5, 16, 9)$ is parametrized by $(2, 5)$.

Proof. We can reduce the equation (1) to the more familiar equation

$$x^2 + 3y^2 = z^2 \tag{2}$$

by the substitution

$$
\begin{aligned}
x &= \frac{b-a}{2} \\
y &= \frac{a+b}{2} \\
z &= c
\end{aligned}
$$

Then we have

$$
\begin{aligned}
a &= y - x \\
b &= x + y
\end{aligned}
$$

The integral solutions of ternary forms can be given in parametric form, as shown in [5], p. 345, Corollary 6.3.6. Conveniently, Cohen also worked out a special case including (2) in Corollary 6.3.15, p. 353 of [5]. According to that corollary, the general integral solution (x, y) of $x^2+3y^2 = z^2$ with x and y coprime is given by either

$$
\begin{aligned}
x &= \pm(s^2 - 3t^2) \\
y &= 2st \\
z &= \pm(s^2 + 3t^2)
\end{aligned}
\tag{3}
$$

or

$$
\begin{aligned}
x &= \pm(s^2 + t^2 + 4st) \\
y &= s^2 - t^2 \\
z &= \pm 2(s^2 + t^2 + st)
\end{aligned}
\tag{4}
$$

for coprime integers s and t of opposite parity (i.e., one odd and one even); the $\pm$ signs are independent. If the second parametrization (4) holds, then since $a = y - x$ and $b = y + x$, we have either

$$
\begin{aligned}
a &= -2t^2 - 4st \\
b &= 2s^2 + 4st
\end{aligned}
$$

or vice-versa. But then both a and b are even, while we are interested only in solutions in which a and b are relatively prime. Hence we may ignore the second parametrization. Then a and b are given by $y \pm x$ from the first parametrization (3):

$$
\begin{aligned}
a &= 2st + s^2 - 3t^2 \\
b &= 2st - s^2 + 3t^2
\end{aligned}
$$

or vice-versa. Since we are only interested in solutions with $0 < a < b$, we want only positive solutions for x and y. The condition $y > 0$ tells that $2st > 0$, so s and t must have the same sign (which we can take to be positive). The condition $x > 0$ requires (and is equivalent to) choosing the plus or minus sign in (3) according as $s > \sqrt{3}t$ or not. The condition $a > 0$ is equivalent to $x < y$. If $s > \sqrt{3}t$ then we take the positive sign, so $x < y$ is equivalent to $s^2 - 3t^2 < 2st$, which is equivalent to $s < 3t$:

$$
\begin{aligned}
s^2 - 3t^2 &< 2st \\
s^2 - 2st + t^2 - 4t^2 &< 0 \\
(s-t)^2 - (2t)^2 &< 0 \\
(s - t - 2t)(s - t + 2t) &< 0 \\
(s - 3t)(s + t) &< 0 \\
s &< 3t \qquad \text{since } s > 0
\end{aligned}
$$

Therefore we can have $\sqrt{3}t < s < 3t$. Alternately if we take the minus sign in (3), which means $0 < s < \sqrt{3}t$ to make $x > 0$, then the condition $x < y$ becomes

$$\begin{aligned} 3t^2 - s^2 &< 2st \\ (2t)^2 - (t+s)^2 &< 0 \\ (2t - (t+s))(2t + (t+s)) &< 0 \\ (t-s)(3t+s) &< 0 \\ t &< s \end{aligned}$$

That completes the proof when a and b are both odd.

When one of a or b is even, we get half-integral solutions x and y of the transformed equation. Multiplying them by 2 we get integral solutions. We parametrize those and then divide by 2 again. For the doubled solution, $z = 2c$ is even, so we have to use the second parametrization, which we rejected when a and b were odd. We then find

$$\begin{aligned} c &= s^2 + t^2 + st \qquad \text{(we must take the + sign)} \\ a &= (y-x)/2 \;=\; -2st - t^2 \\ b &= (y+x)/2 \;=\; 2st + s^2 \end{aligned}$$

or with a and b switched. The condition $a < b$ is equivalent to $y > 0$, which now means $t^2 < s^2$. The condition $a > 0$ (or $b > 0$ if the expressions are switched) tells us $st < 0$, indeed $2st < -t^2$, so s and t have opposite signs. Switching the sign of s or t (whichever one is negative) we can write the parametrization in a form in which can can assume t and s are both positive:

$$\begin{aligned} c &= s^2 + t^2 - st \\ a &= 2st - t^2 \\ b &= s^2 - 2st \end{aligned}$$

Then the condition $b > 0$ (or $a > 0$ if the equations are switched) is equivalent to $t < s$. The condition $a < b$ for the formulas as displayed is $2st - t^2 < s^2 - 2st$, or $2st < (s-t)^2$; otherwise we switch a and b. That completes the proof of the lemma.

Lemma 5 *Suppose a, b, and c are odd coprime integers satisfying $c^2 = a^2 + b^2 + ab$. Then $a + b$ is congruent to 0 mod 8, so if $a \equiv 3$ mod 8 then $b \equiv 5$ and if $a \equiv 1$ then $b \equiv 7$ mod 8.*

Examples. (49, 575, 601) has $575 \equiv 3$ mod 4. The hypothesis that a and b are both odd is not superfluous, as is shown by $(a, b, c) = (5, 16, 19)$, where $a \equiv 1$ mod 8 and $(a, b, c) = (819, 1600, 2131)$, where $a \equiv 3$ mod 8.

Proof. Suppose a and b are both odd. Then by Lemma 4, we have $a = 2st + s^2 - 3t^2$ and $b = 2st - s^2 + 3t^2$, or vice-versa. Then $a + b = 4st$ is congruent to 0 mod 8, since one of s and t is even. That completes the proof of the lemma.

We could not, however, manage to use the parametrization of the solutions to prove anything useful when one of a or b is even.

3 The d matrix when a, b, and c are integers

We cannot, at present, rule out the existence of N-tilings by tiles (a, b, c) with integer side lengths, $a \neq b$, and a 120° angle, although no examples are known. In this section, we wish to rule them out for as many values of N as we can. Since we know the vertex splitting has to be $(3, 3, 3)$, there are only a few shapes of ABC to consider. If we assume $\alpha < \beta$ there are only four possible shapes:

- ABC is equilateral, or
- ABC has angles 2α, 2β, and $\alpha + \beta = \pi/3$, or
- ABC is isosceles with base angles α
- ABC has angles α, $\alpha + \beta$, and $\alpha + 2\beta$

In the following two lemmas, we strive to place some restrictions on the possible boundary behavior of a tiling. That is, to place some restrictions on the possible entries of the **d** matrix. We find one restriction based on geometry, and another restriction based on linear algebra, viewing the **d** matrix as a transformation on a vector space.

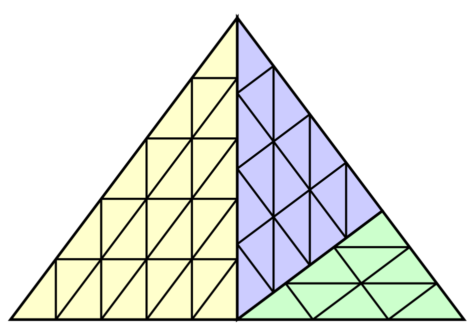
Lemma 6 *Let T be a triangle with an angle $\gamma > \pi/2$ and integer sides a, b, and c. Suppose there is an N-tiling of triangle ABC by T, and ABC is not similar to T, and each of the angles of ABC is less than γ. Then every side of ABC has some c edges (i.e. the right hand column of the* **d** *matrix has only nonzero entries).*

Proof. Suppose some side U of ABC has no c edges. Then every tile with an edge on U has its a or b edge on U, and hence has its γ angle on U. Since $\gamma > \pi/2$, there cannot be two γ angles at the same vertex on U. Since each of the angles of ABC is less than γ, there is no γ angle at either end of U. But this contradicts the pigeonhole principle. That completes the proof of the lemma.

Theorem 1 *Let T be any triangle with integer sides a, b, and c. Suppose there is an N-tiling of triangle ABC by T. Then every row of the* **d** *matrix has a zero.*

Remark. We do not need to assume that T has a 120° angle! Here is a figure illustrating the lemma for another shape of tile. As you see, no side has tiles of all three edge lengths. That is not accidental!

Figure 1: A tiling related to a Pythagorean triple $a^2 + b^2 = c^2$.



Proof. Suppose, for proof by contradiction, that the first row of the **d** matrix has all nonzero entries. Then there are nonzero numbers (the entries of the first row) p, d, and e such that $pa + db + ec = 0$. We inquire into the rank of the **d** matrix. The kernel of the **d** matrix is not the entire space, since

$$\mathbf{d}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

and the vector on the right is not zero. Consider an integral linear relation $ua + vb + wc = 0$, with u, v, and w integers. (They can be positive or negative or zero, but not all zero). Now suppose some row of the $\mathbf{d}$ matrix contains all nonzero entries. For example, suppose p, d, and e are all nonzero in the first row. Then let

$$\mathbf{v} \;=\; \begin{pmatrix} \frac{au}{p} \\ \frac{vb}{d} \\ \frac{wc}{e} \end{pmatrix}$$

Then

$$\mathbf{dv} \;=\; \begin{pmatrix} p & d & e \\ g & m & f \\ h & \ell & r \end{pmatrix} \begin{pmatrix} \frac{au}{p} \\ \frac{vb}{d} \\ \frac{wc}{e} \end{pmatrix}$$

and the first element of this vector is 0, since

$$p\Big(\frac{au}{p}\Big) + d\Big(\frac{vb}{d}\Big) + e\Big(\frac{wc}{e}\Big) \;=\; au + vb + wc \;=\; 0$$

The point is that $\mathbf{dv}$ lies in the two-dimensional subspace W spanned by $(0,1,0)$ and $(0,0,1)$. Now, suppose there are three linearly independent integral relations $u_ia + v_ib + w_ic = 0$, for $i = 1,2,3$, where we call the relations "linearly independent" if the vectors

$$v_i = \begin{pmatrix} \frac{au_i}{p} \\ \frac{bv_i}{d} \\ \frac{cw_i}{e} \end{pmatrix} \tag{5}$$

are linearly independent. Then $\mathbf{d}$ takes the entire space $\mathbb{R}^2$ into W. That is, however, not the case, since $\mathbf{d}$ takes (a,b,c) into (X,Y,Z), which does not belong to W. If it was not the first row but the second or third row of $\mathbf{d}$ that was nonzero, then W will be a different two-dimensional space that also does not contain (X,Y,Z); there is no loss of generality in assuming that it was the first row of $\mathbf{d}$ that was nonzero.

Next we note that three vectors of the form (5) are linearly independent if and only if the the determinant of the matrix having them for columns is nonzero. Since there cannot be three such linearly independent vectors, the determinant is zero, whenever we have three relations $u_ia + v_ib + w_ic = 0$:

$$\begin{vmatrix} \frac{au_1}{p} & \frac{au_2}{d} & \frac{au_3}{e} \\ \frac{bv_1}{p} & \frac{bv_2}{d} & \frac{bv_3}{e} \\ \frac{cw_1}{p} & \frac{cw_2}{d} & \frac{cw_3}{e} \end{vmatrix} = 0$$

This determinant is the sum of six terms, each of which contains the constant $1/(pde)$, so it is zero if and only if the determinant of the elements without those denominators is zero:

$$\begin{vmatrix} au_1 & au_2 & au_3 \\ bv_1 & bv_2 & bv_3 \\ cw_1 & cw_2 & cw_3 \end{vmatrix} = 0$$

That is, (a,b,c) is in the kernel of

$$\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}$$

That kernel $\mathcal{D}$ consists precisely of vectors (u,v,w) that are orthogonal to (a,b,c). Now $\mathcal{D}$ contains, for example, the linearly independent vectors $(b,-a,0)$ and $(0,b,-c)$, and since $\mathcal{D}$ does not contain the orthogonal vector (a,b,c), $\mathcal{D}$ is two-dimensional and hence generated by these two vectors. Now consider the third vector $(c, 0, -a)$, which is also in $\mathcal{D}$. It must be a linear combination of the first two. That is, there exist constants λ and μ such that

$$\begin{aligned} \begin{pmatrix} c \\ 0 \\ -a \end{pmatrix} &= \lambda \begin{pmatrix} b \\ -a \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ b \\ -c \end{pmatrix} \\ &= \begin{pmatrix} \lambda b \\ -\lambda a + \mu b \\ -\mu c \end{pmatrix} \end{aligned}$$

Then $c = \lambda b$ and $\lambda a = \mu b$ and $a = \mu c$. We have

$$\begin{aligned} \lambda a &= \mu b \\ \lambda \mu c &= \mu b \qquad \text{since } a = \mu c \end{aligned}$$

Since $\mu = a/c$ we have $\mu \neq 0$, so we can divide by μ:

$$\lambda c = b$$

But we also have $c = \lambda b$. Hence $c = \lambda c = \lambda(\lambda c) = \lambda^2 c$. Since $c \neq 0$ we can divide the equation $c = \lambda^2 c$ by c, obtaining $\lambda^2 = 1$. Since $c = \lambda b$ and both c and b are positive, we have $\lambda = 1$. But then $c = \lambda b = b$, contradicting $b < c$. That completes the proof of the lemma.

Lemma 7 *Let T be a triangle with a 120° angle and integer sides a, b, and c. Suppose there is an N-tiling of triangle ABC by T, and ABC is not similar to T. Then no side of ABC has both a and b edges on it.*

Proof. Suppose, for proof by contradiction, that some side of ABC has both a and b edges on it. Then the corresponding row of the **d** matrix has nonzero entries in column 1 and column 2. By Theorem 1, the third column must have a zero entry, but that contradicts Lemma 6. That completes the proof.

4 Tilings of an equilateral triangle

So far, we have worked only with the law of cosines equation and the **d** matrix. Our other main tool is the "area equation", according to which N times the area of the tile is the area of ABC. To work with that equation when a and b are not squarefree, the following concepts will be helpful.

Definition 1 *The* **squarefree part** *of x is the product of (one power each of) the primes that divide x to exactly an odd power.*

I could not find the following concept in number theory books, so I gave it a name.

Definition 2 *The* **square divider** *of x is the product of the prime powers p^j where p^{2j} or p^{2j-1} is the exact power of p that divides x.*

Examples. If x is squarefree, then the square divider of x is just x, and the squarefree part of x is also just x. The square divider of 80 is 20. The squarefree part of of 80 is 5.

The following lemma gives the basic properties of the squarefree part and square divider:

Lemma 8 *(i) If x divides y^2, then the square divider of x divides y.*

(ii) If s is the square divider of x then s^2/x is the squarefree part of x.

(iii) If $Nx = y^2$ then N is a square times the squarefree part of x.

Proof. Ad (i): Let p^{2j} or p^{2j-1} be a prime power dividing x. Then p^{2j} or p^{2j-1} divides y^2, so p^j divides y. But the product of these prime powers is the square divider of x, by definition.

Ad (ii): Primes appearing to an even power in x occur to the same power in s^2 and in x, so they do not occur at all in s^2/x. Primes appearing to an odd power in x occur one more time in s^2 than in x, so they occur just once in s^2/x.

Ad (iii): Suppose $Nx = y^2$. Then x divides y^2, so the square divider s of x divides y, i.e. $y = ks$ for some k. Then $y^2 = k^2s^2 = Nx$. Then $N = k^2s^2/x$, which by (ii) is k^2 times the squarefree part of x. That completes the proof of the lemma.

Lemma 9 *Let T be a triangle with a $120°$ angle and integer sides (a, b, c). Suppose there is an N-tiling of an equilateral triangle by T. Let d be the squarefree part and s the square divider of ab, and X the length of a side of ABC. Then for some integer $k \geq 2$, we have $N = k^2d$ and $X = ks$. (In case ab is squarefree, we have $d = s = ab$.)*

Proof. Suppose ABC is equilateral and N-tiled by (a, b, c). By the area equation, $Nab = X^2$, where X is the side length of ABC. The relationship between d and s is that $s^2 = abd$. By Lemma 8, part (i), s divides X; let k be the integer such that $ks = X$. Then the area equation gives us

$$\begin{aligned} Nab &= X^2 \\ N\left(\frac{s^2}{d}\right) &= X^2 \\ N\left(\frac{s^2}{d}\right) &= (ks)^2 \\ N &= dk^2 \end{aligned}$$

That completes the proof of the lemma.

Lemma 10 *Let T be a triangle with a $120°$ angle and integer sides (a, b, c). Suppose there is an N-tiling of some triangle (not necessarily equilateral) by T. Let d be the squarefree part and s the square divider of ab, and X the length of a side of ABC. Then we do not have $X = s$.*

Proof. We continue with the same notation as in the previous lemma. It suffices to show $k \neq 1$, that is $N \neq d$, since then $k \geq 2$ implies $N \geq 4d$.

According to Lemma 7, X can be written in the form $uc + vb$ or the form $uc + va$ for some integers $u > 0$ and $v \geq 0$. If we do not assume $a < b$, we can without loss of generality assume $X = uc + vb$. Assume, for proof by contradiction, that $X = s$. For intelligibility we first give the proof under the assumption that ab is squarefree, then remove that assumption. If ab is squarefree we have $X = s = ab$. Then

$$\begin{aligned} ab &= X \\ &= uc + vb \end{aligned}$$

Taking the equation mod b we have $u \equiv 0 \bmod b$. But $u > 0$, so $u = \ell b$ for some $\ell \geq 1$. Then $X = b(\ell c + v)$. But $X = ab$, so $a = \ell c + v$. Then $a \geq c$, which is a contradiction.

Now we give the proof without assuming that ab is squarefree. Then let s_1 and s_2 be the square dividers of a and b, respectively, and d_1 and d_2 the squarefree parts of a and b. Then $d = d_1d_2$ and $s = s_1s_2$. Now we have

$$\begin{aligned} s &= X \\ &= uc + vb \end{aligned}$$

Instead of taking the equation mod b, we take it mod s_2. Since $s = s_1 s_2$ we get zero on the left. Since s_2 divides b, we get $u \equiv 0 \bmod s_2$. Hence

$$u = \ell s_2 \qquad \text{for some } \ell \geq 1$$

Then

$$\begin{aligned} X &= uc + vb \\ s_1 s_2 &= uc + vb \qquad \text{since } X = s = s_1 s_2 \\ s_1 s_2 &= \ell s_2 c + vb \qquad \text{since } u = \ell s_2 \\ &= \ell s_2 c + v s_2^2 / d_2 \qquad \text{since } b = s_2^2 / d_2 \\ &= s_2(\ell c + v s_2 / d_2) \end{aligned}$$

Dividing by s_2 we have

$$s_1 = \ell c + v/d_2$$

The last term v/d_2 is not necessarily an integer, but no matter–it only needs to be nonnegative. We have

$$\begin{aligned} a &\geq s_1 \qquad \text{since } s_1 \text{ divides } a \\ &\geq \ell c + v/d_2 \\ &\geq c \qquad \text{since } \ell \geq 1 \text{ and } v/d_2 \geq 0 \end{aligned}$$

But $a \geq c$ is a contradiction, since $a < c$. This contradiction shows that $X \neq s$. That completes the proof of the lemma.

Lemma 11 *Let T be a triangle with a 120° angle and integer sides (a, b, c). Suppose there is an N-tiling of an equilateral triangle by T. Let d be the squarefree part and s the square divider of ab, and X the length of a side of ABC. Then $N = k^2 d$ with $k \geq 2$; in particular $N \geq 4d$.*

Example. There is no 15-tiling of an equilateral triangle by the tile $(3, 5, 7)$; indeed if $(3, 5, 7)$ N-tiles an equilateral triangle, then $N \geq 60$.

Proof. By Lemma 9, we have $N = k^2 d$ and $X = ks$, for some integer k, where as before d is the squarefree part of ab and s is the square divider of ab. It only remains to prove $k \geq 2$. Since k is an integer and d and N are positive, the only other possibility is $k = 1$. But if $k = 1$ then $X = s$, contradicting Lemma 10. That completes the proof of the lemma.

Lemma 12 *Let N be a positive integer, and let T be a non-isosceles triangle with a 120° angle and sides a, b, c, with c opposite the 120° angle. Suppose there is an N-tiling of an equilateral triangle ABC by T. Then $N \geq c$.*

Remark. This is a key lemma, because it gives a lower bound on N in terms of the size of the tile. Until now, we could not rule out the possibility that there are tilings with relatively small N, but the sides of the tile are huge. Of course, we have a lower bound on N of sorts, in that the squarefree part of ab divides N, but why couldn't a and b both be gigantic squares, or at least have small squarefree part, and N fairly small? The number theory of $c^2 = a^2 + ab + b^2$ alone is probably not sufficient to prevent that. This lemma answers that question, at least for equilateral triangles.

Proof. Recall that the area equation can be written as

$$Nab\frac{\sqrt{3}}{2} = XY \sin\theta$$

where X and Y are two sides of triangle ABC and θ is the angle between those sides. (Each side of the equation is twice the area of ABC.) In the case of an equilateral ABC, $\theta = \pi/3$ and $X = Y$, so we have

$$Nab = X^2.$$

We may assume that a, b, and c have no common factor, and then by Lemma 2, a, b, and c are pairwise relatively prime.

According to Theorem 1 and Lemma 6, we have either $X = pa+ec$ with $e > 0$, or $X = db+ec$ with $e > 0$. If we do not assume $a < b$, then without loss of generality we can assume $X = pa+ec$. Since $Nab = X^2$, a divides X^2. So $X^2 \equiv 0 \bmod a$. But $X^2 = (pa+ec)^2 \equiv e^2c^2 \bmod a$. Then $e^2c^2 \equiv 0 \bmod a$. Since c and a are relatively prime, we have $e^2 \equiv 0 \bmod a$. Now let s be the square divider of a and let t be the square divider of b. Since a divides e^2, Lemma 8 tells us s divides e. By Lemma 9, there is an integer $k \geq 2$ such that $X = kst$ and $N = k^2d$, where d is the squarefree part of ab. We have

$$\begin{aligned} ec &\leq pa + ec \\ &= X \\ &= kst \\ &\leq ket \qquad \text{since } s \text{ divides } e \end{aligned}$$

Thus we have proved $ec \leq ket$. Dividing both sides by e we have $c \leq kt$. Now b divides t^2, so $t^2 \geq b$. Then

$$\begin{aligned} c &\leq kt \\ &\leq k\sqrt{b} \qquad \text{since } t^2 \geq b \\ c^2 &\leq k^2 b \\ &\leq N\left(\frac{b}{d}\right) \qquad \text{since } k^2 d = N \\ N &\geq \frac{c^2 d}{b} \\ &\geq \frac{c^2}{b} \qquad \text{since } d \geq 1 \\ &\geq c \qquad \text{since } c > b \end{aligned}$$

That completes the proof of the lemma.

Lemma 13 *There is no 60-tiling of an equilateral triangle by the tile with sides* $(3, 5, 7)$.

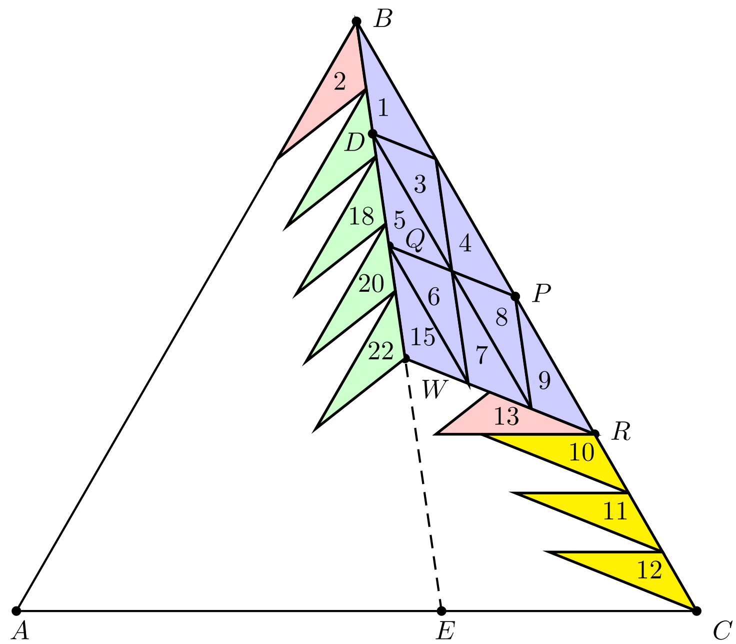
Proof. Suppose ABC is an equilateral triangle tiled by the $(3, 5, 7)$ triangle. For convenience we orient ABC with B at the north, and label A and C so that the tile at vertex B and an edge on AB has its β angle at B. We may suppose A is at the southwest and AC is horizontal (east-west). By the area equation, we have $Nab = X^2$, where X is the length of each side of ABC; since $ab = 15$ and $N = 60$, we must have $X = 30$.

By Lemma 6 we know that every side of ABC has at least one c edge on it; and by Theorem 1, we know that no side has both an a edge and a b edge. We now consider the possibilities for a row of the $\mathbf{d}$ matrix; to fix the notation we consider the first row. Then $X = pa + db + ec$. If $e = 1$ we have $23 = 3p$ or $23 = 5d$, both of which are impossible, so $e \neq 1$. If $e = 2$ we have $16 = 3p$ or $16 = 5d$, both of which are impossible, so $e \neq 2$. If $e = 3$ we have $9 = 3p$, since $9 = 5d$ is impossible. Since $X - 3c = 2 < a$, we cannot have $e > 3$. Hence each row of the $\mathbf{d}$ matrix must be $(3, 0, 3)$; that is, each side of ABC is composed of 3 edges of length a and 3 edges of length c.

Let E be a point on AC such that BE contains the tile boundary between the two tiles at vertex B. Let Tile 1 be the tile on the east of BE at B. Then Tile 1 has its α angle at B. It cannot have its b edge on BC, so it has its c edge there. Then it has its b edge on BE. Let Tile 2 be the tile on the west of BE, with its β angle at B. There are two possible orientations of Tile 2, either with its a or its c edge on AB. Accordingly it may have its a or its c edge on BE. The configuration with the a edge on BE is shown in Fig. 2, but we are not assuming in the proof that the illustrated configuration is the one that occurs. In either case it does not share its southern vertex with Tile 1. Let H be the southern end of the maximal segment lying on BE with its northern end at B. Then H is not the southern vertex D of Tile 1, since either the boundary of Tile 2 extends south of P, or there is $b - a = 2$ remaining on BD south of Tile 2, so whatever tile is west of that part of BD extends south of D on B. Let Tile 3 be south of Tile 1; then Tile 3 shares its a edge with Tile 1. Tile 5 cannot have its γ angle at D because BH passes through D. Therefore Tile 3 has its γ angle at the east, and it forms a parallelogram with Tile 1. Let Tile 4 be east of Tile 3; then Tile 4 has its α angle at the north, and since it cannot have its b edge on BC, it must have its c edge on BC. Let Tile 5 be west of Tile 3; then Tile 5 has its α angle to the north, but there are two possible orientations of Tile 5.

The situation is illustrated in Fig. 2. The figure also shows a number of tiles that have not yet been discussed, and it shows Tile 2 in a particular orientation, one of two possible orientations.

Figure 2: No 60-tiling of equilateral ABC by $(3, 5, 7)$



Then the southern edges of Tiles 4 and 5 form a line segment of length $2a$; let Q be the west end of that segment, the southwest vertex of Tile 5, which lies on BE. Let P be the east end of that segment, which lies on BC.

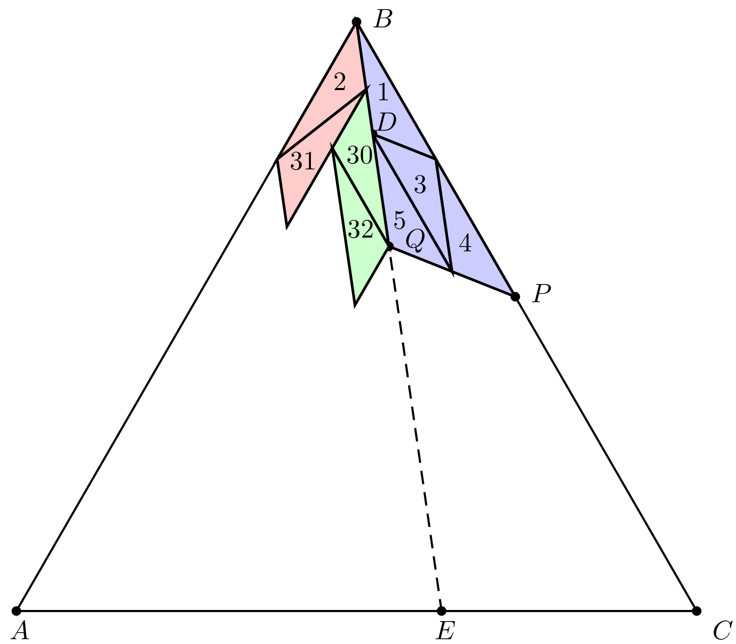
Let Tile 6 be the tile south of Tile 5, sharing an edge or part of an edge with Tile 5, and sharing vertex Q or else Q lies on an edge of Tile 6. We claim that actually Tile 6 shares an a edge exactly with Tile 5 and has its γ angle to the east.

We distinguish two cases: Either there is, or there is not, a tile west of BH with an edge on BH and a vertex at Q. If there is, let it be Tile 30. Since BQ has length $2b = 10$, and Tile 2 has either an a or c edge on BE, there are two subcases:

- Subcase 1A, Tile 2 has its a edge on BE and Tile 30 has its c edge on BE, or
- Subcase 1B, Tile 2 has its c edge on BE and Tile 30 has its a edge there.

We take up Subcase 1A. The situation is illustrated in Fig. 3. There is a tile, say Tile 31, south

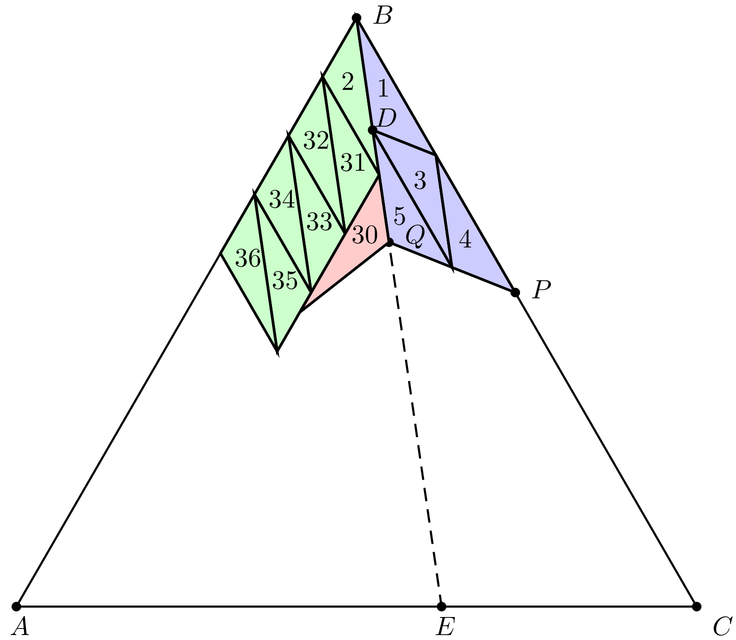
Figure 3: Subcase 1A, Tiles 2 and 30 are as illustrated.



of Tile 2; this tile (and there is only one) must share its b edge with Tile 2. It cannot have its γ angle on BE, since B continues south to at least D. Hence it has its γ angle on AB and forms a parallelogram with Tile 2. Then it has its α angle on BE. Therefore Tile 30, which has its c edge on BE, must have its β angle to the north, as the angle between Tile 31 and BE is exactly β and cannot be filled by two or more angles. Tile 31 then extends west of Tile 30's northwest vertex, so a tile west of Tile 30, say Tile 32, must share that vertex and an edge with Tile 30. If Tile 32 has a vertex at Q then it shares the entire western edge of Tile 30 (which is of length b). It cannot have its γ angle at the north, because Tile 31 is there, so it has its γ angle at Q. Now there are two γ angles at Q as well as either an α or a β angle, so Tile 6 must have its a edge against Tile 5 and its γ edge to the east, as desired. Therefore we may assume, without loss of generality, that Tile 32 does not have a vertex at Q. Then the edge that it shares with Tile 30 is either shorter than b or longer than b, since b is the length of the west edge of Tile 30. If it is shorter, then some other tile shares the rest of the west edge of Tile 30 and extends east of BE at Q. If it is longer, then the edge of Tile 32 itself extends east of BE at Q. Either way, Tile 6 cannot have any part west of Q and cannot have its γ angle at Q. That disposes of Subcase 1A.

We take up Subcase 1B, in which Tile 2 has its c edge on BE and Tile 30 has its a edge on BE. Consider the tile(s) southwest of Tile 2 and sharing an edge with Tile 2. These cannot extend east of the southeast corner of Tile 2, because Tile 30 is there; nor can they extend west of Tile 2, because the boundary AB is there. Hence there is only one such tile, say Tile 31, and it shares the b edge of Tile 2. Since Tile 2 has its γ angle on AB, and $\gamma > \pi/2$, Tile 31 cannot have its γ angle on AB. Therefore Tile 31 has its γ angle at the southeast vertex of Tile 2, which is the northern vertex of Tile 30. Hence Tile 30 cannot have its γ angle there. Hence the γ angle of Tile 30 is at Q. The situation is illustrated in Fig. 4.

Figure 4: Subcase 1B, Tiles 2 and 30 are as illustrated.



The western boundary of Tile 30 is parallel to AB and is of length $c = 7$. The southwest boundary of Tile 2 is parallel to BC. It is now forced that the next six tiles southwest of Tile 2 and west of Tile 30 must be placed in a lattice tiling as shown in the figure. But this will make four a edges on AB, contradiction, since there must be just three a edges on each side of ABC. That disposes of Subcase 1B.

Together these two subcases dispose of Case 1, in which there is a tile west of BQ with a vertex at Q. We now take up Case 2, in which there is no tile west of BQ with a vertex at Q. Then Q lies on an edge of a tile west of BH, and that edge extends south of Q. Then there are two more tiles, Tiles 6 and 8, south of Tiles 5 and 4 respectively, sharing their a edges, since $2a$ cannot be made up of other edge lengths, and Tile 6 does not have its γ angle at Q, since the angle EQP available to Tile 6 is equal to only $\pi/3$. By the pigeonhole principle, Tiles 6 and 8 both have their γ angles to the east. Let Tile 7 be between Tiles 6 and 8, and Tile 9 east of Tile 8. Then Tile 9 has its c edge on BC, and we have used up our allotted three c edges on BC. Hence the rest of BC is made of three a edges, belong to tiles 10, 11, and 12 respectively. Tile 12 has a vertex at C, and hence it cannot have its γ angle to the south. By the pigeonhole principle, all three of Tiles 10, 11, and 12 have their γ angles to the north. Let Tile 13 be between Tile 9 and Tile 10; then Tile 13 has its α angle at its vertex R on BC (the vertex it shares with Tiles 9 and 10), and its northern edge must contain (at least part of) the southern boundary of Tile 7, forcing Tile 7 to share the vertices of its neighbors Tiles 6 and 8.

(One of the two possible orientations of Tile 13 has been illustrated in the figure.) Now let Tile 14 be the next tile along the southern boundary of Tiles 7 and 9 west of Tile 13. (Tile 14 is not shown in Fig. 2.) Let R be the endpoint on BC of the southern boundary of Tiles 7 and 9. Let W be the point where the southern boundary of Tiles 7 and 9, if extended westward, meets BE. Let Tile 15 be southwest of Tile 6. Tile 15 cannot extend below RW since Tile 13 or Tile 14 is there, extending west of the southern vertex of Tile 6. The only way to fit in Tile 15 is if Tile 15 shares the c edge of Tile 6 and forms a parallelogram with Tile 6. Hence the southern endpoint H of the maximal segment BH does not lie strictly north of W.

What is the length of the two edges of Tiles 13 and 14 along RW? Tile 13 has a b or c edge, and we do not know the edge of Tile 14; but whatever it is, the sum of the two edges is at least $b + a = 8$, and if it is not that, then it is at least $c + a = 10$ or $2b = 10$. The length from point Q (on BC) to line BE along the southern boundary of Tiles 7 and 9 is $3a = 9$. If the edges of Tiles 7 and 9 on RW add up to 8, there is not enough room to fit another tile edge east of W; so whether they add to 8 or to at least 10, the west endpoint of the maximal segment along RW is west of BE, and the southern endpoint H of the maximal segment BH must be W.

But as we have seen, on the west side of BH, there is at least one c or a edge, belonging to Tile 2 at the north end of BH. That leaves either $3b - a = 12$ or $3b - c = 8$ along the west side of BH south of Tile 2. It is not possible to make 8 from a, b, and c edges; thus Tile 2 has its a edge on BE (as shown in Fig. 2, but until now not proved to be necessarily the case), and so do the next four tiles with edges on BE to the south, so that one the west, the segment BH is composed of five a edges. Then by the pigeonhole principle, all these five tiles west of BH have their γ angles to the south. The southernmost of these four then blocks the extension of RW west of W, contradiction.

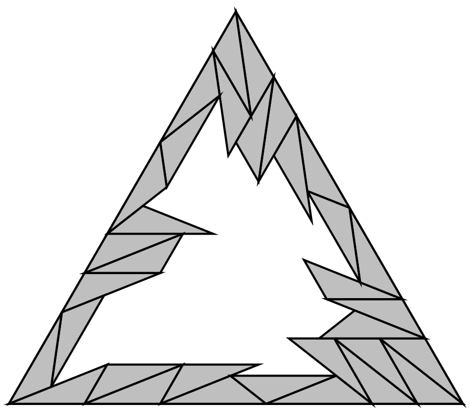
This contradiction has been reached under several assumptions. We have assumed that Tile 5 has the illustrated orientation, and we have assumed that QP does not extend west of BE, and that Tile 6 does not have its γ angle at Q.

Tile 2 has its b edge on the south and its γ angle on BE. Let Tile 15 be the tile south of Tile 2, and Tiles 16 and 18 the next two tiles west of BH with their a edges on BH. Tile 15 has its γ angle on BH to the south, since there is no room for the γ angle at the north; then by the pigeonhole principle, so to Tiles 17 and 19 have their γ angles to the south. But Tile 19 cannot have its γ angle to the south, since the tile boundary at H extends west of BH, and would enter the interior of Tile 19 if Tile 19 had a γ angle at H. This is a contradiction. That contradiction shows that Tile 5 cannot, after all, have its c edge against that of Tile 3.

Therefore, instead, Tile 5 has its b edge against the c edge of Tile 3, and has its c edge on B. Let R be the southern vertex of Tile 5 on BE, and let U be the vertex of Tile 5 lying on the west boundary of Tile 3. Then the length of BR is $b + c = 5 + 7 = 12$. Let Tile 6 be southeast of Tile 5, and let Tile 7 be the tile between Tile 6 and Tile 3. Then Tile 6 shares 2 units of boundary with Tile 3 and entends on south. Hence Tile 8, south of Tile 4, shares its a edge with Tile 4 and forms a parallelogram with Tile 4. Tile 9, east of Tile 8, has its c edge on BC, since its a edge is opposite its α angle, which is to the north. Now we have used up the three c edges on BC, so south of Tile 9 we have Tile 10, sharing a vertex on BC with Tile 9 and having its α angle there, and then Tiles 11, 13, and 14 have their a edges on BC with their γ angles to the north, since Tile 14 cannot have its γ angle to the south, since its southeast vertex is C. Let V be the southern vertex of Tile 8, which lies on the interior of the northern boundary of Tile 10. Then UV has length $2 + 7 = 9$, so must be composed of three a edges of tiles lying west of UV. Each of those three tiles therefore has a γ angle on UV. But there cannot be a γ angle west of UV at U other than the one belonging to Tile 5, and there cannot be a γ angle at UV at V, since the angle between UV and the northern boundary of Tile 10 is β. This contradicts the pigeonhole principle. That completes the proof of the lemma.

Remark. We first tried to prove the preceding lemma by hand. When we failed, we then tried to prove it with the help of a computer program that conducts a depth-first search for a boundary tiling and then tries to complete it. See Fig. 5. There are, however, many thousands of boundary tilings, enough to cause technical difficulties with this approach; and when we decided to modify the search to proceed from the top of the triangle down the dividing line between the two top tiles, we found that a contradiction could already be obtained by hand relatively soon in the search. Therefore we present a computer-free proof. There is a certain tension between the need to complete a maximal segment along the line between the two top tiles and the requirement not to use any b edges on the boundary of ABC (see the proof), but still the proof seems not to be very general.

Figure 5: A boundary tiling that cannot be completed



Lemma 14 *Let T be a triangle with integer side lengths $(a, b, c) = (4, 6, 19)$. Suppose there is an N-tiling of an equilateral triangle ABC by T. Suppose $N = k^2 d$ and $X = ks$ where X is some side of ABC, and d and s are the squarefree part and square divider of ab. Then $N \geq 135$.*

Proof. The squarefree part of ab is 5. The square divider of ab is $s = 20$. So $N = 5k^2$ and $X = 20k$ for some integer k, where X is the length of a side of the tiled triangle ABC. The cases to consider are $k = 2, 3, 4$, corresponding to $N = 20, 45, 80$. (Next after that is 135.)

In each case, we first investigate what $\mathbf{d}$ matrices are possible, given that each side of ABC has at least one c edge and not both a and b edges.

Case 1, $N = 20$, $k = 2$. Then $X = 40$. Since each side of ABC has at least one c edge, and $c = 19$, if we use one c edge that leaves 21, which is not a multiple of 5 and not a multiple of 16. If we use two c edges that leaves only 2, which is impossible. Hence Case 1 is ruled out.

Case 2, $N = 45$, $k = 3$, $X = 60$. If we use j edges of length c, that leaves $60 - 19j$, which reduces mod 5 to j, and hence is not zero for $j = 1$ to 4; but $4 \cdot 19 > 60$, so this can never be zero. On the other hand $60 - 19j$ reduces mod 16 to $12 - 3j$, which is not zero for $j = 1, 2, 3$, and $j = 4$ is already impossible, so Case 2 is ruled out.

Case 3, $N = 80$, $k = 4$, $X = 80$. If we use j c edges that leaves $80 - 19j$. Mod 5 this is congruent to j, so it is not zero for j between 1 and 4, inclusive, but those are all the possible values of j. Mod 16, $80 - 19j$ is congruent to $-3j$; since 3 is relatively prime to 16, this is zero only when j is divisible by 16. Hence there are no possible $\mathbf{d}$ matrices for $N = 80$. That completes the proof of the lemma.

Lemma 15 *Let T be a triangle with integer side lengths $(a, b, c) = (7, 8, 13)$. Suppose there is an N-tiling of an equilateral triangle ABC by T. Suppose $N = k^2 d$ and $X = ks$ where X is some side of ABC, and d and s are the squarefree part and square divider of ab. Then $N \geq 224$.*

Remark. The hypotheses are fulfilled for equilateral ABC, but we shall see below that they are also fulfilled for isosceles ABC, so we want to state this lemma in sufficient generality to cover both cases.

Proof. The squarefree part of ab is 14. The square divider of ab is $s = 28$. So that $N = 14k^2$ and $X = 28k$ for some integer k, where X is the length of some side of the tiled triangle ABC. The cases to consider are $k = 2, 3$, corresponding to $N = 56, 126$. (Next after that is 224.)

We first investigate what **d** matrices are possible, given that each side of ABC has at least one c edge and not both a and b edges.

Suppose we use e edges of length $c = 13$ and p edges of length a. Then $X = 28k = 7p + 13e$. Mod 7 we have $e = 0$. Since $e \geq 1$ we have $e \geq 7$. When $k = 2$, we have $X = 56$, so $7c = 91 > X$. When $k = 3$ we have $X = 3 \cdot 28 = 84$, so again $7c > X$. Hence we cannot compose X of c and a edges. Suppose we use e edges of length c and d edges of length b. Then $X = 28k = 8d + 13e$. Mod 8 we have $e = 0$, since 13 and 8 are relatively prime. Hence $e \geq 8$; but $8c = 104 > X$. Hence we cannot compose X of c and b edges, either. That completes the proof of the lemma.

Lemma 16 *Let T be a triangle with integer side lengths $(a, b, c) = (9, 56, 61)$. Suppose there is an N-tiling of an equilateral triangle ABC by T. Suppose $N = k^2 d$ and $X = ks$ where X is some side of ABC, and d and s are the squarefree part and square divider of ab. Then $N \geq 135$.*

Proof. The squarefree part of ab is 14. The square divider of ab is $s = 3 \cdot 7 \cdot 4 = 84$. So $N = 14k^2$ and $X = 84k$ for some integer k, where X is the length of a side of the tiled triangle ABC. The cases to consider are $k = 2, 3$, corresponding to $N = 56$ and 126.

We first investigate what **d** matrices are possible, given that each side of ABC has at least one c edge and not both a and b edges.

Suppose we use e edges of length $c = 61$ and p edges of length a. Then $X = 84k = 9p + 61e$. Mod 9 we have $3k = 7e$. When $k = 2$ we have $6 = 7e$ mod 9. Multiplying both sides by 4 we have $e = 6$ mod 9, so $e \geq 6$. Then $6c = 366 > X = 168$, so this is impossible. When $k = 3$ we have $e = 0$ mod 9, so $e \geq 9$, and $9c = 549 > X = 252$, again impossible. If, on the other hand, we use e edges of length c and d edges of length b, we have $X = 84k = 56d + 61e$. Mod 7 we have $0 = 5e$, so $e = 0$ mod 7; so $e \geq 7$. But $7c = 427 > X = 84k$, which is 168 if $k = 2$ and 252 if $k = 3$. That completes the proof of the lemma.

Lemma 17 *Let T be a triangle with integer side lengths $(a, b, c) = (32, 45, 67)$. Suppose there is an N-tiling of an equilateral triangle ABC by T. Suppose $N = k^2 d$ and $X = ks$ where X is some side of ABC, and d and s are the squarefree part and square divider of ab. Then $N \geq 224$.*

Proof. The squarefree part of ab is 10. The square divider of ab is $s = 8 \cdot 3 \cdot 5 = 120$. So $N = 10k^2$ and $X = 120k$ for some integer $k \geq 2$. The cases to consider are $k = 2$ and 3, corresponding to $N = 40$ and 90. (The number 224 in the theorem corresponds to $k = 4$.) Suppose X is made of p edges of length a and e edges of length c. Then

$$\begin{aligned} X &= pa + ec \\ 120k &= 32p + 67e \end{aligned}$$

Mod 32 we have

$$24k \equiv 3e \mod 32$$

Since 3 is relatively prime to 32 we have

$$e \equiv 8k \mod 32$$

With $k = 2$ we have $e \equiv 16$ and with $k = 3$ we have $e \equiv 24$. Then $e \geq 16$ in either case; but $16c = 16 \cdot 67 = 1072$, which exceeds X, since $X = 120k \leq 360$ since $k \leq 3$. Hence the case $X = pa + ec$ is impossible.

Therefore instead,

$$\begin{aligned} X &= db + ec \\ 120k &= 45b + 67e \end{aligned}$$

Mod 45 we have

$$30k \equiv 22e \mod 45$$

Since 2 is relatively prime to 45, we can multiply both sides by -2 mod 45, obtaining

$$\begin{aligned} e &\equiv -60 \mod 45 \\ &\equiv 30 \mod 45 \end{aligned}$$

Hence $e \geq 30$. But we already saw that $e \geq 16$ is impossible. That completes the proof of the lemma.

Theorem 2 *Let N be a positive integer, and let T be a non-isosceles triangle with a 120° angle. Suppose there is an N-tiling of an equilateral triangle ABC by T. Then $N \geq 135$.*

Proof. Let a, b, and c be the side lengths of the tile, as usual. By Lemma 12, $N \geq c$, and by Lemma 9, N has the form $k^2 d$ where d is the squarefree part of ab, and $k \geq 2$. Hence, if there are tilings with $N \leq 135$, then we have $c \leq 135$. Since we have $c \leq 3a^2 + 1$ by Lemma 3, it suffices to examine a finite number of possible tiles (a, b, c). We need only examine those for which $4d \leq 135$, since $N \geq 4d$. Here is a complete list of the tiles (a, b, c) in question, showing exactly the cases when $c \leq 135$ and $4d \leq 135$.

```
(a, b, c)             4d

(3, 5, 7)             60
(5, 16, 19)           20
(7, 8, 13)            56
(9, 56, 61)           56
(32, 45, 67)          40
```

This table was computed by a C program, which is given in the Appendix, so the reader can check its correctness or run it.

We have already checked in a series of lemmas that there are no tilings with $N < 135$ for the tiles listed. Specifically:

Lemma 13 for $(3, 5, 7)$.
Lemma 14 for $(5, 16, 19)$.
Lemma 15 for $(7, 8, 13)$.
Lemma 16 for $(9, 56, 61)$.
Lemma 17 for $(32, 45, 67)$.

Remark. The case $N = 135$ arises with the tile $(3, 5, 7)$, where the squarefree part of ab is 15 and $k = 3$. There are now more possibilities for the **d** matrix than arose with that tile when $k = 2$ and $N = 60$. We were able to dispose of $N = 60$ by hand, but to do something similar for $N = 135$ seems a bit daunting; it should be possible by computer, though.

5 Tilings of an isosceles triangle

Lemma 18 *Let T be a triangle with a 120° angle, and two other angles α and β, and sides a, b, and c. Then $\cos\beta = (b+2a)/(2c)$ and $\cos\alpha = (a+2b)/(2c)$.*

Proof. By the law of cosines we have $c^2 = a^2 + ab + b^2$. By the law of sines we have

$$\sin\beta = \left(\frac{b}{c}\right)\sin\gamma = \left(\frac{b}{c}\right)\frac{\sqrt{3}}{2}.$$

We have

$$\begin{aligned} 2c\cos\beta &= 2c\sqrt{1-\sin^2\beta} \\ &= 2c\sqrt{1-\left(\frac{b}{c}\right)^2\frac{3}{4}} \\ &= \sqrt{4c^2-3b^2} \\ &= \sqrt{4(a^2+b^2+ab)-3b^2} \\ &= \sqrt{4a^2+4ab+b^2)} \\ &= \sqrt{(2a+b)^2} \\ &= 2a+b \end{aligned}$$

Similarly with a and b interchanged, and α and β interchanged. That completes the proof of the lemma.

Lemma 19 *Let α and β be the small angles of a triangle with a 120° angle. Then*

$$\sin\left(\frac{\pi}{3}+2\beta\right) = \frac{a(a+2b)}{c^2}\frac{\sqrt{3}}{2}.$$

Proof.

$$\begin{aligned} \sin\left(\frac{\pi}{3}+2\beta\right) &= (\sin\frac{\pi}{3}\cos 2\beta + \cos\frac{\pi}{3}\sin 2\beta) \\ &= \left(\frac{\sqrt{3}}{2}(1-2\sin^2\beta) + \left(\frac{1}{2}\right)2\sin\beta\cos\beta\right) \end{aligned}$$

By the law of sines, $\sin\beta = (b/c)\sin\gamma = (b/c)\sqrt{3}/2$.

$$\begin{aligned} \sin\left(\frac{\pi}{3}+2\beta\right) &= \left(\frac{\sqrt{3}}{2}(1-\frac{3}{2}\left(\frac{b}{c}\right)^2) + \frac{\sqrt{3}}{2}\left(\frac{b}{c}\right)\cos\beta\right) \\ &= \frac{\sqrt{3}}{4c^2}\left((2c^2-3b^2)+2bc\cos\beta\right) \end{aligned}$$

By Lemma 18 we have $2c\cos\beta = 2a+b$. Putting that in, we have

$$\sin\left(\frac{\pi}{3}+2\beta\right) = \frac{\sqrt{3}}{4c^2}\left((2c^2-3b^2)+b(2a+b)\right)$$

Substituting for c^2 from $c^2 = a^2+ab+b^2$, we have

$$\begin{aligned} \sin\left(\frac{\pi}{3}+2\beta\right) &= \frac{\sqrt{3}}{4c^2}\left((2(a^2+ab+b^2)-3b^2+b(2a+b)\right) \\ &= \frac{\sqrt{3}}{4c^2}(2a^2+4ab) \\ &= \frac{\sqrt{3}}{2c^2}(a(a+2b)) \end{aligned}$$

That completes the proof of the lemma.

Lemma 20 *Let T be a tile with integer side lengths a, b, and c, and one 120° angle. Suppose the triangle ABC is isosceles with base angles α. Suppose ABC is N-tiled by T. Assume α is the angle opposite a, but do not assume $a < b$. Then*

$$Nbc^2 = X^2(a + 2b).$$

Remark. This equation is the analogue, for isosceles ABC, of the equation $Nab = X^2$ for equilateral ABC.

Proof. Dropping the assumption $\alpha < \beta$, we may suppose without loss of generality that the base angles are α. Suppose angles A and B are equal to α, and the opposite sides are each equal to X; let Z be the side opposite angle C, which is $\pi/3 + 2\beta$ (since $\pi = 2\alpha + \pi/3 + 2\beta$). Then the area equation tells us

$$Nab \sin\left(\frac{\pi}{3}\right) = X^2 \sin\left(\frac{\pi}{3} + 2\beta\right)$$

By Lemma 19 we have

$$\sin\left(\frac{\pi}{3} + 2\beta\right) = \frac{a(a + 2b)}{c^2}\frac{\sqrt{3}}{2}$$

Putting that in we have

$$Nab \sin\left(\frac{\pi}{3}\right) = X^2 \frac{a(a + 2b)}{c^2}\frac{\sqrt{3}}{2}$$

Since $\sin(\pi/3) = \sqrt{3}/2$, that term cancels, and we have

$$Nab = X^2 \frac{a(a + 2b)}{c^2}$$

Multiplying by c^2 we have

$$Nbc^2 = X^2(a + 2b)$$

That completes the proof of the lemma.

Lemma 21 *Suppose $c^2 = a^2 + ab + b^2$, and a, b, and c have no common factor. Then $a + b$ and c are relatively prime.*

Proof. We have

$$\begin{aligned} c^2 &= a^2 + ab + b^2 \\ &= (a + b)^2 - ab \end{aligned}$$

Assume, for proof by contradiction, that p is a prime dividing both c and $a + b$. Then p divides ab, so p divides a or p divides b. But a, b, and c are relatively prime, by Lemma 2, so p cannot divide either a or b. That completes the proof of the lemma.

Lemma 22 *Suppose $c^2 = a^2 + ab + b^2$, and there is no common factor of a, b, and c. Then $a + 2b$ and $2a + b$ are relatively prime to c, and $a + 2b$ is relatively prime to b, and only the prime 2 can divide both a and $a + 2b$. Similarly with a and b interchanged.*

Proof. We have

$$\begin{aligned} c^2 &= a^2 + ab + b^2 \\ &= a^2 + 4ab + 4b^2 - 3ab - 3b^2 \\ &= (a + 2b)^2 - 3b(a + 2b) + 3b^2 \end{aligned}$$

For proof by contradiction, suppose that p is a prime that divides both c and $a+2b$. Then p divides $3b^2$ as well. By Lemma 21, p does not divide b. Hence $p = 3$. Dividing both sides by 3, we have

$$\frac{c^2}{3} \quad = \quad \frac{(a+2b)^2}{3} - b(a+2b) + b^2$$

Now 3 still divides all the terms except b^2; hence 3 divides b^2 as well, and hence 3 divides b. Hence 3 both divides b and does not divide b. That contradiction proves that $a+2b$ and c are relatively prime.

By Lemma 2, a and b are relatively prime. It follows that $a+2b$ and b are relatively prime. Now to prove that $a+2b$ and a are relatively prime. Since a and b are relatively prime, if some prime p divides both $a+2b$ and a, we must have $p = 2$. That completes the proof of the lemma.

Lemma 23 *Let N be a positive integer, and let T be a non-isosceles triangle with a 120° angle and sides a, b, c, with c opposite the 120° angle. Suppose there is an N-tiling of an isosceles triangle ABC by T. Then $(a+2b)$ divides N.*

Proof. Let X be the length of the two equal sides of ABC. By Lemma 20 we have

$$Nbc^2 = X^2(a+2b)$$

By Lemma 22, $a+2b$ is relatively prime to bc^2. Hence $a+2b$ divides N. That completes the proof of the lemma.

Lemma 24 *Let N be a positive integer, and let T be a non-isosceles triangle with a 120° angle and sides a, b, c, with c opposite the 120° angle. Suppose there is an N-tiling of an isosceles triangle ABC by T. Let X be the length of the two equal sides of ABC. Let d and s be the squarefree part and square divider of $b(a+2b)$, respectively. Then for some integer k we have*

$$\begin{aligned} N &= k^2 d \\ X &= \frac{kcs}{a+2b} \\ Z &= ks \end{aligned}$$

Remark. These formulas are similar to the formulas in the equilateral case, but d and s are the squarefree part and square divider of different expressions.

Proof. We start with the equation from Lemma 20:

$$Nbc^2 \quad = \quad X^2(a+2b)$$

Multiply both sides by $(a+2b)$:

$$Nbc^2(a+2b) \quad = \quad X^2(a+2b)^2.$$

Let s be the square divider of $b(a+2b)$, as mentioned in the statement of the lemma. Since c is relatively prime to $b(a+2b)$ (by Lemma 22), cs is the square divider of $bc^2(a+2b)$. Since $bc^2(a+2b)$ divides $X^2(a+2b)^2$, by Lemma 8, we have cs divides $X(a+2b)$. Then we can define the integer k to be the quotient:

$$k := \frac{X(a+2b)}{cs}.$$

Then

$$X(a+2b) = kcs$$

and we have

$$\begin{aligned} Nbc^2(a+2b) &= X^2(a+2b)^2 \\ &= (kcs)^2 \end{aligned}$$

Dividing both sides by c^2 we have

$$Nb(a+2b) \quad = \quad k^2 s^2$$

By Lemma 8 and the definition of s, we have

$$b(a+2b) = \frac{s^2}{d}.$$

Putting that into the previous equation we have

$$\frac{Ns^2}{d} \quad = \quad k^2 s^2$$

Multiplying by d and dividing by s^2, we have

$$N = k^2 d.$$

It remains to derive the formula for Z. By the law of sines we have

$$\begin{aligned}
Z &= X \sin(\pi/3 + 2\beta) \frac{1}{\sin\alpha} \\
&= X \frac{a(a+2b)}{c^2} \frac{\sqrt{3}}{2\sin\alpha} && \text{by Lemma 19} \\
&= X \frac{a(a+2b)}{c^2} \frac{\sqrt{3}}{2(a/c)\frac{\sqrt{3}}{2}} \\
&= X \frac{a+2b}{c} \\
&= \frac{kcs}{a+2b} \frac{a+2b}{c} \\
&= ks
\end{aligned}$$

That completes the proof of the lemma.

Lemma 25 *Suppose ABC is isosceles with base angles α, and is N-tiled by a non-isosceles tile with a 120° angle. If the integer k in Lemma 24 is 1, then either*

(i) $a+2b$ and b are both squarefree, and $N = b(a+2b)$, and $X = bc$, or

(ii) $2a < b$ and X can be written as $X = ua + vc$ with $v > 0$ and $u \ge 0$. (We did not assume $a < b$.)

Remark. We tried (but failed) to prove that $k = 1$ is impossible, as for the equilateral case. This is what came of the attempt. It cuts down the number of cases to be considered later.

Proof. With notation as in the previous lemma, recall that d and s are (respectively) the squarefree part and the square divider of $b(a+2b)$, and we have

$$\begin{aligned}
Nbc^2 &= X^2(a+2b) \\
N &= k^2 d \\
X &= \frac{kcs}{a+2b}
\end{aligned}$$

Suppose, for proof by contradiction, that $k = 1$. Then $N = k^2 d$ become $N = d$, and putting that into the first equation we have

$$dbc^2 \quad = \quad X^2(a+2b) \tag{6}$$

By Lemma 22, $(a+2b)$ is relatively prime to bc^2, so $a+2b$ divides d. But d is the product of the squarefree part of $a+2b$ and the squarefree part of b. The latter is relatively prime to $a+2b$, so $a+2b$ divides its own squarefree part. Hence $a+2b$ is squarefree. Define δ to be the squarefree part of b. Then $d = (a+2b)\delta$. Putting that into (6), we have

$$(a+2b)\delta bc^2 \quad = \quad X^2(a+2b)$$

Dividing both sides by $a+2b$ we have

$$\delta bc^2 \quad = \quad X^2$$

Let σ be the square divider of b. Then $b = \sigma^2/\delta$, so $\delta b = \sigma^2$. Putting that into the previous equation we have

$$\sigma^2 c^2 = X^2$$

and hence $X = \sigma c$. By Lemma 6 and Theorem 1, we have either $X = ub + vc$ or $X = ua + vc$, with $u \geq 0$ and $v > 0$.

Case 1: Suppose $X = ub + vc$. Then

$$ub + vc \quad = \quad \sigma c$$

Taking this equation mod σ, and noting that $b \equiv 0 \bmod \sigma$ since σ is the square divider of b, we see that $vc \equiv 0 \bmod \sigma$. Since c and b are relatively prime, σ divides v. Let ℓ be the integer such that $v = \ell\sigma$; then since $v > 0$ we have $\ell > 0$ and

$$ub + \ell\sigma c \quad = \quad \sigma c$$

Since $u \geq 0$ and $\ell > 0$, this is possible only if $u = 0$ and $\ell = 1$. Hence the two equal sides of ABC are composed only of c tile edges. Since the base angles of ABC are α, the b sides of the tiles in the corner must lie on the base of ABC.

We have shown that whenever $X = ub + vc$ with $v > 0$ then $u = 0$ and $v = \sigma$; that is, we have shown it for any possible u and v, not only for the u and v that occur in the tiling. Now, if b is not squarefree, then $\sigma > b$, so we could write $X = bc + (\sigma - b)c = uc + vc$ with $u = c$ and $v = \sigma - b$. Since this is not possible, b is squarefree and $\sigma = b$. Then $X = bc$. Then the equation $Nbc^2 = X^2(a+2b)$ becomes $Nbc^2 = (bc)^2(a+2b)$, and dividing by bc^2 we have $N = b(a+2b)$. (We can also derive that by pointing out that $N = k^2 d = d$ and since $b(a+2b)$ is squarefree, $d = b(a+2b)$.) But this is the conclusion of the lemma. Thus we have disposed of Case 1.

Case 2, $X = ua + vc$, with $v > 0$. Since $X = \sigma c$ we have

$$\begin{aligned} \sigma c &= ua + vc \\ (\sigma - v)c &= ua \end{aligned}$$

Since c and a are relatively prime, c divides u and a divides $\sigma - v$. Let L be the integer such that $cL = u$. Then

$$(La + v)c \quad = \quad \sigma c \ \leq \ bc \qquad \text{since } \sigma \leq b$$

Dividing by c we have $La + v \leq b$. We may assume $u > 0$ since if $u = 0$, Case 1 applies. Therefore $L - u/c > 0$ and $a < b$. Moreover $2a \leq b$: since $\sigma c = au + vc \geq 2ac$, so $b \geq \sigma \geq 2a$. That completes the proof of the lemma.

Lemma 26 *Let T be the tile with sides $(3, 5, 7)$, and let ABC be the isosceles triangle with two sides 35 and one side 65. Then there is no 65-tiling of ABC by T.*

Remark. The area of ABC is 65 times the area of the tile, so no other value of N is possible.

Proof. Here $a + 2b = 13$ and $a + 2b$ and b are both squarefree. Then $b(a + 2b) = 65 = N$, so $k = 1$. We have $X = \sigma c = bc = 5 \cdot 7 = 35$, so we fall under Case 1 of Lemma 25. There is no other way to write $35 = ub + vc$ with $v > 0$, since v would have to be equal to 0 mod 5. Hence all the tiles along the two sides of length X have their c sides on the boundary, i.e. $X = 5c$. The base Z of ABC (which for definiteness we take to be AC) is given by $Z = b(a + 2b) = 65$.

The two tiles at A and C have their b edges on AC, since they have their α angles at the vertices and their c edges on AB and BC. Hence Z must have the form $ub + vc$, rather than the form $ua + vc$, with $v > 0$. What are the possibilities for u and v? We have $5u + 7v = 65$. Mod 7 we have $5u \equiv 2$. Multiplying by 3 we have $u \equiv 6$ mod 7. $u = 6$ is the only possibility as $u = 13$ is already too large. Then $v = 5$ works. Thus along AC we have 6 edges of length b and 5 edges of length c.

Therefore there cannot be four tiles in a row starting from both A and C with their b edges on AC. We may assume, without loss of generality, that there are not four tiles starting from A; otherwise, change "east" for "west" and relabel A and C.

Now, we analyze the tiling moving left to right from A towards C (picturing B at the north and AC horizontal, east-west, with A at the west). We say a tile is of Type I if it has its c edge parallel to AB and its b edge parallel to AC. We say a tile is of Type II if it has its c edge parallel to AC and its b edge parallel to AB. Now consider lines RW with R on AC and W on AB, parallel to the a edge of Tile 1, the tile at A. We call that direction "Direction 1". Consider the easternmost such line RW with the property that all tiles wholly or partially west of it are of Type I or Type II. We claim that W is not west of B. Suppose, for proof by contradiction, that W occurs on AB west of B. No tiles with an edge on AB are of Type II, because those tiles all have their c edges on AB. Therefore there is at least one tile with its a edge on RW, namely the one at W. Let P be the southernmost point on RW such that PW lies on tile boundaries. Since Type II tiles do not have any edges in the direction of PW, all the tiles west of PW are Type I and have their a edges on PW. By the definition of P, there is a tile east of PW with an edge on PW and a vertex at P (else PW could be extended past P on tile boundaries). We claim that all the tiles east of PW with an edge on PW have their a edges on PW also. If not, then there is a relation of the form $ja = ua + vb + wc$, where j is the number of tiles west of PW with an edge on PW and the right side represents the tile edges on the east side of PW. In that case, $j \geq 4$, because the smallest possible such relation is $4a = b + c$, i.e. $12 = 5 + 7$. For PW to have length 4 and W to be west of B, we would have to have $R = P$, and there would have to be a quadratic tiling of ARW. But then there would be four b edges on AC starting from A, contradiction. Hence, all the tiles east of PW with an edge on PW have their a edges on PW also. Fig. 6 illustrates the situation.

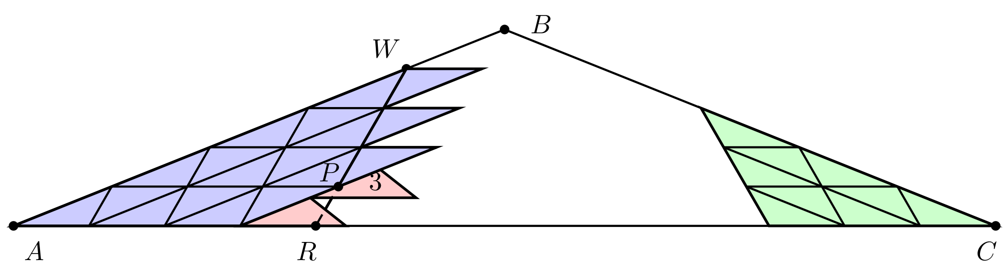


Figure 6: W must actually be B. The notches have to be filled, and RW moves east.

P might lie on AC; but if not, then south of P and at least partially west of RW we have a Type II tile, say Tile 3. We claim that this tile does not have a vertex at P, but instead P lies on the interior of the northern edge of Tile 3. Suppose, for proof by contradiction, that Tile 3 does have a vertex at P. Let Tile 4 be the tile north of Tile 3. If Tile 4 is also of Type II, let Tile 5 be the tile north of Tile 4. Since all the tiles west of RW are either Type I or Type II, at most three of them have a vertex at P, so Tile 5 (if Tile 4 is of Type II) is certainly a Type I tile with its a edge on PW. Let V be the western edge of the boundary PV between Type I and Type II tiles. (So PV either separates Tile 3 from Tile 4, or Tile 4 from Tile 5.) Then PV is composed on one side of only c edges, and on the other side of only b edges. Hence its length is at least the least common multiple of b and c; but b and c are relatively prime, and in this case $bc = 35$. Hence there are at least five tiles on each side of PV. But PV is parallel to either AB or AC, and since W is west of B and P is south of W, there is not enough room in ABC to accommodate a sufficiently long maximal segment. This contradiction shows that Tile 3 does not have a vertex at P. Therefore the tile boundary PV extends east of P, either parallel to AB or parallel to AC.

Now let Tile 6 be the tile east of PW with an edge on W and a vertex at P. If P lies on AC, then Tile 6 does not have its γ angle at P. On the other hand, if P does not lie on AC, then we have shown that PV extends east of V, so in that case also Tile 6 does not have its γ angle at P. Since each of the tiles east of PW with an edge on PW has its a edge on PW, each of those tiles has a γ angle on PW. Since $\gamma > \pi/2$, the pigeonhole principle implies that those tiles all are oriented the same way. Since Tile 6 cannot have its γ angle to the south, all those tiles have their γ angles to the north. Then they are all of Type I.

Those tiles then form "notches" between them, whose sides are parallel to AB and AC. The angles at the vertices on PW that those notches form are all β angles. Hence the tiles that fill those notches are all of Type I or Type II.

The same argument applies with WP replaced by any maximal segment EF of the tiling that lies on RW; namely, the tiles east of EF with an edge on EF all have to have their a edges on EF, since EF has length less than 12, and they cannot have their γ angles to the south, so they form notches that have to be filled by Type I or Type II tiles. Hence a line parallel to RW, but slightly east of RW, will still have the property that all tiles partially or wholly west of it are of Type I or Type II. But that contradicts the definition of RW. This contradiction shows that in fact $W = B$.

The same argument can be applied with "west" and "east" interchanged (and A and C interchanged). Let "Type III" and "Type IV" be the analogues of "Type I" and "Type II", i.e. tiles with their c sides parallel to BC and their b sides parallel to AC (Type III) or vice-versa (Type IV). The result of the argument is a point S on RC (playing the role of R) and a point Q (playing the role of P) on BS, such that all tiles partially or wholly east of BQ are of Type III or Type IV. The situation is illustrated in Fig. 7.

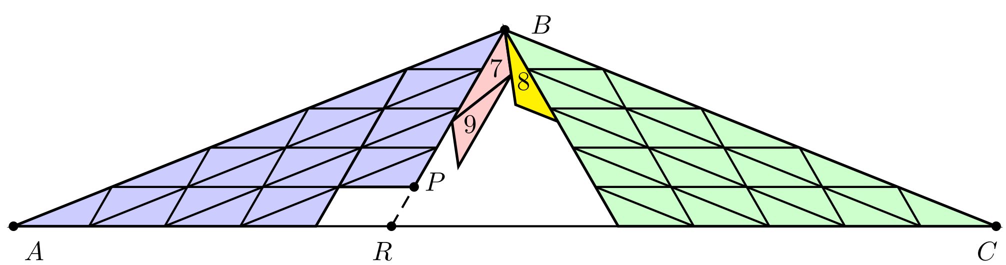


Figure 7: There is no 65-tiling of this triangle

As before, if P does not lie on AC, then the Type II tile southwest of BP must have its northern boundary extending east of P, since even when $R = P$ there is still not room for a maximal segment of length $bc = 35$ west of BP. Therefore the tile east of BP with an edge on BP and a vertex at P does not have its γ angle at P. Therefore not all the tiles east of BP with an edge on BP have their a edges on BP, since if they did, they would all have a γ angle on BP, but that would contradict the pigeonhole principle, since they cannot have a γ angle at B or at P. Therefore BP has length at least 12, since $b + c = 4a = 12$ is the shortest possible segment having all a edges on one side and another combination of edges on the other side. Therefore there must be at least four tiles west of BP with their a edge on BP. If all the tiles east of BP had their a edges on BP, they would all have to have their γ angles to the north. But that is not possible for the tile with a vertex at B, since then the northern boundary of that tile would be horizontal (exactly east-west), while in fact BC has a negative slope. Hence, it is not the case that all the tiles east of BP with an edge on BP have their a edges on BP.

Therefore there are tiles east of BP with b and c edges on BP. In this specific triangle that means BP has length 12 or 15. In case P lies on AC, then BP has length 15 and there are three tiles east of BP, one with a c edge on BP, one with a b edge on BP, and the third with an a edge on BP. In case P does not lie on AC, then BP has length 12, and there are two tiles east of BP, one with a c edge on BP and the other with a b edge on BP. The case when P does not lie on AC is illustrated in Fig. 7.

Let Tile 7 be the tile east of BP with a vertex at B and an edge on BP. Tile 7 cannot have its γ angle at B, as just discussed; so it has either its β angle or its α angle there. Since the vertex splitting of the tiling has to be $(3, 3, 0)$, and there are α angles at A and C, there are three β angles and one α angle at B. So far, two β angles are accounted for (the tiles with their edges on AB and BC, respectively). Hence one of the two remaining tiles has its β angle at B. If it is not Tile 7, then let us change "west" to "east". After that reflection, and renaming the vertices and tiles, it will be the case that Tile 7 has its β angle at B. Hence, without loss of generality, we may assume that Tile 7 has its β angle at B. Hence, it does not have its b edge on BP.

Since Tile 7 does not have its b edge on BP, it has its c edge or its a edge there. We first take the case when Tile 7 has its c edge on BP. Let U be the southeast vertex of Tile 7. Then BU is the a edge of Tile 7. Let Tile 8 be the tile east of Tile 7. Then Tile 8 has its α angle at B. Hence it does not have its a side on BU. Hence the western boundary of Tile 8 extends south of U. Let Tile 9 be the tile south of Tile 7. Then the northern edge of Tile 9 lies along the southern boundary of Tile 7, which is of length b and terminates at both ends in transverse tile boundaries. Hence Tile 9 must share its b edge with Tile 7 (since b is not a multiple of a). Tile 9 then has its α and γ angles at the vertices it shares with Tile 7. It cannot have its γ angle at U, since the western boundary of Tile 8 prevents it. Then it has its α angle at U and its γ angle to the west. Hence Tile 9 forms a parallelogram with Tile 7. Let V be the western vertex shared by Tile 7 and Tile 9. Then V lies on BP, which extends to the southwest of V. The angle between PV and Tile 9 is β, since Tile 7 and Tile 9 have their α and γ angles at V. Let Tile 10 be the tile filling that angle; there can only be one tile there since β is not a multiple of α. Then Tile 10 has an edge on BP, which as we have proved already must be its b edge; but its β angle lies at the vertex V on BP, so the b edge must be opposite V, and not on BP. This is a contradiction (which is why Tile 10 is not shown in the figure). That completes the proof in case Tile 7 has its c edge on BP.

Therefore we may assume Tile 7 has its a edge on BP. Then let E be the southwest vertex of Tile 7; that point lies on BP at a distance of a from B, and Tile 7 has its γ angle there, so the angle remaining at E east of BP and west of Tile 7 is β. Then we can make the same argument as an in the previous paragraph, using EP instead of BP. This time it is slightly easier, since in order that EP have length 12, we must have P on AC, so it is immediate that the tile east of BP at P does not have its γ angle at P. The rest of the argument is unchanged, except for replacing B by E. That completes the proof of the lemma.

Lemma 27 *Let ABC be isosceles, and suppose (a, b, c) are the sides of a triangle T with a 120° angle. Suppose $(b-2)a < b+c$, and suppose also that $b(a+2b)$ is squarefree and $N = b(a+2b)$. Then there is no N-tiling of ABC by T.*

Remarks. This lemma applies to $(3, 5, 7)$ with $N = 65$. We would not expect the inequality in the lemma to hold for large a and b, so it seems fortuitous that it worked for $(3, 5, 7)$. The conditions that $b(a + 2b)$ is squarefree and $N = b(a + 2b)$ are consequences of $k = 1$ in the terminology used above.

Proof. The stated hypotheses are sufficient to carry out the proof of Lemma 26. Rather than repeat the entire proof, we explain how these hypotheses were used. First, assuming b and $a+2b$ are squarefree, the condition $N = b(a + 2b)$ implies $k = 1$, so $\sigma = b$ and the length X of AB is $k\sigma c = bc$. Then only c edges can be used on AB. Then the tile at A had to have a b edge on AC, so the length Z of AC had to be of the form $ub + vc$ rather than $ua + vc$. Second, the base AC of ABC has length $Z = b(a + 2b)$, because $k = 1$ and $b(a + b)$ is squarefree, and when Z is written in the form $ub + vc$, we had $Z = 65 = 6b + 5c$, so $u \leq 6$ (in fact in our case $u = 6$ was the only possibility, but we only used $u \leq 6$). The significance of 6 here was that one side or the other (east or west) of AC has at most 3 (half of six) b edges in a row proceeding from the corner of ABC. Hence we cannot get a boundary of length $12 = b + c$ composed of four a edges as the eastern boundary of some quadratic tiling of a triangle with its northwest boundary on AB until the northern vertex W of that triangle is B. That is, $(b+c)/a$ (which is always an integer) is the number of tiles needed for the west side of a maximal segment with b and c edges on the right. Since $X = bc$, b is the number of tiles along AB. Then we needed $\lceil u/2 \rceil < (b + c)/a$. In the case of $(3, 5, 7)$ that was enough, as we only had to rule out the possibility of a quadratic tiling with four tiles on each side. To make the argument work more generally, we consider that the number of a edges on RW might rise as far as $b - 1$ by the time R reaches b; so it is at most $b - 2$ while R is west of B; so we need $(b - 2)a < b + c$. In our case that was $3 \cdot 3 < 12$, so it worked. The argument about the value of u in $Z = ub + vc$ is not actually needed, if we assume $(b - 1)a < b + c$.

Also, in orienting the triangles just east of RW, we needed to know that there was insufficient room west of RW for a maximal segment composed of b and c edges; since b and c are relative prime, that means that a segment of length bc won't fit in a direction parallel to AB or AC west of BP. That follows automatically since bc is the length of AB, so the entire region west of BP lies inside a circle of radius bc with center anywhere on BR.

In summary: the argument works if $k = 1$, b and $a + 2b$ are squarefree, and whenever $(b - 2)a < b + c$. That completes the proof of the lemma.

Theorem 3 *Let T be a non-isosceles triangle with a 120° angle, with sides (a, b, c) having no common factor. Suppose there is an N-tiling of an isosceles triangle ABC with base angles α by T. Then $a + 2b$ divides N, and $N \geq 130$.*

Remark. This numerical bound is implied by certain facts summarized at the beginning of the proof, which should be convenient if one wants to improve the bound.

Proof. We start by listing the facts we will use. If there is a tiling, then let a, b, and c be the side lengths of the triangle, as usual. We may suppose without loss of generality that a, b, and c have no common factor. Let d and s be the squarefree part and the square divider of $b(a + 2b)$, respectively. Let X be the length of the two equal sides of ABC.

(i) $N = k^2 d$ and $X = ksc/(a + 2b)$ (by Lemma 24).

(ii) $Nbc^2 = X^2(a + 2b)$. (by Lemma 20).

(iii) If $k == 1$ and $X = bc$, then both $a + 2b$ and b are squarefree, and $N = b(a + 2b)$, (by Lemma 25). In addition we must have $(b - 2)a \geq b + c$, by Lemma 27.

(iv) If $k = 1$ and $X \neq bc$, then $X = sc$, and X can be written in the form $X = ua + vc$ with $u \geq 0$ and $v > 0$, and $2a < b$ (by Lemma 25).

Suppose, for proof by contradiction, that there is such a tiling.

By Lemma 22, $a + 2b$ is relatively prime to c and b, and hence is also relatively prime to the squarefree part d of ab. Hence $a + 2b$ divides N, establishing the first claim of the theorem.

It remains to establish the numerical bound $N \geq 135$. We wrote a C program to find all solutions of $c^2 = a^2 + ab + b^2$ with $a + 2b \leq m$ (where for the theorem we take $m = 135$). The program computes d and finds the values of k such that $N = k^2 d \leq m$. Then it rejects that candidate (a, b) if $a+2b$ does not divide N. After that, it rejects the ones that fail the conditions listed in (i) to (iv) above.

For convenience, the program assumes $a < b$, but then it also checks for isosceles triangles with base angle β, by switching a and b, so nothing is lost. The result is that there are exactly two more possibilities to check with $N \leq 135$. These are

```
(3, 5, 7)   with base angle beta and N = 132
(5, 16, 19) with base angle beta and N = 130
```

Since our conclusion is only $N \geq 130$, we do not need to check those cases; if we did so we could advance to $N \geq 135$.

We note that Lemma 27 was used only for the cases $N = 33$ and $N = 65$ with $(a, b, c) = (3, 5, 7)$. For completeness, we reprint the output of the C program that computed the table. The output is sufficiently detailed that the reader can check the calculations if desired. For tiles with no explicit rejection statement, that means there are no k such that $N = k^2 d \leq 135$. In the output, "the 65-lemma" means Lemma 27. It would not have been impossible to make these calculations by hand–just tedious and more error-prone than programming.

```
Trying (3,5,7) with base angles alpha
     sqfree(b(a+2b)) = 65
     Trying k=1         Rejecting, by the 65-lemma.
Trying (3,5,7) with base angles beta
     sqfree(a(b+2a)) = 33
     Trying k=1         Rejecting, by the 65-lemma.
     Trying k=2
Possible: (3, 5, 7) with base angle beta and N = 132
Trying (5,16,19) with base angles alpha
     sqfree(b(a+2b)) = 37
     Trying k=1         Rejecting, because b is not squarefree.
Trying (5,16,19) with base angles beta
     sqfree(a(b+2a)) = 130
     Trying k=1
Possible: (5, 16, 19) with base angle beta and N = 130
Trying (7,8,13) with base angles alpha
     sqfree(b(a+2b)) = 46
     Trying k=1         Rejecting, because b is not squarefree.
Trying (7,8,13) with base angles beta
     sqfree(a(b+2a)) = 154
Trying (7,33,37) with base angles alpha
     sqfree(b(a+2b)) = 2409
Trying (7,33,37) with base angles beta
     sqfree(a(b+2a)) = 329
Trying (9,56,61) with base angles alpha
     sqfree(b(a+2b)) = 14
     Trying k=1  Rejecting, because a+2b=121 doesn't divide N = 14
     Trying k=2  Rejecting, because a+2b=121 doesn't divide N = 56
     Trying k=3  Rejecting, because a+2b=121 doesn't divide N = 126
Trying (9,56,61) with base angles beta
     sqfree(a(b+2a)) = 74
     Trying k=1         Rejecting, because a is not squarefree
Trying (11,24,31) with base angles alpha
     sqfree(b(a+2b)) = 354
Trying (11,24,31) with base angles beta
```

```
     sqfree(a(b+2a)) = 506
Trying (11,85,91) with base angles alpha
     sqfree(b(a+2b)) = 15385
Trying (11,85,91) with base angles beta
     sqfree(a(b+2a)) = 1177
Trying (13,35,43) with base angles alpha
     sqfree(b(a+2b)) = 2905
Trying (13,35,43) with base angles beta
     sqfree(a(b+2a)) = 793
Trying (16,39,49) with base angles alpha
     sqfree(b(a+2b)) = 3666
Trying (16,39,49) with base angles beta
     sqfree(a(b+2a)) = 71
     Trying k=1         Rejecting, because a is not squarefree
Trying (17,63,73) with base angles alpha
     sqfree(b(a+2b)) = 1001
Trying (17,63,73) with base angles beta
     sqfree(a(b+2a)) = 1649
Trying (19,80,91) with base angles alpha
     sqfree(b(a+2b)) = 895
Trying (19,80,91) with base angles beta
     sqfree(a(b+2a)) = 2242
Trying (32,45,67) with base angles alpha
     sqfree(b(a+2b)) = 610
Trying (32,45,67) with base angles beta
     sqfree(a(b+2a)) = 218
Trying (40,51,79) with base angles alpha
     sqfree(b(a+2b)) = 7242
Trying (40,51,79) with base angles beta
     sqfree(a(b+2a)) = 1310
```

That completes the proof of the theorem.

6 The case when ABC has angles 2α and 2β

Lemma 28 *Let T be a triangle with a 120° angle, and other angles α and β, and integer sides a, b, and c with no common divisor. Let ABC be a triangle with angle $A = 2\alpha$. If there is an N-tiling of triangle ABC, then $(a + 2b)$ divides Nc^2, and*

$$Nbc^2 = YZ(a + 2b)$$

where Y and Z are the two sides of ABC adjacent to the 2α angle.

Remark. We do not assume $\alpha < \beta$ or equivalently $a < b$.

Proof. Twice the area of the tile is $Nbc \sin\alpha$. Twice the area of ABC is $YZ \sin 2\alpha$. Assuming, for proof by contradiction, that there is an N-tiling of ABC by this tile, we have the area equation

$$Nbc \sin\alpha = YZ \sin 2\alpha.$$

where Y and Z are the lengths of the sides adjacent to angle A. Since a, b, and c are integers, Y and Z are also integers. Using the double-angle formula for $\sin 2\alpha$, the area equation becomes

$$Nbc \sin\alpha \quad = \quad YZ 2 \sin\alpha \cos\alpha$$

Canceling $\sin\alpha$ and multiplying by c we have

$$Nbc^2 \quad = \quad 2YZc\cos\alpha \tag{7}$$

By Lemma 18, we have

$$2c\cos\alpha = a + 2b.$$

Substituting this value on the right side of the previous equation, we have

$$Nbc^2 = YZ(a + 2b).$$

Hence Nbc^2 is divisible by $a + 2b$. By Lemma 2, a and b are relatively prime. Unless a is even and b is odd, that implies that also $a + 2b$ is relatively prime to b. Hence $a + 2b$ divides Nc^2. Therefore we may suppose a is even and b is odd. In that case c is odd, since a, b, and c are relatively prime, so N is even, since the right side of $Nbc^2 = YZ(a + 2b)$ is even. Then b and $a/2 + b$ are relatively prime, so $a/2 + b$ divides $(N/2)c^2$. Hence $a + 2b$ divides Nc^2 also in this case. That completes the proof of the lemma.

Lemma 29 *Suppose $c^2 = a^2 + ab + b^2$ for integers a, b, and c with no common factor. Then $a + 2b$ and $2a + b$ are relatively prime.*

Proof. Let $P = a + 2b$ and $Q = b + 2a$. If prime p divides P and Q, then it divides $2Q - P$, which is $3a$, and $2P - Q$, which is $3b$. But a and b are relatively prime, by Lemma 2, so $p = 3$. Suppose, then, that 3 divides both $a + 2b$ and $b + 2a$. Then

$$\begin{aligned} 0 &\equiv (a + 2b)^2 + (b + 2a)^2 \qquad \text{mod } 3 \\ &= 5a^2 + 5b^2 + 8ab \\ &= 5c^2 + 3ab\text{since } c^2 = a^2 + ab + b^2 \\ &\equiv 2c^2 \qquad \text{mod } 3 \end{aligned}$$

Multiplying by 2 we have $c^2 \equiv 0$; hence $c \equiv 0 \bmod 3$. Then 3 divides c. But now 3 divides both c and $a + 2b$, contradicting Lemma 22. Hence $(a + 2b)$ and $(b + 2a)$ are indeed relatively prime. That completes the proof of the lemma.

Lemma 30 *Let T be a triangle with a 120° angle, and two other angles α and β, and integer sides a, b, and c. Let ABC be a triangle with one angle 2α. If there is an N-tiling of triangle ABC, then N is divisible by $(a + 2b)$.*

Proof. We may assume that a, b, and c have no common factor, and hence (by Lemma 2) are relatively prime. By Lemma 28, we have

$$Nbc^2 = YZ(a + 2b).$$

By Lemma 22, $a + 2b$ is relatively prime to bc^2. Hence $a + 2b$ divides N. That completes the proof of the lemma.

Lemma 31 *Suppose T is a non-isosceles triangle with a 120° angle, and sides (a, b, c) with no common factor. Suppose triangle ABC has an angle 2α and an angle 2β. Suppose there is an N-tiling of ABC by T. Then N is divisible by $(a + 2b)(2a + b)$, which is at least 143.*

Proof. Let (a, b, c) with no common factor be the sides of T. By Lemma 30, $(a+2b)$ and $(b+2a)$ both divide N. By Lemma 29, these two numbers are relatively prime. Since they both divide N, their product divides N. That proves the first claim of the lemma.

It remains to establish the numerical bound. The smallest solution (a, b, c) of $c^2 = a^2+ab+b^2$ is $(3, 5, 7)$. For this case we have $(a + 2b)(b + 2a) = 143$. All other solutions have larger a or b. That completes the proof of the lemma.

The previous lemma does establish a lower bound on N, but it does not give us formulas in terms of some parameter for X, Y, and Z, which we need in order to construct specific examples of open tiling problems for ABC of this shape. The following theorem provides the required formulas.

Theorem 4 *Let T be a triangle with a 120° angle and integer side lengths, and other angles α and β. Let ABC be a triangle with one angle 2α and one angle 2β. Let X be the length of the side opposite 2α, Z the length of the side opposite 2β, and Y the length of the third side (so $X < Y < Z$ if $\alpha < \beta$). Then for some integer ℓ, we have*

$$\begin{aligned} N &= \ell^2(2a+b)(a+2b) \\ X &= \ell a(a+2b) \\ Y &= \ell c^2 \\ Z &= \ell b(2a+b) \end{aligned}$$

Moreover $N \geq 143$.

Remark. Then the least unknown case is when $(a, b, c) = (3, 5, 7)$, $N = 143$, and $(X, Y, Z) = (39, 49, 55)$.

Proof. The third angle of ABC is $\pi/3$. By Lemma 28, applied first to the 2α angle and then to the 2β angle, we have

$$\begin{aligned} Nbc^2 &= YZ(a+2b) \qquad (8) \\ Nac^2 &= XY(2a+b) \end{aligned}$$

Dividing these equations we have

$$\frac{b}{a} = \frac{Z}{X}\frac{a+2b}{2a+b}$$

Solving for Z we have

$$Z = X\frac{b}{a}\frac{2a+b}{a+2b} \qquad (9)$$

The area equation using the $\pi/3$ vertex is

$$Nab\sin(2\pi/3) = XZ\sin(\pi/3)$$

Since $\sin(2\pi/3) = \sin(\pi/3)$, we can cancel those terms:

$$Nab = XZ$$

Substituting for Y from (9) we have

$$Nab = X^2\frac{b}{a}\frac{2a+b}{a+2b}$$

Clearing denominators and canceling b, we have

$$Na^2(a+2b) = X^2(2a+b) \qquad (10)$$

By Lemma 31, there is an integer k such that

$$N = k(a+2b)(2a+b).$$

Substituting this value for N in the previous equation we have

$$\begin{aligned} k(a+2b)(2a+b)a^2(a+2b) &= X^2(2a+b) \\ ka^2(a+2b)^2 &= X^2 \end{aligned}$$

It follows that k is a square, say $k = \ell^2$. Then

$$\begin{aligned} X &= \ell a(a+2b) \\ N &= \ell^2(a+2b)(2a+b) \end{aligned}$$

as claimed in the statement of the lemma. To find Z we use (9):

$$\begin{aligned} Z &= X\frac{b}{a}\frac{2a+b}{a+2b} \\ &= \ell a(a+2b)\frac{b}{a}\frac{2a+b}{a+2b} \\ &= \ell b(2a+b) \end{aligned}$$

as claimed in the statement of the lemma. To find Y we use (8):

$$\begin{aligned} Y &= \frac{Nbc^2}{Y(a+2b)} \\ &= \frac{\ell^2(a+2b)(2a+b)bc^2}{Y(a+2b)} \\ &= \frac{\ell^2(2a+b)bc^2}{Y} \\ &= \frac{\ell^2(2a+b)bc^2}{\ell b(2a+b)} \\ &= \ell c^2. \end{aligned}$$

The smallest possible solution arises when $(a,b,c) = (3,5,7)$, and we get $N = 143$, which was already proved in the previous lemma anyway. That completes the proof of the lemma.

7 The case when ABC has angles α, $\pi/3$, and $\alpha + 2\beta$

Lemma 32 *Let T be a triangle with a 120° angle, and two other angles α and β, and integer side lengths a, b, and c. Let ABC be a triangle with angle $A = \alpha$. Then $Nbc = YZ$, where Y and Z are the lengths of the sides adjacent to angle A.*

Proof. By the law of sines we have

$$\sin\alpha = \left(\frac{a}{c}\right)\sin\gamma.$$

The area equation is

$$\begin{aligned} Nab\sin\gamma &= YZ\sin\alpha \\ &= YZ\left(\frac{a}{c}\right)\sin\gamma \end{aligned}$$

Dividing both sides by $a\sin\gamma$ and multiplying by c we have

$$Nbc = YZ$$

That completes the proof of the lemma.

Lemma 33 *Let T be a triangle with a 120° angle, and two other angles α and β, and integer sides a, b, and c with no common factor. (Assume $a < c$ and $b < c$ but do not assume $a < b$.) Let ABC be a triangle with angle $C = \alpha + 2\beta$. Let X and Y be the lengths of the sides forming angle C, and Z the side opposite C. Then $Nabc = XY(a+b)$ and $a+b$ divides N.*

Proof. The area equation is

$$\begin{aligned} Nab\sin\gamma &= XY\sin(\alpha+2\beta) \\ &= XY\sin(\pi/3+\beta) \\ &= XY(\sin\frac{\pi}{3}\cos\beta+\cos\frac{\pi}{3}\sin\beta) \end{aligned}$$

By the law of sines we have $\sin\beta = (b/c)\sin\gamma$. Hence

$$Nab\sin\gamma = XY(\sin\frac{\pi}{3}\cos\beta+\cos\frac{\pi}{3}\Big(\frac{b}{c}\Big)\sin\gamma)$$

Dividing both sides by $\sin\gamma = \sin\pi/3$, and putting in $\cos\pi/3 = 1/2$, we have

$$Nab = XY(\cos\beta+\frac{b}{2c})$$

By Lemma 18 we have $\cos\beta = (2a+b)/(2c)$. Hence

$$\begin{aligned} Nab &= XY\Big(\frac{2a+b}{2c}+\frac{b}{2c}\Big) \\ &= 2XY\Big(\frac{a+b}{c}\Big) \end{aligned}$$

Multiplying both sides by c we have

$$Nabc = 2XY(a+b)$$

$$Nabc = XY(a+b).$$

Since a, b, and c have no common factor, they are relatively prime by Lemma 2. Also c is relatively prime to $a+b$ by Lemma 21. Therefore, the fact that $a+b$ divides $Nabc$ implies that $a+b$ divides N. That completes the proof of the lemma.

Lemma 34 *Let T be a triangle with a 120° angle, and two other angles α and β, and integer sides a, b, and c with no common factor. (Assume $a<c$ and $b<c$ but do not assume $a<b$.) Let ABC be a triangle with angle $A=\alpha$ and angle $C=\alpha+2\beta$. Let X and Y be the lengths of the sides forming angle C, and Z the side opposite C. Then $Nabc = XY$.*

Let d and s be the squarefree part and square divider of $b(a+b)$. Then there exists an integer k such that

$$\begin{aligned} N &= k^2 d \\ X &= \frac{kas}{a+b} \\ Y &= \frac{kcs}{a+b} \\ Z &= ks = (a+b)\frac{Y}{c} = (a+b)\frac{X}{a} \end{aligned}$$

Proof. By Lemma 33, we have $Nabc = XY(a+b)$. By Lemma 32, $Nbc = YZ$. The third angle of ABC, namely B, is $\alpha+\beta = \pi/3$, so the area equation using angle B is $XZ\sin(\pi/3) = Nab\sin(2\pi/3)$. Since $\sin(\pi/3) = \sin(2\pi/3)$ we have $XZ = Nab$. Dividing $YZ = Nbc$ by $XZ = Nab$ we find $Y/X = c/a$.

Since $Nbc = YZ$ we have $Z = Nbc/Y$. Since $Nabc = XY(a+b)$ we have $Z = Nbc/Y = X(a+b)/a$ (which is one of the claims of the lemma). Since $Y/X = c/a$ we have $Y = Xc/a$ and $X = Ya/c$, so $Z = X(a+b)/a = Y(a+b)/c$ (which is another claim of the lemma). Substituting $Z = Y(a+b)/c$ into $Nbc = YZ$, and multiplying by c, we have

$$Nbc^2 = Y^2(a+b)$$

Since b is relatively prime to $a+b$, the squarefree part of b divides N. Since $a+b$ also divides N, the squarefree part d of $b(a+b)$ divides N too. Define

$$J := \frac{N}{d}.$$

Then J is an integer, and

$$\begin{aligned} Nbc^2 &= Y^2(a+b) \\ Jdbc^2 &= Y^2(a+b) \\ J &= \left(\frac{Y}{c}\right)^2\left(\frac{a+b}{db}\right) \end{aligned} \tag{11}$$

By Lemma 8, $db(a+b) = s^2$, where s is the square divider of $b(a+b)$. So we have

$$J = \left(\frac{Y(a+b)}{cs}\right)^2$$

Define

$$k := \frac{Y(a+b)}{cs}.$$

Then $J = k^2$. Since J is an integer, and k is rational, k is also an integer. Substituting k^2 for J in (11) we have

$$k^2dbc^2 = Y^2(a+b)$$

Multiplying both sides by b we have

$$k^2b^2dc^2 = Y^2(a+b)b$$

By Lemma 8 we have $(a+b)b = s^2/d$. Putting this in on the right side we have

$$\begin{aligned} k^2b^2c^2d &= Y^2s^2/d \\ Y^2 &= k^2b^2c^2d^2/s^2 \\ Y &= kbcd/s \end{aligned}$$

Since $s^2/d = b(a+b)$, we have $d/s^2 = 1/(b(a+b))$, so $d/s = s/(b(a+b))$. Hence

$$Y = \frac{kcs}{a+b}$$

Since $Y/X = c/a$ we have

$$X = \frac{kas}{a+b}.$$

Since $J = N/d$ by definition, and $J = k^2$, we have $N = k^2d$. Since $Z = X(a+b)/a$, and $X = kas/(a+b)$, we have

$$Z = ks$$

That completes the proof of the lemma.

Theorem 5 *Let T be a triangle with a 120° angle, and two other angles α and β, and integer sides a, b, and c. Let ABC be a triangle with angle $A = \alpha$ and angle $B = \alpha + \beta$. Let N be a positive integer. If there is an N-tiling of triangle ABC, then $N \geq 96$.*

Remark. The case $N = 96$ can probably be ruled out without the aid of a computer, leaving the next case $N = 160$.

Proof. Suppose there is an N-tiling of ABC. Let d be the squarefree part of $b(a+b)$. Then we know the following:

- N is a square times d, by Lemma 34.
- $a + b$ divides N, by Lemma 33.
- X, Y, and Z can each be written in the form $ua + vc$ or $ub + vc$ with $u \geq 0$ and $v > 0$, by Theorem 1 and Lemma 6.

Computation (see the Appendix for the program) reveals that the only possibilities satisfying these three conditions with $N \leq 160$ are when $(a, b, c) = (3, 5, 7)$, with $k = 4$ and $N = 160$, or when $(a, b, c) = (5, 3, 7)$, with $k = 4$ and $N = 96$. These cases are excluded by the statement of the theorem, and will have to be treated to increase the lower bound from 96 to some larger number in the future. That completes the proof of the theorem.

8 The case when ABC has angles α and 2α

Theorem 6 *Let T be a non-isosceles triangle with angles α, β, and γ, where we suppose $\alpha < \gamma$ and $\beta < \gamma$ but we do not suppose $\alpha < \beta$. Let a, b, and c be the sides opposite α, β, and γ. Suppose triangle ABC has angle $A = \alpha$ and angle $B = 2\alpha$. Let d and s be the squarefree part and square divider of $3a$, respectively, and let $\delta = 2$ if a is even and 1 if a is odd. Let sides (X, Y, Z) be opposite angles $(\alpha, 2\alpha, 3\beta)$ respectively. Then for some integer $k \geq 1$, we have*

$$
\begin{aligned}
N &= 3k^2(a + 2b)(a + b) \\
X &= kc^2 \\
Y &= kc(a + 2b) \\
Z &= 3kb(a + b)
\end{aligned}
$$

and we have the lower bound

$$N \geq 264.$$

Proof. Let X be the length of BC (opposite $A = \alpha$), let Y be the length of AB (opposite $C = 2\alpha$), and let Z be the length of AC (opposite $B = 3\beta$). We will need the sines of all three angles of ABC, so we begin by working those out. By the law of sines we have

$$\sin \alpha = \left(\frac{a}{c}\right) \frac{\sqrt{3}}{2} \tag{12}$$

We have

$$
\begin{aligned}
\sin 2\alpha &= 2 \sin \alpha \cos \alpha \\
&= 2 \frac{a}{c} \frac{\sqrt{3}}{2} \cos \alpha \qquad \text{by (12)} \\
&= 2 \frac{a}{c} \frac{\sqrt{3}}{2} \frac{a + 2b}{2c} \qquad \text{by Lemma 18)} \\
\sin 2\alpha &= \frac{a(a + 2b)}{c^2} \frac{\sqrt{3}}{2} \qquad (13)
\end{aligned}
$$

Now we calculate $\sin 3\beta$:

$$
\begin{aligned}
\sin 3\beta &= 3 \sin \beta - 4 \sin^3 \beta \\
&= 3\left(\frac{b}{c} \frac{\sqrt{3}}{2}\right) - 4\left(\frac{b}{c} \frac{\sqrt{3}}{2}\right)^3 \\
&= \left(3\left(\frac{b}{c}\right) - 4\left(\frac{b}{c}\right)^3 \left(\frac{3}{4}\right)\right) \frac{\sqrt{3}}{2}
\end{aligned}
$$

$$
\begin{aligned}
&= \Big(3\Big(\frac{b}{c}\Big) - 3\Big(\frac{b}{c}\Big)^3\Big)\frac{\sqrt{3}}{2} \\
&= \Big(\frac{3bc^2 - 3b^3}{c^3}\Big)\frac{\sqrt{3}}{2} \\
&= \Big(\frac{3b}{c^3}\Big)(c^2 - b^2)\frac{\sqrt{3}}{2} \\
&= \Big(\frac{3b}{c^3}\Big)a^2 + b^2 + ab - b^2)\frac{\sqrt{3}}{2} \\
&= \Big(\frac{3b}{c^3}\Big)(a^2 + ab)\frac{\sqrt{3}}{2}
\end{aligned}
$$

$$\sin 3\beta = \frac{3ab(a+b)}{c^3}\frac{\sqrt{3}}{2} \tag{14}$$

Now we will write the area equation three times, using each vertex in succession. We start at the 2α vertex:

$$
\begin{aligned}
Nab\frac{\sqrt{3}}{2} &= XZ\sin 2\alpha \\
Nab\frac{\sqrt{3}}{2} &= XZ\frac{a(a+2b)}{c^2}\frac{\sqrt{3}}{2} \qquad \text{by (13)} \\
Nbc^2 &= XZ(a+2b) \qquad (15)
\end{aligned}
$$

Next we write the area equation at the α vertex:

$$
\begin{aligned}
Nab\frac{\sqrt{3}}{2} &= ZY\sin\alpha \\
&= ZY\frac{a}{c}\frac{\sqrt{3}}{2} \qquad \text{by (12)} \\
Nbc &= ZY \qquad (16)
\end{aligned}
$$

Finally we write the area equation at the 3β vertex:

$$
\begin{aligned}
Nab\frac{\sqrt{3}}{2} &= XY\sin 3\beta \\
Nc^3 &= 3XY(a+b) \qquad (17)
\end{aligned}
$$

Solving (16) for Z we have

$$Z = \frac{Nbc}{Y} \tag{18}$$

Putting that value into (15) we have

$$Nbc^2 = X(a+2b)\frac{Nbc}{Y}$$

Solving for Y we have

$$Y = X\Big(\frac{a+2b}{c}\Big) \tag{19}$$

Putting that into (17) and multiplying by c, we have

$$Nc^4 = 3X^2(a+2b)(a+b) \tag{20}$$

By Lemma 22, $a+2b$ is relatively prime to c, and by Lemma 21, $a+b$ is relatively prime to c. Therefore c^4 divides $3X^2$. If 3 does not divide c then c^2 divides X^2. We claim that even if 3 does divide c, c^2 still divides X^2. Suppose 3 divides c, and let 3^j be the exact power of 3 in c, so $j > 0$. Then 3^{4j} divides $3X^2$, so 3^{4j-1} (which has a positive power of 3) divides X^2, so 3^{2j} divides X. It follows that c^2 divides X.

Define k to be the integer such that

$$X = kc^2.$$

Then we have

$$\begin{aligned} Nc^4 &= 3X^2(a+2b)(a+b) && \text{by (20)} \\ &= 3k^2c^4(a+2b)(a+b) && \text{since } X = kc^2 \end{aligned}$$

Dividing by c^4 we have

$$N = 3k^2(a+2b)(a+b) \tag{21}$$

Now to derive formulas for Y and Z. From (19) we have

$$\begin{aligned} Y &= X\Big(\frac{a+2b}{c}\Big) \\ &= kc(a+2b) && \text{since } X = kc^2 \end{aligned}$$

which is the formula for Y in the statement of the lemma.

From the law of sines we have

$$\begin{aligned} Z &= X\Big(\frac{\sin 3\beta}{\sin\alpha}\Big) \\ &= kc^2\Big(\frac{\sin 3\beta}{\sin\alpha}\Big) && \text{since } X = kc^2 \\ &= kc^2\frac{3ab(a+b)}{c^3}\frac{\sqrt{3}}{2}\frac{1}{\sin\alpha} && \text{by (14)} \\ &= \frac{3kab(a+b)}{c}\frac{c}{a} \text{ by (12)} \\ Z &= 3kb(a+b) \end{aligned}$$

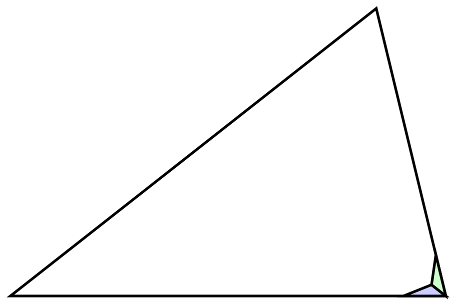
which is the formula for Z in the statement of the lemma.

Now to get a lower bound on N. We use (21); the smallest value will arise when $(a,b,c) = (5,3,7)$, since then $(a+2b)$ will be 11, while when $(a,b,c) = (3,5,7)$ it is 13. Then when $k = 1$ we have $N = 3 \cdot 11 \cdot 8 = 264$. That completes the proof of the theorem.

Example. We now exhibit the triangle corresponding to the least N that is not ruled out by the theorem, $N = 264$. Then according to the proof, we have $(a,b,c) = (5,3,7)$. Then we have $k = 1$, $a+2b = 11$, $a+b = 8$, yielding

$$\begin{aligned} X &= 7^2 = 49 \\ Y &= 7 \cdot 11 = 77 \\ Z &= 3 \cdot 3 \cdot 8 = 72 \end{aligned}$$

Figure 8: Can this triangle be 264-tiled by tiles like the two shown?



9 The case when ABC has angles α and 2β

Theorem 7 *Let T be a non-isosceles triangle with angles α, β, and γ, where we suppose $\alpha < \gamma$ and $\beta < \gamma$ but we do not suppose $\alpha < \beta$. Let a, b, and c be the sides opposite α, β, and γ. Suppose triangle ABC has angle $A = \alpha$ and angle $B = 2\beta$. Let (X, Y, Z) be opposite $(2\alpha, 2\alpha + \beta, 2\beta)$ respectively. Then*

(i) for some integer k, $N = 4k^2(a + b)(b + 2a)$, and

(ii) $X = 2ack$, $Y = 2kc(a + b)$, and $Z = 2kb(2a + b)$, and

(iii) $N \geq 342$

Proof. Let X be the length of BC (opposite $A = 2\alpha$), let Y be the length of AC (opposite $B = 2\alpha + \beta$), and Z be the length of AB (opposite $C = 2\beta$). We write the area equation three times, using each vertex in succession. We start at the 2β vertex:

$$Nab\frac{\sqrt{3}}{2} = XY \sin 2\beta$$

We have

$$\begin{aligned} \sin 2\beta &= 2 \sin \beta \cos \beta \\ &= 2\frac{b}{c}\frac{\sqrt{3}}{2} \cos \beta \\ &= 2\frac{b}{c}\frac{\sqrt{3}}{2}\frac{2a + b}{2c} \qquad \text{by Lemma 18} \\ &= \frac{b}{c}\frac{\sqrt{3}}{2}\frac{2a + b}{c} \end{aligned}$$

The area equation becomes

$$\begin{aligned} Nab\frac{\sqrt{3}}{2} &= XY\frac{b}{c}\frac{2a + b}{c}\frac{\sqrt{3}}{2} \\ Nac\frac{\sqrt{3}}{2} &= XY\frac{2a + b}{c}\frac{\sqrt{3}}{2} \\ Nac^2 &= XY(2a + b) \end{aligned}$$

We worked out the area equation at the α vertex already in the previous section, but just to verify that it did not depend on any assumptions not in effect now, we derive it again:

$$Nab\frac{\sqrt{3}}{2} = ZY \sin \alpha$$

$$\begin{aligned} &= ZY\frac{a}{c}\frac{\sqrt{3}}{2} \\ Nbc &= ZY \end{aligned}$$

Finally we write the area equation at the $2\alpha + \beta$ vertex:

$$\begin{aligned} Nab\frac{\sqrt{3}}{2} &= ZX\sin(2\alpha+\beta) \\ &= ZX\sin(\pi/3+\alpha) \\ &= ZX(\sin(\pi/3\cos\alpha + \cos(\pi/3)\sin\alpha) \\ &= ZX\Big(\frac{\sqrt{3}}{2}\cos\alpha + \frac{1}{2}\Big(\frac{a}{c}\Big)\frac{\sqrt{3}}{2}\Big) \\ Nab &= XZ\Big(\cos\alpha + \frac{1}{2}\Big(\frac{a}{c}\Big)\Big) \\ &= XZ\Big(\frac{a+2b}{2c} + \frac{1}{2}\Big(\frac{a}{c}\Big)\Big)\text{by Lemma 18} \\ Nabc &= XZ\Big(\frac{a+2b}{2} + \frac{a}{2}\Big) \\ Nabc &= XZ(a+b) \end{aligned}$$

Summarizing our work so far, we have

$$Nac^2 = XY(2a+b) \tag{22}$$
$$Nbc = ZY \tag{23}$$
$$Nabc = XZ(a+b) \tag{24}$$

Solving (23) for Z we have

$$Z = \frac{Nbc}{Y}. \tag{25}$$

Putting that into (24) we have

$$Nabc = X\Big(\frac{Nbc}{Y}\Big)(a+b)$$

Solving (22) for Y we have

$$Y = \frac{Nac^2}{X(2a+b)}. \tag{26}$$

Putting that into the previous equation we have

$$\begin{aligned} Nabc &= X(a+b)\Big(\frac{NbcX(2a+b)}{Nac^2}\Big) \\ &= X^2(a+b)(2a+b)\frac{Nbc}{Nac^2} \\ Na^2c^2 &= X^2(a+b)(2a+b) \end{aligned}$$

By Lemma 21, $a+b$ is relatively prime to a, b, and c; and by Lemma 22, $2a+b$ is relatively prime to a and 2 is the only prime that can divide both $2a+b$ and b. If $2a+b$ is even, then b is even, so a is odd, so $2a+b \equiv 2 \bmod 4$. That is, $2a+b$ is only divisible by one power of 2 at most. Also $a+b$ is odd. Therefore $2a^2c^2$ divides X^2. On the other hand, if $2a+b$ is odd, then b is odd, so a is even, so $a+b$ is odd. In that case X must be even, since there is a 2 on the left. So in that case also $2a^2c^2$ divides X^2. Then X is even, and $2ac$ divides X. Let k be the integer such that $X = 2ack$. Then

$$\begin{aligned} Na^2c^2 &= 4k^2a^2c^2(a+b)(2a+b) \\ N &= 4k^2(a+b)(2a+b) \end{aligned}$$

That completes the proof of part (i) of the lemma.

To prove part (ii), first observe that $X = 2ack$ by the definition of k. By (26) we have

$$Y = \frac{Nac^2}{X(2a+b)}$$

Putting in the values of X and Nac^2, we have

$$\begin{aligned} Y &= \frac{4k^2ac^2(a+b)(2a+b)}{2ack(2a+b)} \\ &= 2kc(a+b) \end{aligned}$$

which is the formula for Y given in part (ii). By (25) we have

$$\begin{aligned} Z &= \frac{Nbc}{Y} \\ &= \frac{4k^2bc(a+b)(2a+b)}{Y} \\ &= \frac{4k^2bc(a+b)(2a+b)}{2kc(a+b)} \\ &= 2kb(2a+b) \end{aligned}$$

That completes the proof of part (ii) of the lemma.

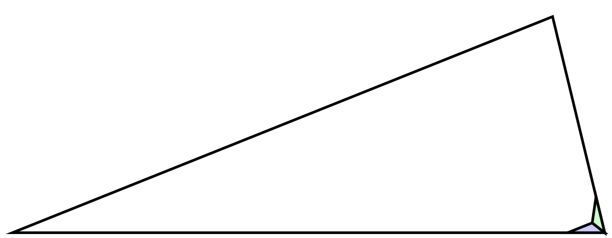
To prove part (iii), we just need to replace (a, b, c) in part (i) by the smallest possible values. These arise from the solution $(3, 5, 7)$, but since we did not assume $a < b$, we must try both $(a, b, c) = (3, 5, 7)$ and $(a, b) = (5, 3, 7)$. The former gives $4(a+b)(2a+b) = 4 \cdot 8 \cdot 11 = 352$. The latter gives $4(a+b)(2a+b) = 4 \cdot 8 \cdot 13 = 416$. Hence $N \geq 352$. That completes the proof of the theorem.

Remark. The least unknown case arises with $k = 1$ and $(a, b, c) = (3, 5, 7)$. Then

$$\begin{aligned} N &= 4 \cdot 8 \cdot 11 = 352 \\ X &= 2ac = 42 \\ Y &= 2c(a+b) = 112 \\ Z &= 2b(2a+b) = 110 \end{aligned}$$

Fig. 9 illustrates this triangle, showing two sample tiles. The triangle is very nearly isosceles, which makes sense, since $2\alpha + \beta = 81.8^\circ$ is not very different from $2\beta = 76.4^\circ$.

Figure 9: Can this triangle be 352-tiled by tiles like the two shown?



10 Main theorem of this paper

The following theorem summarizes the work in this paper on tilings in which the tile has integer side lengths and a 120° angle. A more comprehensive theorem, combining this theorem with previous work, is given in another section below.

Theorem 8 *Let T be a non-isosceles triangle with a 120° angle, and two other angles α and β, and integer sides a, b, and c. Suppose there is an N-tiling of triangle ABC by T, and ABC is not similar to T. Then $N \geq 110$. More specifically, we have the following lower bounds on N, according to the shape of ABC:*

(i) If ABC is equilateral, then $N \geq 135$.

(ii) if ABC is isosceles (but not equilateral), then $N \geq 130$.

(iii) if ABC has an angle 2α and an angle 2β, then $N \geq 141$.

(iv) if ABC has an α angle and an angle 2α, then $N \geq 479$.

(v) if ABC has an α angle and an angle 2β, then $N \geq 110$.

(vi) if ABC has an α angle and an angle $\pi/3 = \alpha + \beta$, then $N \geq 96$

Remark. In each case, there are examples of triangles ABC for which we do not know if there is an N-tiling by $(3, 5, 7)$ for N equal to the stated bound.

Proof. We have already proved these bounds in each specific case, with one section of the paper devoted to each case. It remains only to check that these cases are exhaustive. Suppose there is an N-tiling of some triangle ABC by T. Then the vertex splitting is given by $(3, 3, 0)$, as shown in [2]. That is, there are three α angles and three β angles at the vertices of ABC.

Case 1, in which there are no vertices with just one tile. Then there are at least two tiles at each of the three vertices, and hence, there are exactly three two tiles at each vertex. If no vertex has two α angles then each vertex has one α and one β, so ABC is equilateral, i.e (i) applies. On the other hand, if some vertex does have two α angles, then one of the other two vertices must have two β angles, so (iii) applies.

Case 2, in which there is a vertex with just one tile. Since we did not assume $\alpha < \beta$, we can assume that this vertex has an α angle. Then there are five angles belonging to the other two vertices. If one of the other two vertices has a single β angle then ABC is similar to T, which is ruled out by hypothesis. If one of the other two vertices has a single α angle, then ABC is isosceles, i.e. (ii) applies. Otherwise, the five angles are divided two and three between the other two vertices. Consider the vertex with two tiles. If they are both α angles then (iv) applies; if they are both β angles then (v) applies. If one is α and one is β then (vi) applies.

That completes the proof of the theorem.

11 Conclusions and Open Problems

In this paper we studied the question of tiling a triangle ABC by a tile T with a 120° angle, such that T is not similar to ABC. We could neither find such a tiling, nor prove that no such tilings exist. Instead, we found the following restrictions on the possibilities for such a tiling:

- The vertex splitting is $(3, 3, 0)$. That means that there are in total six tiles with vertices at A, B, or C, three with angle α and three with angle β at a vertex of ABC.
- The tile is similar to a triangle with integer side lengths.
- α is not a rational multiple of π.
- $N \geq 110$, with specific lower bounds on N for each possible shape of ABC, for example $N \geq 135$ for equilateral ABC.

- There are specific examples of triangles ABC of each possible shape such that we do not know if there is an N-tiling of ABC by $(3,5,7)$, for the N mentioned in the bounds.

It seems that a final proof of the nonexistence of such tilings will require proof techniques beyond those in this paper. There seems to be nothing analogous to the "tiling equation" that we used in [3], because the tilings studied here generally cannot be 2-colored. The tiling equation was used (in cases where tilings do not exist) to show that there must be a maximal segment too long to fit in the triangle, but we have not been able to prove something similar for the tilings considered in this paper.

For the record, we state the open problem:

Open Problem 1 *Let $\gamma = 2\pi/3$, and $\alpha + \beta = \pi/3$. Let T be a triangle with angles α, β, and γ. Show that no triangle ABC can be N-tiled by T unless ABC is similar to T, or find such a tiling. (Any such tiling will have to satisfy the restrictions listed above, in particular $N > 110$.)*

Remarks. The main theorem of [4] shows that if such a tiling exists, we can (by clearing denominators) assume that the tile has integer side lengths. The simplest unsolved case is when ABC is equilateral, and the sides of the tile are $(a, b, c) = (3, 5, 7)$.

For completeness, we also state some results of the series of papers (including this paper and [1, 2, 3]. Our main result implies that that if there is any N-tiling, then N is either a square, or a sum of two squares, or is 2,3, or 6 times a square, or twice a sum of two squares, or is a counterexample to the conjectures just stated. Since we have proved $N > 110$ for such a counterexample, we can rule out N-tilings for many N. For example, there are no N-tilings for $N = 7$, 11, 19, 31, or 41. The following theorem gives more information about the possibilities for the shapes of the tile and the tiled triangle.

Theorem 9 (Main Theorem) *Suppose triangle ABC is N-tiled by triangle T. Suppose ABC is not similar to T and T is not a right triangle. Then one of the following conclusions holds:*

(i) ABC is equilateral, T is isosceles with base angles $\pi/6$, and N is three times a square, or

(ii) $3\alpha + 2\beta = \pi$, where α and β are the two smallest angles of the tile, in either order, and α is not a rational multiple of π, and $\sin(\alpha/2)$ is rational (which implies that the sides of the tile have rational ratios), and two of the angles of ABC are 2α and β, or

(iii) T has a 120° angle and has integer side lengths (up to a scale factor), the vertex splitting is $(3, 3, 0)$ (i.e., there three α and three β angles of tiles at the vertices of ABC) and $N \geq 110$. (No such tilings are known as of May, 2012.)

Theorems covering the case when ABC is similar to T and the case when T is a right triangle are in [1]. Theorems covering the case in conclusion (ii) are in [3], where a necessary and sufficient condition on N is given for an N-tiling to exist in that case. These are the "triquadratic tilings." These theorems, together with the non-existence theorems in [2], give a complete characterization of the possible triples (ABC, T, N) such that ABC can be N-tiled by T, except for the cases mentioned in Open Problem 1.

Open Problem 2 *If one cannot solve Open Problem 1, then at least extend the numerical lower bound on the smallest N for a presently unknown N-tiling, by examining in detail the particular cases of triangles mentioned in the text of the paper as the least unsolved cases for the various possible shapes of ABC, for example $N = 135$ and $T = (3, 5, 7)$ when ABC is equilateral.*

We summarize the open problems with smallest N for each shape in the following table. In each case, the areas match and it is possible to compose the sides of ABC as sums of tile edges.

N	(a,b,c)	(A,B,C)	(X,Y,Z)
96	$(3,5,7)$	$(\alpha,\pi/3,\alpha+2\beta)$	$(30,42,48)$
130	$(5,16,19)$	$(\beta,\beta,\pi/3+2\alpha)$	$(95,95,130)$
132	$(3,5,7)$	$(\beta,\beta,\pi/3+2\alpha)$	$(42,42,66)$
135	$(3,5,7)$	$(\pi/3,\pi/3,\pi/3)$	$(60,60,60)$
143	$(3,5,7)$	$(2\alpha,\pi/3,2\beta)$	$(39,49,55)$
352	$(3,5,7)$	$(\alpha,2\beta,2\alpha+\beta)$	$(42,112,110)$
962	$(3,5,7)$	$(\alpha,2\alpha,3\beta)$	$(49,91,370)$

These cases, especially the case $N = 96$, might be possible to treat by hand, and should be possible to treat with a carefully written computer program. On the other hand, the search space will become unmanageably large for value of N around 1000, if not sooner. Moreover, if a program finds a tiling, then that is an unequivocal result, but if it fails to find a tiling, then we have the traditional difficulty with computer proofs: how do we ensure that the search was correct and exhaustive? While we made some use of computer programs in this paper, that use was dispensable, in the sense that the computations in question can be made or checked by hand relatively easily.

Appendix: some C programs

In this appendix, we reprint some of the C programs we used, in case the reader wishes to check their correctness, or run them. Readers who wish to run them should be aware that cutting and pasting from pdf files may cause some trouble; at the very least you will have to remove the page numbers. Here is the C program that computes the table in Theorem 2.

```
long long sqfree(long long n)
// return the squarefree part of n
{ int p;
  long long ans = 1;
  for(p=2;p<=n;p++)
      if(n %p == 0)
         { int count = 0;
           while(1)
             { n = n/p;
               ++count;
               if(n % p)
                  break;
            }
           if(count & 1)
             ans *= p;
         }
  return ans;
}

int gcd(int n, int m)
  { return m == 0 ? abs(n) : gcd(m,n%m);
  }
void equilateralboundtable(int n)
{  int a,b,c,d;
   for(a=1;a<=n;a++)
     { for(b=a+1;b<=(3*a*a+1)/2;b++)
           { if(gcd(a,b) != 1) continue;
             int t = a*a + b*b +a*b;
```

```
            c = (int) (sqrt((double) t + 0.001));
            if(c*c == t)
               { d = sqfree(a*b);
                 if(4*d <= 135 && c <= 135)
                   printf("(%d, %d, %d)         %d\n",a,b,c,4*d);
               }
          }
    }
}

int main()
{ equilateralboundtable(135);
  return 1;
}
```

Here is the C program that makes the computation used in Theorem 5. Some of the print statements are commented out; you can uncomment them to see every detail. The function `sqfree` called here is given in the program above, so it is not repeated.

```
int sqdiv(int x)
// return the square divisor of a nonzero integer x
{ int d = sqfree(x);
  return (int) sqrt((double) x*d + 0.0001);
}

void aux23(int a, int b, int c, int d, int m)
// preconditions:  c^2 = a^2 + b^2 + ab and d = sqfree((b*(a+b))
// see if we can reject this solution (i.e show N >= m),
// and if we cannot, print it out.
{ int k,X,Y,Z,s,v,t;
 // printf("\nTrying (%d, %d, %d)", a,b,c);
  int NN;
  for(k=1; k*k*d <= m; k++)
     {  NN = k*k*d;
       if(NN %(a+b) != 0)
           continue;

       printf("\nTrying  (%d, %d, %d) with k = %d and N = %d", a,b,c,k, NN);
       s = sqdiv(b*(a+b));
       X = k*a*s/(a+b);
       Y = k*c*s/(a+b);
       Z = k*s;
       printf(" and (X,Y,Z) = (%d, %d, %d)", X, Y, Z);
       for(v=2;v*c<=X;v++)
          { t = X-v*c;
            if(t % a == 0 ||  t % b == 0)
              break;
          }
       if(v*c > X)
          {  //  printf("\n  Rejecting: ");
             //  printf("X can't be written as ua+vc or ub + vc");
            continue;
          }
       for(v=2;v*c<=Y;v++) // 2 because there are at least 2 c edges
```

```
          { t = Y-v*c;
            if(t % a == 0 ||  t % b == 0)
              break;
          }
       if(v*c > Y)
          { //   printf("\n  Rejecting: ");
            //   printf("Y can't be written as ua+vc or ub + vc");
            continue;
          }
      for(v=2;v*c<=Z;v++)  // 2 because there are at least 2 c edges
          { t = Z-v*c;
            if(t % a == 0 ||  t % b == 0)
              break;
          }
       if(v*c > Z)
          { //   printf("\n  Rejecting: " );
            //   printf("Z can't be written as ua+vc or ub + vc");
            continue;
          }
       // We can also reject if neither Y nor Z can be written as
       // ub + vc, even if both can be written as ua + vc.

        for(v=2;v*c<=X;v++) // 2 because there are at least 2 c edges
          { t = X-v*c;
            if(t % b == 0)
              break;
          }
       printf("\nv = %d and Y = %d",v, Y);
       if(v*c > X)
          {  // then X can't be written as ub + vc
            for(v=1;v*c<=Y;v++)
               { t = Y-v*c;
                 if( t % b == 0)
                   break;
               }
            if(v*c > Y)
               {  // printf("\n  Rejecting: ");
                  // printf("neither X nor Y can be written as  ub + vc");
                  continue;
               }
          }

       printf("\n   Possible!");
     }
  }

void alphaandalphaplusbetatable(int m)
{  int a,b,c,d;
   for(a=1;a<=m;a++)
     { for(b=a+1;b<=(3*a*a+1)/2;b++)
          { if(gcd(a,b) != 1) continue;
            if(a+b >= m) continue;   // because a+b divides N
```

```
            int t = a*a + b*b +a*b;
            c = (int) (sqrt((double) t + 0.001));
            if(c*c == t)
               { d = sqrfree(b*(a+b));
                 if(d <= m)
                    aux23(a,b,c,d,m);
                 d = sqfree(a*(a+b));
                if(d <= m)
                    aux23(b,a,c,d,m);
               }
          }
     }
}


int main()
{ alphaandalphaplusbetatable(160);
  return 1;
}
```

References

[1] Michael Beeson. Triangle tiling I: the tile is similar to ABC or has a right angle. To appear, available on the author's website.

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[5] Henri Cohen. *Number Theory, Volume 1: Tools and Diophantine Equations*. Springer, 2007.