Triangle Tiling IV:
A non-isosceles tile with a 120 degree angle

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Abstract

An N-tiling of triangle ABC by triangle T is a way of writing ABC as a union of N triangles congruent to T, overlapping only at their boundaries. The triangle T is the "tile". The tile may or may not be similar to ABC. We wish to understand possible tilings by completely characterizing the triples (ABC, T, N) such that ABC can be N-tiled by T. In particular, this understanding should enable us to specify for which N there exists a tile T and a triangle ABC that is N-tiled by T; or given N, to determine which tiles and triangles can be used for N-tilings; or given ABC, to determine which tiles and N can be used to N-tile ABC. This is our fourth paper on this subject. In this paper, we take up the last remaining case: when ABC is not similar to T, and T has one angle equal to 120°, and T is not isosceles (although ABC can be isosceles or even equilateral).

Here is our result: If there is such an N-tiling, then the smallest angle of the tile is not a rational multiple of π. In total there are six tiles with vertices at the vertices of ABC. If the sides of the tile are (a, b, c), then there must be at least one edge relation of the form $jb = ua + vc$ or $ja = ub + vc$, with j, u, and v all positive. The ratios a/c and b/c are rational, so that after rescaling we can assume the tile has integer sides, which by virtue of the law of cosines satisfy $c^2 = a^2 + b^2 + ab$. A simple unsolved specific case is when ABC is equilateral and $(a, b, c) = (3, 5, 7)$. The techniques used in this paper, for the reduction to the integer-sides case, involve linear algebra, elementary field theory and algebraic number theory, as well as geometrical arguments. Quite different methods are required when the sides of the tile are all integers.

1 Introduction

For a general introduction to the problem of triangle tiling, see [1]. This paper is entirely concerned with the non-existence of tilings, rather than their existence. In [1] we enumerated the known tilings; it is our aim to prove that these families exhaust all the possible tilings, or at least, exhaust all the triples (ABC, N, T) such that ABC can be N-tiled by tile T.

As it turns out, one of the most difficult cases to analyze is the case of a tiling in which T is not similar to ABC, and T has $\gamma = 2\pi/3$ and $\alpha < \beta$. In analyzing this case, we alternated several times between trying to construct such a tiling (using paper copies of the tile) and trying to prove no such tiling exists. Although there are some quite interesting ways of fitting together tiles of this shape, one never seems to be able to make a triangle. After Herculean efforts, we have still not been able to rule out the existence of such tilings. The main theorem of this paper is that, if there is such a tiling, then the tile is similar to a tile whose sides are all integers, such as $(3, 5, 7)$. Several different arguments are used. The proof involves the "vertex splitting"; let P and Q be the numbers of α and β, and γ angles (respectively) occurring (in total) at the vertices of ABC. Then the cases divide according as α is a rational multiple of π or not, and according as $(P, Q) = (3, 3)$ or not, and the case $(P, Q) = (3, 3)$ and α is not a rational multiple of π divides further according to the values of P and Q. None of the cases is very simple. The case when α is a rational multiple of π divides according as $\alpha = \pi/12$, $\pi/9$, or $2\pi/15$; all other cases can be disposed of easily.

In [1], we introduced the **d** matrix and the **d** matrix equation,

$$\mathbf{d}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

where a, b, and c are the sides of the tile, and X, Y, and Z are the lengths of the sides of ABC, in order of size. The angles of ABC are, in order of size, A, B, and C, so $X = \overline{BC}$, $Y = \overline{AC}$, and $Z = \overline{AB}$. We keep this convention even if some the angles are equal. The **d** matrix has nonnegative integer entries, describing how the sides of ABC are composed of edges of tiles.

The d matrix is used in almost all our proofs. To avoid having every page filled with cumbersome subscript notation $\mathbf{d}_{ij}$ for the entries of the matrix, we introduce letters for the entries. While this eliminates subscripts, it does require the reader to remember which element is denoted by which letter. Here, for reference, we define

$$\mathbf{d} = \begin{pmatrix} p & d & e \\ g & m & f \\ h & \ell & r \end{pmatrix}$$

2 A little number theory

The following lemma identifies those relatively few rational multiples of π that have rational tangents or whose sine and cosine satisfy a polynomial of low degree over $\mathbb{Q}$. The lemma and its proof are of course well-known, but it is short and may help to make the paper more self-contained.

Lemma 1 *Let $\zeta = e^{i\theta}$ be algebraic of degree d over $\mathbb{Q}$, where θ is a rational multiple of π, say $\theta = 2m\pi/n$, where m and n have no common factor.*

Then $d = \varphi(n)$, where φ is the Euler totient function. In particular if $d = 4$, which is the case when $\tan\theta$ is rational and $\sin\theta$ is not, then n is 5, 8, 10, or 12; and if $d = 8$ then n is 15, 16, 20, 24, or 30.

Remark. For example, if $\theta = \pi/6$, we have $\sin\theta = 1/2$, which is of degree 1 over $\mathbb{Q}$. Since $\cos\theta = \sqrt{3}/2$, the number $\zeta = e^{i\theta}$ is in $\mathbb{Q}(i, \sqrt{3})$, which is of degree 4 over $\mathbb{Q}$. The number ζ is a 12-th root of unity, i.e. n in the theorem is 12 in this case; so the minimal polynomial of ζ is of degree $\varphi(12) = 4$. This example shows that the theorem is best possible.

Remark. The hypothesis that θ is a rational multiple of π cannot be dropped. For example, $x^4 - 2x^3 + x^2 - 2x + 1$ has two roots on the unit circle and two off the unit circle.

Proof. Let f be a polynomial with rational coefficients of degree d satisfied by ζ. Since $\zeta = e^{i2m\pi/n}$, ζ is an n-th root of unity, so its minimal polynomial has degree $d = \varphi(n)$, where φ is the Euler totient function. Therefore $\varphi(n) \le d$. If $\tan\theta$ is rational and $\sin\theta$ is not, then $\sin\theta$ has degree 2 over $\mathbb{Q}$, so ζ has degree 2 over $Q(i)$, so ζ has degree 4 over $\mathbb{Q}$. The stated values of n for the cases $d = 4$ and $d = 8$ follow from the well-known formula for $\varphi(n)$. That completes the proof of (ii) assuming (i).

Corollary 1 *If $\sin\theta$ or $\cos\theta$ is rational, and $\theta < \pi$ is a rational multiple of π, then θ is a multiple of $2\pi/n$ where n is 5, 8, 10, or 12.*

Proof. Let $\zeta = \cos\theta + i\sin\theta = e^{i\theta}$. Under the stated hypotheses, the degree of $\mathbb{Q}(\zeta)$ over $\mathbb{Q}$ is 2 or 4. Hence, by the lemma, θ is a multiple of $2\pi/n$, where $n = 5$, 8, 10, or 12 (if the degree is 4) or $n = 3$ or 6 (if the degree is 3). But the cases 3 and 6 are superfluous, since then θ is already a multiple of $2\pi/12$.

We will need the following well-known number theoretical results, which we state here at the outset for reference later.

The following lemma only involves trigonometry in its statement, but involves some algebra in the proof; strangely, it crops up in two quite different places in the paper.

Lemma 2 *Suppose $\alpha + \beta = \pi/3$, and $(\sin\alpha)/\sin\beta$ is rational. Then $(\sin 2\alpha)/\sin 2\beta$ is also rational, and*

$$\frac{\cos\alpha}{\cos\beta} = \frac{a+2b}{2a+b}$$

where $a = \sin\alpha$ and $b = \sin\beta$.

Proof. Suppose, for proof by contradiction, that $(\sin 2\alpha)/\sin 2\beta$ is rational. Let $a = \sin\alpha$ and $b = \sin\beta$. Let T be a triangle with one side a and one side b and an angle of 120° between those sides, and let c be the opposite side. Then by the law of cosines we have

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos(2\pi/3) \\ &= a^2 + b^2 + ab \qquad \text{since } \cos(2\pi/3) = -1/2 \\ 0 &= a^2 + ab + b^2 - c^2 \end{aligned}$$

If we choose $c = \sin(2\pi/3) = \sqrt{3}/2$, then we have $a = \sin\alpha$ and $b = \sin\beta$ by the law of sines in triangle T. Then $c^2 = 3/4$. Setting $x = a/b$ we have

$$x^2 + x + 1 = \frac{3}{4}\left(\frac{1}{b^2}\right) \tag{1}$$

We also have

$$\begin{aligned} \frac{\sin 2\alpha}{\sin 2\beta} &= \frac{\sin\alpha\cos\alpha}{\sin\beta\cos\beta} \\ &= \frac{a}{b}\frac{\cos\alpha}{\cos\beta} \end{aligned}$$

Hence $(\sin(2\alpha)/\sin 2\beta)$ is rational if and only if $(\cos\alpha)/\cos\beta$ is rational. Define

$$\begin{aligned} \xi &:= \frac{\cos\alpha}{\cos\beta} \\ \xi^2 &= \frac{\cos^2\alpha}{\cos^2\beta} \\ &= \frac{1-a^2}{1-b^2} \\ 1 - a^2 &= (1-b^2)\xi^2 \end{aligned}$$

Dividing both sides by b^2 we have

$$\begin{aligned} \frac{1}{b^2} - x^2 &= \left(\frac{1}{b^2} - 1\right)\xi^2 \\ \frac{1}{b^2}(1-\xi^2) &= x^2 - \xi^2 \\ \frac{1}{b^2} &= \frac{x^2-\xi^2}{1-\xi^2} \end{aligned}$$

Substituting the right side of this equation for $1/b^2$ in (1) we have

$$x^2 + x + 1 \quad = \quad \frac{3}{4}\Big(\frac{x^2 - \xi^2}{1 - \xi^2}\Big)$$

Multiplying by $4(1 - \xi^2)$ a we have

$$\begin{aligned} 4(1 - \xi^2)(x^2 + x + 1) &= 3(x^2 - \xi^2) \\ 4(x^2 + x + 1) - 3x^2 &= \xi^2(4(x^2 + x + 1) - 3) \\ x^2 + 4x + 4 &= \xi^2(4x^2 + 4x + 1) \\ (x + 2)^2 &= \xi^2(2x + 1)^2 \\ \xi &= \frac{x + 2}{2x + 1} \end{aligned}$$

Since x is rational, so is ξ. Replacing x by its definition a/b, we have the formula in the lemma. That completes the proof of the lemma.

3 $\gamma = 2\pi/3$ and $(P, Q) \neq (3, 3)$

Lemma 3 *Suppose ABC is N-tiled by a triangle T whose largest angle γ is $2\pi/3$. Suppose ABC is not similar to T, and T is not isosceles, i.e. is not the tile used in the equilateral 3-tiling. Let P and Q be the total numbers of α and β angles (respectively) at the vertices of ABC. Suppose $(P, Q) \neq (3, 3)$. Then*

(i) $P \neq Q$ and $\alpha = \frac{\pi}{3}(Q - 3)/(Q - P)$ and

(ii) Either $\alpha = \pi/9$, or $\alpha = \pi/12$, or $\alpha = 2\pi/15$, and

(iii) When $P < Q$ and $\alpha = \pi/9$ we have $(P, Q) = (1, 4)$, and if $P < Q$ and $\alpha = \pi/12$ we have $(P, Q) = (0, 4)$, and

(iv) In case $\alpha = 2\pi/15$, triangle ABC is isosceles with base angles either β or 2β, and $(P, Q) = (0, 5)$.

Proof. Since ABC is not similar to T, we have vertex splitting. Since T is not isosceles, $\alpha < \beta$. Let P and Q be the total number of α angles and β angles, respectively, at vertices of ABC. The number R of γ angles is either zero or 1, since $2\gamma = 4\pi/3 > \pi$. If $R = 0$ we have $P\alpha + Q\beta = \pi$, and if $R = 1$ we have $P\alpha + Q\beta = \pi/3$. Because ABC is not similar to T, at least two angles of ABC are split, so $P + Q + R \geq 5$. Since $R \leq 1$, we have $P + Q \geq 4$.

Fix any vertex V of the tiling, and let n, m, and ℓ be the number of α, β, and γ angles at V. If $\ell = 2$ then either $m = n = 1$ or $m = 0$ and $n > 2$. If $\ell = 1$ then at least two more angles are required. Thus, unless $\ell = 3$, there are at least as many α and β angles at V as γ angles. Since at the vertices of ABC, there are more α and β angles than γ angles, this cannot be the case at every vertex, since altogether there must be N of each. This proves that at some vertex V we have $\ell = 3$, i.e. three γ angles meet. Let R be the number of γ angles at the vertices of ABC. We have

$$\begin{pmatrix} 0 & 0 & 3 \\ 1 & 1 & 1 \\ P & Q & R \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \frac{2\pi}{3} \end{pmatrix} = \begin{pmatrix} 2\pi \\ \pi \\ \pi \end{pmatrix} \tag{2}$$

The determinant of the matrix on the left is $3(Q - P)$. Suppose, for proof by contradiction, that $P = Q$. Then $R = 0$, since if $R = 1$ we have $\pi/3 = P\alpha + Q\beta = P(\alpha + \beta) = P\pi/3$, so $P = Q = 1$, contradicting $P + Q \geq 4$. Now that we know $R = 0$ we have $\pi = P\alpha + Q\beta = P(\alpha + \beta) = P(\pi/3)$ since $\alpha + \beta = \pi/3$. Hence $\pi = P(\pi/3)$; hence $P = 3$. Since $P = Q$ we have $P = Q = 3$. But that contradicts the hypothesis. That contradiction shows that $P \neq Q$.

Therefore the determinant of the matrix equation (2) is not zero, and we can solve for α by Cramer's rule:

$$\begin{aligned}
\alpha &= \frac{\pi}{3(Q-P)} \begin{pmatrix} 2 & 0 & 3 \\ 1 & 1 & 1 \\ 1 & Q & R \end{pmatrix} \\
&= \frac{\pi}{3}\frac{2R+Q-3}{Q-P} \qquad (3)
\end{aligned}$$

Since $P \neq Q$, there are only two possibilities, $Q < P$ and $Q > P$. Consider the case $Q < P$. We will show that α must be $\pi/9$ or $\pi/12$. The meaning of $Q < P$ is that there are fewer β angles than α angles at the vertices of ABC (taken together). Since each copy of the tile has one β and one α vertex, there must exist a vertex of the tiling at which there occur more β angles than α angles. Let V be such a vertex. Then at least one β angle occurs at V. The angle sum at V is either π or 2π. Suppose first that it is π. If a γ angle occurs at V, then there is room for only one β and one α angle in addition, contradicting the fact that more β angles than α angles occur at V. Therefore no γ angles occur at V, and at least two β angles.

There cannot be exactly two β angles at V, since $\pi - 2\beta = \pi/3 + 2\alpha > 2\alpha$, so more than two α angles must occur at V, contradiction. If exactly three β angles occur, then there is room for only three α angles (since $\alpha + \beta = \pi/3$); but that contradicts the fact that more β than α angles occur at V. If four β angles occur, then let $k < 4$ be the number of α angles at V; then we have

$$\begin{aligned}
4\beta + k\alpha &= \pi \\
(4-k)\beta + k(\alpha+\beta) &= \pi \\
(4-k)\beta + k\frac{\pi}{3} &-= \pi \qquad \text{since } \alpha+\beta = \pi/3 \\
(4-k)\beta = (3-k)\frac{\pi}{3}
\end{aligned}$$

Let us check the possibilities for k. With $k = 3$ we have $\beta = 0$, a contradiction. With $k = 2$ we have $\alpha = \beta = \pi/6$, contradicting the hypothesis that T is not isosceles. With $k = 1$ we have $\beta = 2\pi/9$ and $\alpha = \pi/9$,as desired. With $k = 0$ we have $\beta = \pi/4$ and $\alpha = \pi/12$, as desired. contradicting the assumption that $\alpha \neq \pi/12$. Hence, if the angle sum at V is π, the conclusion of the lemma is verified.

If, on the other hand, the vertex V has angle sum 2π, and has two γ angles, that leaves $2\pi/3$ to be composed of α and β angles. That could be done with two α and two β angles, but that is not more α than β angles; it cannot be done with fewer than two β angles; and if three β angles occur, with zero α angles, then their sum is $2\pi/3 = 3\beta = \pi - 3\alpha$, which implies $\alpha = \pi/9$. If three β angles occur with one α angle, then

$$\begin{aligned}
3\beta + \alpha &= 2\pi/3 \\
3(\pi/3 - \alpha) + \alpha &= 2\pi/3 \qquad \text{since } \alpha+\beta = \pi/3 \\
\alpha &= \pi/6 \qquad \text{contradiction, since } \alpha < \pi/6 < \beta
\end{aligned}$$

If three β angles occur with two α angles, then

$$\begin{aligned}
3\beta + 2\alpha &= 2\pi/3 \\
3(\pi/3 - \alpha) + 2\alpha &= 2\pi/3 \qquad \text{since } \alpha+\beta = \pi/3 \\
\alpha &= \pi/3 \qquad \text{contradiction, since } \alpha < \pi/6 < \beta
\end{aligned}$$

But since there are more β angles than α angles at V, this exhausts the possibilities. That completes the proof of the claim that if $Q < P$ then $\alpha = \pi/9$ or $\alpha = \pi/12$.

Turning to the case $Q > P$, since $\alpha < \pi/6$, we have by (3) that

$$\frac{\pi}{3}\frac{2R+Q-3}{Q-P} < \frac{\pi}{6}.$$

Hence $Q - P > 2(2R+Q-3)$. Hence $4R+P+Q < 6$. Since $P+Q \geq 4$, the case $R = 1$ is ruled out, and we can assume $R = 0$. Then $P + Q < 6$. Hence $P + Q = 4$ or $P + Q = 5$. Since $\alpha > 0$ we have $Q > 3$ by (3), so the possibilities for (P, Q) are $(0, 4)$, $(1, 4)$, $(0, 5)$. The corresponding values of α can be computed from (3), and β can then be found from $\alpha + \beta = \pi/3$. The results are shown in the following table:

P	Q	α	α in degrees	β	β in degrees
0	4	$\pi/12$	15	$\pi/4$	45
1	4	$\pi/9$	20	$2\pi/9$	40
0	5	$2\pi/15$	24	$\pi/5$	36

Note that in case $(P, Q) = (0, 4)$, the triangle ABC must be a right isosceles triangle, since with four β angles to make three vertices of ABC, two of them must be β and the other one $2\beta = \pi/2$. In case $(P, Q) = (0, 5)$, five β angles are distributed among three vertices of ABC, so the triangle ABC must also be isosceles, but there are two possibilities: either the base angles are β and the vertex angle is 3β, or the base angles are 2β and the vertex angle is β. That is, the base angles are either 36° or 72°. That completes the proof of the lemma.

Now the reader might expect us to rule out the three remaining values of α and be done with this section. It is not quite simple, because there are many possible shapes of ABC to consider. Strangely, we have to divide the argument into cases not only by the value of α, but by whether ABC is or is not isosceles.

4 $\gamma = 2\pi/3$ and $\alpha = \pi/12$ and ABC isosceles

Lemma 4 *Suppose that triangle ABC is isosceles. Then ABC cannot be N-tiled by a triangle T whose largest angle is $\gamma = 2\pi/3$ and smallest angle is $\alpha = \pi/12$.*

Proof. Let $a = \sin\alpha$. We will work in the field $\mathbb{Q}(a)$, so it will be necessary to establish some identities that determine arithmetic in $\mathbb{Q}(a)$. One can calculate a as follows:

$$\begin{aligned} a &= \sin\frac{\pi}{12} \\ &= \sin\frac{\pi/6}{2} \\ &= \sqrt{\frac{1-\cos\frac{\pi}{6}}{2}} \\ &= \sqrt{\frac{1}{2}-\frac{\sqrt{3}}{4}} \\ &= \frac{\sqrt{3}-1}{2\sqrt{2}} \end{aligned}$$

The last step can be verified by squaring both sides and simplifying. Squaring and solving for $\sqrt{3}$ we have

$$\sqrt{3} = 2 - 4a^2 \tag{4}$$

Squaring and subtracting 3 we find $16a^4 - 16a^2 + 1 = 0$. Hence

$$a^4 = a^2 - \frac{1}{16} \tag{5}$$

From the equation for a it now follows that both $\sqrt{2}$ and $\sqrt{3}$ belong to $\mathbb{Q}(a)$, which therefore has degree 4 over $\mathbb{Q}$, since $\sqrt{3}$ does not belong to $\mathbb{Q}(\sqrt{2})$. Hence the fourth-degree polynomial above is the minimal polynomial of a.

In triangle T, we have $\beta = \pi/3 - \alpha = \pi/4$. Hence $b = \sin\beta = 1/\sqrt{2}$ already belongs to $\mathbb{Q}(a)$. We need to find an explicit expression for b in powers of a.

$$\begin{aligned}
a &= \frac{\sqrt{3}-1}{2\sqrt{2}} \\
&= \frac{1}{\sqrt{2}}\frac{\sqrt{3}-1}{2} \\
&= b\frac{\sqrt{3}-1}{2} \\
b &= \frac{2a}{\sqrt{3}-1} \\
&= \frac{2a}{\sqrt{3}-1}\frac{\sqrt{3}+1}{\sqrt{3}+1} \\
&= a(\sqrt{3}+1) \\
&= a(2-4a^2+1) \qquad \text{by (4)} \\
&= a(3-4a^2)
\end{aligned}$$

with the final result

$$b = 3a - 4a^3 \tag{6}$$

The third side of the tile is $c = \sin(2\pi/3) = \sqrt{3}/2 = 1 - 2a^2$.

With these algebraic preliminaries in order, we now consider an isosceles triangle ABC, with base BC and vertex angle at A. The possibilities for the base angles of ABC are $k\alpha$, for $1 \le k \le 5$, and the corresponding possibilities for the vertex angle A are the even multiples of α up to 10α. In other words, since $\pi/2 = 6\alpha$, the vertex angle A is either $\pi/2$, or $\pi/2 \pm 2\alpha$, or $\pi/2 \pm 4\alpha$. That is, the angle at A is $\pi/2 + 2J\alpha$, for $-2 \le J \le 2$. Fix this number J.

Let X be the length of sides AC and BC. Then the area of triangle ABC is given by

$$\begin{aligned}
\mathcal{A}_{ABC} &= \frac{1}{2}X^2\sin(\pi/2 + 2J\alpha) \\
&= \frac{1}{2}X^2\cos(2J\alpha)
\end{aligned}$$

Next we calculate the area of the tile T. We have $a = \sin\alpha$, $b = \sin\beta$, and by the cross-product formula for the area of a triangle we then have

$$\begin{aligned}
\mathcal{A}_T &= \frac{1}{2}\sin\alpha\sin\beta\sin\gamma \\
&= \frac{1}{4}ab \qquad \text{since } \sin\gamma = \sin\tfrac{2\pi}{2} = \tfrac{1}{2} \\
&= \frac{1}{2}a(3a-4a^3) \qquad \text{by(6)} \\
&= \frac{3}{2}a^2 - 2a^4 \\
&= \frac{3}{2}a^2 - 2(a^2 - \frac{1}{16}) \qquad \text{by (5)} \\
&= \frac{1}{8} - \frac{3}{2}a^2
\end{aligned}$$

Since ABC is tiled by N copies of the tile T, we have

$$N\mathcal{A}_T \quad = \quad \mathcal{A}_{ABC}$$

$$N\Big(\frac{1}{8}-\frac{3}{2}a^2\Big) \quad = \quad \frac{1}{2}X^2\cos(2J\alpha)$$

Let p, q, and r be the numbers of a sides, b sides, and c sides of tiles on side AB. In other words, p,q,r is the row of the $\mathbf{d}$ matrix corresponding to side AB. Then $X = pa+qb+rc$. Using the expressions derived in (6) and (4) we have

$$\begin{aligned} X &= pa+qb+rc \\ &= pa+q(3a-4a^3)+r(1-2a^2) \\ &= r+(p+3q)a-2ra^2-4qa^3 \end{aligned}$$

Squaring both sides we have

$$\begin{aligned} X^2 &= (r+(p+3q)a-2ra^2-4qa^3)^2 \\ X^2 &= r^2+2r(p+3q)a+a^2((p+3q)^2-4r^2)-a^3(8qr+4r(p+3q))+a^4(4r^2-8q(p+3q)) \end{aligned}$$

This expression for X^2 does not depend on the shape of ABC, but only the shape of T. Now we bring the shape of ABC into the equations. Using (5) to eliminate a^4 we find

$$\begin{aligned} X^2 &= r^2+2r(p+3q)a+a^2((p+3q)^2-4r^2)-a^3(8qr+4r(p+3q))+(a^2-\frac{1}{16})(4r^2-8q(p+3q)) \\ &= \frac{3}{4}r^2+\frac{1}{2}q(p+3q)+2r(p+3q)a+a^2((p+3q)^2-8q(p+3q))-a^3(8qr+4r(p+3q)) \\ &= \frac{3}{4}r^2+\frac{1}{2}q(p+3q)+2r(p+3q)a+a^2(p+3q)(p-5q)-a^3(8qr+4r(p+3q)) \end{aligned}$$

Assume, for proof by contradiction, that $J=0$, i.e ABC has a right angle at A. Then

$$\begin{aligned} 2N\mathcal{A}_T &= 2\mathcal{A}_{ABC} \\ 2N\Big(\frac{1}{8}-\frac{3}{2}a^2\Big) &= X^2 \\ \frac{N}{4}-3Na^2 &= \frac{3}{4}r^2+\frac{1}{2}q(p+3q)+2r(p+3q)a+a^2(p+3q)(p-5q)-a^3(8qr+4r(p+3q)) \\ &= \frac{3}{4}r^2+\frac{1}{2}q(p+3q)+2r(p+3q)a+a^2(p+3q)(p-5q)-a^3(8qr+4rp+12rq) \end{aligned}$$

Since $\{1,a,a^2,a^3\}$ is a basis for $\mathbb{Q}(a)$, the coefficients of like powers of a are equal on both sides of this equation. The linear term tells us that $r(p+3q)=0$. Hence either $r=0$ or both p and q are zero, since p and q are nonnegative. If both p and q are zero, then the quadratic term is zero on the right, but is $-3N$ on the left, contradiction. Hence $r=0$. Equating the coefficients of the quadratic term on both sides we have

$$3N = (p+3q)(5q-p)$$

Equating coefficients of the constant terms we have

$$\frac{N}{4} = \frac{1}{2}q(p+3q)$$

Multiplying by 12 we have

$$3N = 6q(p+3q)$$

Equating the two expressions we have derived for $3N$ we have

$$(p+3q)(5q-p) = 6q(p+3q)$$

We have already proved $p+3q \neq 0$, so we can cancel it, obtaining $5q-p=6q$, which is impossible, since $p \geq 0$. This contradiction shows that the assumption $J=0$ is untenable, i.e. no such tiling exists with a right angle at vertex A.

Assume, for proof by contradiction, that $J = \pm 1$, which corresponds to vertex angle $\pi/2 \pm 2\alpha$. Then

$$\begin{aligned}
2N\mathcal{A}_T &= X^2 \sin(\frac{\pi}{2} \pm 2\alpha) \\
\frac{N}{4} - 3Na^2 &= X^2 \cos 2\alpha \\
&= X^2 \cos\frac{\pi}{6} \\
&= X^2 \frac{\sqrt{3}}{2} \\
&= X^2(1-2a^2) \qquad \text{by (4)}
\end{aligned}$$

Now we put in for X^2 the expression calculated above:

$$\begin{aligned}
\frac{N}{4} - 3Na^2 &= \Big\{\frac{3}{4}r^2 + \frac{1}{2}q(p+3q) + 2r(p+3q)a + a^2(p+3q)(p-5q) \\
&\quad -a^3(8qr + 4r(p+3q))\Big\}(1-2a^2) \\
&= \frac{3}{4}r^2 + \frac{1}{2}q(p+3q) + 2r(p+3q)a + a^2((p+3q)(p-5q) - \frac{3}{2}r^2 - q(p+3q)) \\
&\quad -a^3((8qr + 4r(p+3q)) + 4r(p+3q)) - 2a^4(p+3q)(p-5q) + 2a^5(8qr + 4r(p+3q))
\end{aligned}$$

Now we use (5) to eliminate a^4 and a^5

$$\begin{aligned}
\frac{N}{4} - 3Na^2 &= \frac{3}{4}r^2 + \frac{1}{2}q(p+3q) + 2r(p+3q)a + a^2((p+3q)(p-5q) - \frac{3}{2}r^2 - q(p+3q)) \\
&\quad -a^3((8qr + 4r(p+3q)) + 4r(p+3q)) - 2(a^2 - \frac{1}{16})(p+3q)(p-5q) \\
&\quad +2(a^3 - \frac{a}{16})(8qr + 4r(p+3q))
\end{aligned}$$

Equating coefficients of the linear term we have

$$\begin{aligned}
0 &= 2r(p+3q) - \frac{1}{8}(8qr + 4r(p+3q)) \\
&= 2r(p+3q) - qr - \frac{1}{2}r(p+3q) \\
&= \frac{3}{2}r(p+q)
\end{aligned}$$

Hence either $r = 0$ or both p and q are zero. Assume, for proof by contradiction, that both p and q are zero. Then equating the constant coefficients, we have $N = 3r^2$. Equating the coefficients of a^2 we have

$$3N = -\frac{3}{2}r^2$$

Substituting $N = 3r^2$ we have $-9r^2 = -(3/2)r^2$; hence $r = 0$. But we cannot have $r = 0$ when $p = q = 0$, since $r + p + q$ is the number of tiles along one side of ABC, and hence is positive. This contradiction proves that not both p and q are zero. Therefore $r = 0$. Substituting $r = 0$ and collecting like powers of a we have

$$\begin{aligned}
\frac{N}{4} - 3Na^2 &= \frac{1}{2}q(p+3q) + a^2((p+3q)(p-5q) - q(p+3q)) - 2(a^2 - \frac{1}{16})(p+3q)(p-5q) \\
&= \frac{1}{2}q(p+3q) + \frac{1}{8}(p+3q)(p-5q) + a^2((p+3q)(p-5q) - 2(p+3q)(p-5q)) \\
&= \frac{1}{8}(p+3q)(p-q) - a^2(p+3q)(p-5q)
\end{aligned}$$

Multiplying by 2 and equating the coefficients of like powers of a we have

$$\begin{aligned} 2N &= (p+3q)(p-q) \\ 6N &= (p+3q)(p-5q) \end{aligned}$$

Subtracting the second equation from three times the first, we have

$$\begin{aligned} 0 &= (p+3q)(3(p-q)-(p-5q)) \\ &= 2(p+3q)(p+q) \end{aligned}$$

But since p and q are nonnegative, and not both zero, this is a contradiction. That completes the proof in case $J = 1$.

Therefore the one remaining possibility, $J = 2$, must be the case, and we have

$$\begin{aligned} 2N\mathcal{A}_T &= X^2 \sin(\frac{\pi}{2} \pm 4\alpha) \\ \frac{N}{4} - 3Na^2 &= X^2 \cos 4\alpha \\ &= X^2 \cos \frac{\pi}{3} \\ &= \frac{1}{2}X^2 \\ \frac{N}{2} - 6Na^2 &= X^2 \end{aligned}$$

The right hand side is the same as in case $J = 0$, and we reach a contradiction in the same way. The factor of 2 on the left side makes no difference, as the two expressions we find for $3N$ are both multiplied by 2, and hence can still be equated. That completes the proof of the lemma.

5 $\gamma = 2\pi/3$ and $\alpha = \pi/12$

Lemma 5 *Suppose triangle ABC is N-tiled by a triangle T whose largest angle $\gamma = 2\pi/3$, and suppose ABC is not similar to T. Then $\alpha \neq \pi/12$, where α is the smallest angle of T.*

Proof. Suppose, for proof by contradiction, that triangle ABC is N-tiled by the tile T mentioned in the lemma, and T is not similar to ABC. We start by calculating $\sin\alpha$ and $\cos\alpha$:

$$\begin{aligned} \sin\alpha &= \sin\frac{\pi}{12} \\ &= \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \\ &= \sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4}) \\ &= \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} - \frac{1}{2}\frac{\sqrt{2}}{2} \\ &= \frac{1}{4}(\sqrt{6}-\sqrt{2}) \\ \cos\alpha &= \cos\frac{\pi}{12} \\ &= \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4} \\ &= \frac{1}{2}\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} \\ &= \frac{\sqrt{6}+\sqrt{2}}{4} \end{aligned}$$

We calculate the area of the tile

$$\begin{aligned}
\mathcal{A}_T &= \sin\alpha\sin\beta\sin\gamma/2 \\
&= \sin(\pi/12)\sin(\pi/4)\sin(2\pi/3) \\
&= \frac{\sqrt{6}-\sqrt{2}}{4}\frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} \\
&= \frac{3}{8} - \frac{\sqrt{3}}{8}
\end{aligned}$$

Hence $\mathcal{A}_T$ belongs to $\mathbb{Q}(\sqrt{3})$, a proper subfield of $Q(\zeta)$.

Since $\mathcal{A}_T$ belongs to $\mathbb{Q}(\sqrt{3})$, it is fixed by all elements σ of the Galois group of $\mathbb{Q}(\zeta)$ that fix $\sqrt{3}$. The elements of the Galois group are σ_j for j relatively prime to 24. In particular $j = 13$ will be of interest. The element σj takes ζ to ζ^j. Since $2i\sin(j\alpha) = \zeta^j - \zeta^{-j}$, σ_j takes $i\sin(\alpha)$ to $i\sin(j\alpha)$. Since β and γ are multiples of α, we also have that σ_j takes $i\sin(\beta)$ to $i\sin(j\beta)$ and $i\sin\gamma$ to $i\sin(j\gamma)$. With $j = 13$ then we find that σ_{13} fixes $i\sin\gamma$, σ_{13} fixes i, σ_{13} takes $i\sin\beta$ to $-i\sin\beta$, and $i\sin\alpha$ to $-i\sin\alpha$. Since σ_{13} fixes $\rho = e^{i\pi/3} = \zeta^4$ and σ_{13} fixes i, it fixes $\sqrt{3}$, since $\sqrt{3} = 4i(\rho - \rho^{-1})$. Hence σ_{13} fixes $\mathcal{A}_T$. Since 13 is congruent to 1 mod 12, $\sin(13j\alpha) = \pm\sin(j\alpha)$ for every integer j, so σ_{13} takes $i\sin(j\alpha)$ to $i\sin(j\alpha)$ for j even and to $-i\sin(j\alpha)$ for j odd. Since it fixes i, it takes $\sin(j\alpha)$ to $\pm\sin(j\alpha)$, according as j is even or odd.

Fix two sides U and V of ABC and let θ be the angle between those sides. Since all the angles of ABC are composed of angles α, β, and γ (although we know γ cannot occur, we do not need that fact here), θ is an integer multiple of α, $\theta = j\alpha$ for some j. We then have from the area equation

$$N\mathcal{A}_T = UV\sin\theta/2$$

We have, for some nonnegative integers p, q, and r, that

$$\begin{aligned}
U &= pa + qb + rc \\
&= p\sin\alpha + q\sin\beta + r\sin\gamma
\end{aligned}$$

where not both p and q are zero, because all the sides of ABC have degree 2. For notational simplicity we abbreviate σ_{13} to just σ. We now apply σ to U:

$$\begin{aligned}
U\sigma &= p(\sin\alpha)\sigma + q(\sin\beta)\sigma + r(\sin\gamma)\sigma \\
&= -p\sin\alpha - q\sin\beta + r\sin\gamma
\end{aligned}$$

Since p and q are nonnegative, we have $|U\sigma| \le U$. By the same reasoning we have $|V\sigma| \le V$. Since $\theta = j\alpha$ for some j, we have $(\sin\theta)\sigma = \pm\sin\theta$. Then

$$\begin{aligned}
\mathcal{A}_T &= \mathcal{A}_T\sigma \\
&= (UV\sin\theta)\sigma \\
&= (U\sigma)(V\sigma)(\sin\theta)\sigma \\
&= \pm(U\sigma)(V\sigma)\sin\theta \qquad \text{since } (\sin\theta)\sigma = \pm\sin\theta
\end{aligned}$$

Taking absolute values we have

$$\begin{aligned}
\mathcal{A}_T &= |\mathcal{A}_T| \\
&= |(U\sigma)(V\sigma)\sin\theta| \\
&= |(U\sigma)||(V\sigma)||\sin\theta| \\
&\le UV\sin\theta \qquad \text{since } \sin\theta > 0 \\
&= \mathcal{A}_T
\end{aligned}$$

Therefore equality holds throughout. Hence $|U_\sigma| = U$ and $|V_\sigma| = V$. That implies that, for each of U and V, either $p = q = 0$ (the side is composed of only c edges) or $r = 0$ (there are no c edges). If $r = 0$ then $U\sigma = -p\sin\alpha - q\sin\beta$ is negative, while if $p = q = 0$ then $U\sigma$ is positive. Let J be the integer such that $\theta = J\alpha$. Then $\sin\theta$ changes sign under σ if and only if J is odd. Say that a side of triangle ABC is "of type c" if it is composed entirely of c sides of tiles, i.e. $p = q = 0$ for that side. Suppose J is even. Then $\sin\theta$ does not change sign under σ, so either both of U and V change sign (i.e. neither is of type c), or neither does (i.e. both are of type c). If J is odd, then under σ, exactly one of U and V must change sign under σ, i.e. exactly one is of type c. On the other hand, if J is even, then either both the adjacent sides are of type c, or neither is of type c.

For example, if ABC is similar to T and the tiling is a quadratic tiling, then the long side BC is of type c, and the other two sides are not of type c; one of them has $p = 0$ and the other has $q = 0$.

Now we consider the effect of σ_7. This automorphism changes the sign of i, and takes $2i\sin\alpha = \zeta - \zeta^{-1}$ to $\zeta^7 - \zeta^{-7} = 2i\sin 7\alpha = -2i\cos\alpha$. Hence σ_7 takes $\sin\alpha$ to $\cos\alpha$. In general σ_7 takes $\sin J\alpha$ to $-\sin((7J \bmod 24)\alpha$, so it fixes $\sin\beta$ (where $J = 3$) and changes the sign of $\sin\gamma$ (where $J = 8$.) We have

$$
\begin{aligned}
2N\mathcal{A}_T &= \mathcal{A}_{ABC} \\
N\sin\alpha\sin\beta\sin\gamma &= UV\sin J\alpha
\end{aligned}
$$

Suppose, for proof by contradiction, that angle $A = \alpha$. Then the tile at A has its b side on one side of ABC and its c side on the other. Let U be the side on which there is a c side, and V the one on which there is a b side. Then U is of type c, and since $\sigma = \sigma_7$ changes the sign of $\sin\gamma = c$, $U\sigma = -U$. On the other hand if $V = pa + qb$, then $V\sigma = p\cos\alpha + qb \ge V$, with equality holding if and only if $p = 0$. Canceling $\sin\alpha$ from both sides of the area equation, we have

$$N\sin\beta\sin\gamma = UV$$

Applying σ to both sides, the left hand side changes sign, and we have

$$-N\sin\beta\sin\gamma = -U(V_\sigma)$$

Adding these two equations we find $V = V_\sigma$. Hence $p = 0$, i.e. only b sides of tiles occur on side U. Suppose that T_1 is the copy of T with its b side along U and one end at an endpoint of U where ABC has an angle less than γ; Let V_1 and V_2 be the endpoints of this b side and suppose the γ angle of T_1 is at V_2. Then the b side of T_1 ends at V_2, and since only b sides of tiles occur on DE, there is another tile T_2 on DE sharing vertex V_2, with its b side on DE. Therefore its β angle is not at V_2. Its γ angle cannot be at V_2 since the γ angle of T_1 is there and $2\gamma > \pi$. Therefore T_2 has its α angle at V_2. Hence its γ angle is at the other end of its b side on U; call that vertex V_3. Now we are in the same position with respect to T_2 and vertex V_3 as we formerly were with respect to T_1 and V_2, and by induction there must be a sequence of such pairs of tiles, reaching all the way to the endpoint of U. Therefore the angle at that endpoint is at least γ, so that endpoint is vertex C of ABC. It cannot be the case that angle $C = \gamma$, since that would make ABC similar to T. It cannot be the case that angle $C = \gamma + 2\alpha$, since if it were, then the third angle of ABC would be α, making ABC isosceles, which contradicts Lemma 4. Therefore angle $C = \gamma + \alpha$ and the third angle is 2α.

The length of side AC (which is of type c) is $qb = q\sin\beta$ for some integer q. By the law of sines, since 2α is the angle opposite side AC, the length of side AB is $\sin(\gamma+\alpha)/\sin 2\alpha$ times the length of AC. But since AB is of type c, there is an integer r such that $\overline{AB} = rc$. Equating the two expressions for $\overline{AB}$, we have

$$rc = \frac{\sin(\gamma+\alpha)}{\sin 2\alpha} q\sin\beta$$

$$\begin{aligned}
\frac{r}{q}\sin\gamma\sin 2\alpha &= \sin(\gamma+\alpha)\sin\beta \qquad \text{since } c=\sin\gamma \\
\frac{r}{q}\frac{\sqrt{3}}{2}\sin 2\alpha &= \sin\beta(\sin\gamma\cos\alpha+\cos\gamma\sin\alpha) \\
\frac{r}{q}\sqrt{3}(\sin\alpha\cos\alpha) &= \frac{\sqrt{2}}{2}\Big(\frac{\sqrt{3}}{2}\cos\alpha+\frac{1}{2}\sin\alpha\Big) \qquad \text{since } \sin\beta = 1/\sqrt{2}
\end{aligned}$$

Putting in the values calculated at the beginning of the proof for $\sin\alpha$ and $\cos\alpha$ we have

$$\frac{r}{q}\frac{\sqrt{3}}{2}\frac{\sqrt{6}-\sqrt{2}}{4}\frac{\sqrt{6}+\sqrt{2}}{4} = \frac{\sqrt{2}}{2}\Big(\frac{\sqrt{3}}{2}\frac{\sqrt{6}+\sqrt{2}}{4}+\frac{1}{2}\frac{\sqrt{6}-\sqrt{2}}{4}\Big)$$

Multiplying by 16 and simplifying, we have

$$\begin{aligned}
8\frac{r}{q}\sqrt{3} &= \sqrt{2}(2\sqrt{3}+2\sqrt{6}-\sqrt{2}) \\
&= 2\sqrt{6}+4\sqrt{3}-2
\end{aligned}$$

This is an equation in $\mathbb{Q}(\sqrt{3},\sqrt{2})$, which is of degree 4 with basis $\{\sqrt{2},\sqrt{3},\sqrt{6}\}$. Hence the constant coefficients on each side must be equal; but on the left the constant coefficient is zero and on the right it is -2. This contradiction shows the impossibility of $A=\alpha$.

Now suppose the smallest angle of ABC is $3\alpha=\beta=\pi/4$. Since ABC is not isosceles by Lemma 4, the only possibility for the other angles is $B=4\alpha=\pi/3$ and $C=5\alpha=\gamma-\alpha$. Either side AB or side AC is of type c, since 3 is odd. If side AC, which is opposite angle $B=4\alpha$, is of type c, then for some integer r we have $\overline{AC}=rc$. Then side AB, which is opposite angle 5α, has length $rc\sin(5\alpha)/\sin(4\alpha)$. The area equation becomes

$$\begin{aligned}
N\sin\alpha\sin\beta\sin\gamma &= \sin A\ \overline{AC}\ \overline{AB} \\
&= r^2c^2\sin\beta\frac{\sin 5\alpha}{\sin 4\alpha} \\
N\sin\alpha\sin\gamma\sin 4\alpha &= r^2c^2\sin 5\alpha \\
&= r^2c^2(\sin\alpha\cos\gamma+\cos\alpha\sin\gamma) \\
N\sin\alpha\frac{\sqrt{3}}{2} &= r^2c^2\Big(\frac{\sqrt{3}}{2}\sin\alpha+\frac{1}{2}\cos\alpha\Big) \\
N\frac{\sqrt{6}-\sqrt{2}}{4}\frac{\sqrt{3}}{2} &= \frac{\sqrt{3}}{2}\Big(\frac{\sqrt{6}-\sqrt{2}}{4}+\frac{1}{2}\frac{\sqrt{6}+\sqrt{2}}{4}\Big)
\end{aligned}$$

Multiplying by 8 and simplifying, we have

$$N(2\sqrt{3}-\sqrt{6}) = 4r^2c^2\sqrt{2}$$

This is an equation in $\mathbb{Q}(\sqrt{3},\sqrt{2})$, for which $\{1,\sqrt{2},\sqrt{3},\sqrt{6}\}$ is a basis. Hence the coeffiencts of $\sqrt{6}$ on both sides must be equal, but on the left we have $-N$ and on the right 0. This contradiction shows that it is impossible that angle A is 3α.

The smallest angle of ABC cannot be more than $4\alpha=\pi/3$. By Lemma 4, it cannot be equal to $\pi/3$ either, since that would make ABC equilateral, and hence isosceles. Therefore the smallest angle A is 2α, since α and 3α have already been ruled out. Either sides AC and AB are both of type c, or neither is. First suppose they both are of type c. Then there are two integers r and s such that the lengths of AC and AB are rc and sc, respectively. Then the area equation becomes

$$\begin{aligned}
N\sin\alpha\sin\beta\sin\gamma &= rsc^2\sin 2\alpha \\
&= 2rsc^2\sin\alpha\cos\alpha
\end{aligned}$$

Canceling $\sin\alpha$ and putting in the values of the other trig expressions, we have

$$\begin{aligned} N\frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} &= 2rsc^2\frac{\sqrt{6}+\sqrt{2}}{4} \\ N\frac{\sqrt{6}}{4} &= \frac{rsc^2}{2}(\sqrt{6}+\sqrt{2}) \end{aligned}$$

This is an equation in $\mathbb{Q}(\sqrt{2},\sqrt{3})$, and the coefficients of $\sqrt{2}$ are not equal on the two sides of the equation. This contradiction shows that not both AB and AC can be of type c.

Hence neither AB nor AC is of type c. Therefore both AB and AC are composed entirely of a and b tile edges. Let $\sigma = \sigma_7$ be the automorphism of $\mathbb{Q}(\zeta)$ defined above. Then σ_7 fixes $\sin\beta$ and $\sin 2\alpha$ but takes $\sin\alpha$ to $\cos\alpha$, and changes the sign of γ. Let U be the length of AB and V the length of AC. The area equation is

$$N\sin\alpha\sin\beta\sin\gamma = UV\sin 2\alpha \tag{7}$$

Applying σ we have

$$\begin{aligned} -N\cos\alpha\sin\beta\sin\gamma &= -U\sigma\ V\sigma\sin 2\alpha \\ &= -U\sigma\ V\sigma 2\sin\alpha\cos\alpha \end{aligned}$$

Multiplying both sides by $-\sin\alpha/\cos\alpha$ we have

$$N\sin\alpha\sin\beta\sin\gamma = U\sigma\ V\sigma 2\sin^2\alpha$$

Dividing this equation by (7) we have

$$\begin{aligned} 1 &= \frac{U\sigma\ V\sigma\ 2\sin^2\alpha}{UV\sin 2\alpha} \\ &= \frac{U\sigma\ V\sigma\ 2\sin^2\alpha}{UV2\sin\alpha\cos\alpha} \\ &= \frac{U\sigma\ V\sigma\sin\alpha}{UV\cos\alpha} \end{aligned}$$

Hence

$$\frac{U\sigma\ V\sigma}{UV} = \frac{\cos\alpha}{\sin\alpha}$$

On the other hand, $U = pa + qb$ for some nonnegative integers p and q, and $V = na + mb$ for some nonnegative integers n and m. We have

$$\begin{aligned} U &= p\sin\alpha + q\sin\beta \\ U\sigma &= p\cos\alpha + q\sin\beta \\ \frac{U\sigma}{U} &= \frac{p\cos\alpha + q\sin\beta}{p\sin\alpha + q\sin\beta} \end{aligned}$$

with equality holding if and only if $q = 0$. To make the equations shorter we introduce $d = \cos\alpha$ and and use the already-defined $a = \sin\alpha$, so we have

$$\frac{d}{a} = \frac{U\sigma\ V\sigma}{UV} = \frac{(pd+q)(nd+m)}{(pa+q)(na+m)} \tag{8}$$

This is possible if $n = 0 = q$ or $p = 0 = m$. We claim those are the only conditions under which this equation is solvable. To prove this we cross-multiply and subtract, obtaining

$$a(pd+q)(nd+m) - d(pa+q)(na+m) = 0$$

Expressing a and d in terms of $\zeta = e^{i\pi/12}$ and expanding and simplifying (using a computer algebra system–we used Sage), one finds

$$mq\zeta^{-1} + np\zeta^3 + np\zeta = 0$$

Multiplying by ζ we have

$$mq + np\zeta^2 + np\zeta^4 = 0$$

The field $\mathbb{Q}(\zeta)$ has degree $\varphi(24) = 8$; hence all three coefficients are zero. That is, $mq = 0$ and $np = 0$. Hence either $n = 0$ or $p = 0$, and either $m = 0$ or $q = 0$. If $p = 0$ then by (8) we have

$$\frac{d}{a} = \frac{nd + m}{na + m}$$

which implies $m = 0$. Similarly if $q = 0$ we find $n = 0$. That establishes our claim about the conditions for solvability of (8).

We have proved that one of the two sides U, V is composed entirely of a sides of tiles, and the other is composed entirely of b sides. We may assume that U is the one composed entirely of a sides, i.e. $U = pa = p\sin\alpha$ and $V = nb = n\sin\beta$.

The area equation (7) then becomes

$$N\sin\alpha\sin\beta\sin\gamma \quad = \quad pn\sin\alpha\sin\beta\sin 2\alpha$$

Canceling $\sin\alpha\sin\beta$ we have

$$\begin{aligned} N\sin\gamma &= pn\sin 2\alpha \\ &= 2pn\sin\alpha\cos\alpha \end{aligned}$$

and putting in the values for $\sin\gamma$ and $\sin 2\alpha$, we find

$$\begin{aligned} N\frac{\sqrt{3}}{2} &= pn\frac{\sqrt{6}-\sqrt{2}}{4}\frac{\sqrt{6}+\sqrt{2}}{4} \\ &= \frac{pn}{4} \end{aligned}$$

contradicting the irrationality of $\sqrt{3}$. This contradiction shows that angle A cannot be 2α, and since that was the last remaining possibility, that completes the proof of the lemma.

6 $\gamma = 2\pi/3$ and $\alpha = 2\pi/15$ or $\alpha = \pi/9$

If we argue in the same way as we did for $\alpha = \pi/12$, we will be working in the field generated over $\mathbb{Q}$ by $a = \sin(2\pi/15)$ and $b = \sin(8\pi/15$. The arithmetic in this field is more cumbersome, and the number of possible vertex angles is greater. Instead, we use another method, involving primes in the cyclotomic field. We use, for the first and only time in this paper, concepts going beyond elementary field theory.

Lemma 6 *Suppose that isosceles triangle ABC is N-tiled by a triangle T whose largest angle $\gamma = 2\pi/3$. Then*

(i) $\alpha \neq 2\pi/15$

(ii) If $\alpha = \pi/9$ then the vertex angle of ABC cannot be $3\alpha = \pi/3$ or 5α.

Proof. Define

$$\begin{aligned} \zeta &:= e^{2\pi i/30} \text{ qquad when } \alpha = 2\pi/15 \\ \zeta &:= e^{2\pi i/18} \text{ qquad when } \alpha = \pi/9 \\ \rho &:= e^{i\pi/3} \end{aligned}$$

Then we have $e^{i\alpha} = \zeta^2$ in case (i) and $e^{i\alpha} = \zeta$ in case (ii). Let $K = 5$ when $\alpha = 2\pi/15$ and $K = 3$ when $\alpha = 2\pi/9$. Then we have $\rho = \zeta^K$. Since $\alpha + \beta + 2\pi/3 = \pi$, we have $\beta = \pi/3 - \alpha$. We calculate $\sin\alpha$ and $\sin\beta$ in terms of ζ. First, in case $\alpha = 2\pi/15$ we have

$$\begin{aligned} 2i\sin\alpha &= \zeta^2 - \zeta^{-2} \\ 2i\sin\beta &= e^{i(\pi/3-\alpha)} - e^{-i(\pi/3-\alpha)} \\ &= \rho\zeta^{-2} - \rho^{-1}\zeta^2 \\ &= \zeta^{K-2} - \zeta^{2-K} \end{aligned}$$

In case $\alpha = \pi/9$ we have instead

$$\begin{aligned} 2i\sin\alpha &= \zeta - \zeta^{-1} \\ 2i\sin\beta &= e^{i(\pi/3-\alpha)} - e^{-i(\pi/3-\alpha)} \\ &= \rho\zeta^{-1} - \rho^{-1}\zeta \\ &= \zeta^{K-1} - \zeta^{1-K} \end{aligned}$$

These expressions belong to $\mathbb{Z}[\zeta]$ and hence are algebraic integers of the field $\mathbb{Q}(\zeta)$. Let X, Y, and Z be the sides of triangle ABC in non-decreasing order. Then

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \mathbf{d} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mathbf{d} \begin{pmatrix} \sin\alpha \\ \sin\beta \\ \sin\gamma \end{pmatrix},$$

so $2iX$, $2iY$, and $2iZ$ are algebraic integers of $\mathbb{Q}(\zeta)$.

The area $\mathcal{A}_T$ of the tile can be computed as a cross product

$$\begin{aligned} \mathcal{A}_T &= \frac{1}{2} bc\sin\alpha \\ &= \frac{1}{2}\sin\alpha\sin\beta\sin\gamma \end{aligned}$$

Since the factor on the right are algebraic integers divided by $2i$, it follows that $16i^3\mathcal{A}_T$ belongs to $\mathbb{Z}[\zeta]$.

The area of triangle ABC is N times the area of the tile. That is,

$$N\mathcal{A}_T = \mathcal{A}_{ABC}$$

The area of triangle ABC is given by $\frac{1}{2}U^2\sin\theta$, where U is the length of the two equal sides of ABC, and θ is the vertex angle. Inserting the forms of these expressions given above, and multiplying by 2, we have

$$N\sin\alpha\sin\beta\sin\gamma = U^2\sin\theta \tag{9}$$

Example. To fix the ideas, we pause to give an example. Of course, the point of the theorem is that no example exists, so to give an example we must choose $\alpha = \beta$, which is not covered by the theorem, but is not contradicted by the arguments so far. Consider the $3r^2$ tiling, in which ABC is equilateral and T has $\alpha = \beta = \pi/6$. Then only c sides share the boundary of ABC, so in the formula for $\mathcal{A}_{ABC}$, the coefficients of $\sin\alpha$ and $\sin\beta$ are zero, and the coefficients of $\sin\gamma$ are $3r$, r, r, and r in the four terms, respectively. Hence

$$\begin{aligned} \mathcal{A}_{ABC} &= (r^2\sin^2\gamma/4)\sqrt{3} \\ &= \frac{r^2(\sqrt{3}/2)^2}{4}\sqrt{3} \\ &= \frac{3\sqrt{3}r^2}{1}6 \end{aligned}$$

We have $\mathcal{A}_T = \sin\alpha \sin\beta \sin\gamma/2 = \sqrt{3}/16$. Since $N = 3r^2$ we check our equation: indeed $N\mathcal{A}_T = (3r^2)\sqrt{3}/16 = A_{ABC}$.

Now we return to the consideration of the area equation (9). Except for a factor of $16i$, both sides of (9) are algebraic integers of the field $\mathbb{Q}(\zeta)$, because they belong to $\mathbb{Z}[\zeta]$. Dividing by $U^2 \sin\alpha \sin\beta$ and using $\sin\gamma = \sqrt{3}/2$ we have

$$\frac{\sqrt{3}N}{2U^2} = \frac{\sin\theta}{\sin\alpha\sin\beta} \tag{10}$$

Recall that $2iU$ is an algebraic integer of $\mathbb{Q}(\zeta)$, since U is an integral linear combination of sides of the tile, which are $\sin\alpha$, $\sin\beta$, and $\sin\gamma$, and each of these is an algebraic integer over $2i$. We claim that $2i\sin\theta$ is also an algebraic integer. When $\alpha = 2\pi/15$, have shown in Lemma 3 that triangle ABC is isosceles with vertex angle $\theta = 3\beta = 3\pi/5$, or $\theta = \beta = \pi/5$. We have already shown that $2i\sin\beta$ is an algebraic integer, and $2i\sin(3\beta) = 2i(\zeta^{3K-6} - \zeta^{6-3K})$. On the other hand, when $\alpha = \pi/9$, Lemma 3 shows that the vertex splitting of ABC involves one α angle and four β angles, and here $\beta = \pi/3 - \alpha = 2\pi/9 = 2\alpha$. If ABC is isosceles, then the vertex angle θ is either α or $\alpha + \beta = \pi/3$, and in either case $2i\sin\theta$ is an algebraic integer.

Now suppose that the right hand side of (10) has norm a (positive or negative) power of 2; or more generally, at least the norm contains only an even power of 3, not an odd power. Suppose, in addition, that the rational prime 3 does not split in $\mathbb{Q}(\zeta)$, but ramifies with ramification degree 2. Then there is just one prime $\mathfrak{P} = \sqrt{3}$ lying over the rational prime 3, and $\mathfrak{P}$ divides N to an even power (twice the power of 3 in N), and $\mathfrak{P}$ divides U^2 to an even power (twice the power of $\mathfrak{P}$ in U); and it divides $\sqrt{3}$ just once; hence the power of $\mathfrak{P}$ in the left-hand side is odd. Hence the norm of the left-hand side contains a (positive or negative) power of 3. But that contradicts the assumption that the right-hand side contains only an even power of 3.

A similar argument also works in case the prime $p = 3$ ramifies as $\mathfrak{P}^e$, where e is congruent to 2 mod 4. (We only need the case $e = 6$). Then again $\mathfrak{P}$ divides N to an even power (twice the power of 3 in N), and it divides U^2 to an even power (twice the power of $\mathfrak{P}_i$ in U). Now $\sqrt{3} = \sqrt{\mathfrak{P}^e} = \pm\mathfrak{P}^{e/2}$, and since e is congruent to 2 mod 4, this is an odd power of $\mathfrak{P}$.

We need to know how the rational primes split and ramify in $\mathbb{Q}(\zeta)$. Turning to number theory textbooks for the solution to this problem, we found that many textbooks are content to state the facts only for primes p relatively prime to m, where ζ is a primitive m-th root of unity. But exercise 14, p. 337 of [4], states the facts in full generality: Let p be a rational prime, and let $m = p^k m'$ where p does not divide m', then p factors into $(\mathfrak{P}_1 \ldots \mathfrak{P}_g)^e$ in $Q(\zeta)$, where $fg = \varphi(m')$ and f is the smallest integer such that $p^f \cong 1 \bmod m'$, the norm of $\mathfrak{P}_i$ is p^f, and the ramification degree e is $\varphi(p^k)$.

To prove (i), we apply this result when $\zeta = e^{2\pi i/30}$, so $m = 30$, and $p = 3$. Then $m' = 10$ and $k = 1$, and f is the smallest integer such that 3^f is congruent to 1 mod 10. Hence $f = 4$. Since $\varphi(m') = \varphi(10) = 4$, we have $g = 1$. The ramification degree e is $\varphi(3) = 2$. Hence there is only one $\mathfrak{P}_i = \sqrt{3}$, and its norm is 3^4.

We now show that these assumptions do in fact hold when $\alpha = 2\pi/15$. Then $\beta = \pi/5$, and we have shown in Lemma 3 that triangle ABC is isosceles with vertex angle $\theta = 3\beta = 3\pi/5$, or $\theta = \beta = \pi/5$. We need to compute the norms of $\sin(2\pi/15)$, $\sin(\pi/5)$, and $\sin(3\pi/5)$. We use the definition of the norm of x: it is the product of the images of x under all members of the Galois group. Since the Galois group of $\mathbb{Q}(\zeta)$ consists of the maps σ_j that take ζ onto ζ^j for j relatively prime to 15, we have the following formulas for the norms:

$$\begin{aligned}
\mathcal{N}(2i\sin(2\pi/15)) &= \prod_{(j,30)=1} 2i\sin(4j\pi/30) \;=\; 1 \\
\mathcal{N}(2i\sin(\pi/5)) &= \prod_{(j,30)=1} 2i\sin(6j\pi/30) \;=\; 25 \\
\mathcal{N}(2i\sin(3\pi/5)) &= \prod_{(j,30)=1} 2i\sin(18j\pi/30) \;=\; 25
\end{aligned}$$

Since we know in advance that the norm of an algebraic integer is an integer, it suffices to compute these numbers to a few decimal places using an ordinary scientific calculator (or a computer algebra system or a short computer program), and round to the nearest integer. This computation reveals that $2i\sin(2\pi/15) = 2i\sin\alpha$ has norm 1, and hence is a unit in $\mathbb{Z}[\zeta]$, while $2i\sin(\pi/5) = 2i\sin\beta$ and $2i\sin(3\pi/5) = 2i\sin(3\beta)$ each have norm 25. Since the vertex angle θ is either β or 3β, in either case the norm of $2i\sin\theta$ is 25. Hence the ratio $\sin\theta/\sin\beta$ has norm 1, so it is a unit, and since $\varphi(30) = 8$, the norm of $\sin\alpha = \sin(2\pi/15)$ is $1/2^8 = 1/256$. Hence the norm of the right hand side of (10) is a power of 2, as claimed. We have already shown above that the prime $p = 3$ ramifies with degree 2 and does not split in this $Q(\zeta)$. That completes the proof of (i), i.e. the case $\alpha = 2\pi/15$.

Now we take up the case of $\alpha = \pi/9$. Then $3K = m = 18$ and $\beta = \pi/3 - \pi/9 = \pi/6 = 2\alpha$. First we consider how the prime $p = 3$ splits and ramifies. With $p = 3$ we have $m' = 2$ and $k = 2$, and f is the smallest integer such that 3^f is congruent to 1 mod 2. Hence $f = 1$. Since $fg = \varphi(m') = \varphi(2) = 1$, we have $g = 1$. The ramification degree e is $\varphi(p^k) = \varphi(3^2) = 6$. Hence 3 ramifies as $\mathfrak{P}^6$, where $\mathfrak{P}$ has norm 3. We have shown above that this is sufficient for the proof, provided the norm of the right hand side of (10) contains either no power of 3, or only an even power.

It remains only to compute the norm of the right hand side of (10). We compute the norms of $2i\sin\alpha$ and $2i\sin\beta$ and the two possibilities $\sin(3\alpha)$ and $\sin(5\alpha)$ for $\sin\theta$:

$$\begin{aligned}
\mathcal{N}(2i\sin(\pi/9)) &= 64\prod_{(j,18)=1}\sin(j\pi/9) = -3 \\
\mathcal{N}(2i\sin(\pi/6)) &= 64\prod_{(j,18)=1}\sin(j\pi/6) = 1 \\
\mathcal{N}(2i\sin(\pi/3)) &= 64\prod_{(j,18)=1}\sin(j\pi/3) = -27 \\
\mathcal{N}(2i\sin(5\pi/9)) &= 64\prod_{(j,18)=1}\sin(5j\pi/9) = -3
\end{aligned}$$

These values show that the right hand side of (10) has norm 1 or 9, depending on whether the vertex angle θ is $5\pi/9$ or $\pi/3$. Since both are even powers of 3, the proof of (ii) is comnplete. That completes the proof of the lemma.

Lemma 7 *Suppose triangle ABC is N-tiled by tile T, with $\gamma = 2\pi/3$ and $\alpha = \pi/9$ or $\alpha = \pi/12$. Then the area of the tile is not a rational number, and not a rational multiple of $\sqrt{3}$.*

Proof. We begin by proving that the area of the tile is not a rational number. Since the sides of the tile are given by $a = \sin\alpha$, $b = \sin\beta$, and $c = \sin\gamma$, we can find the area by taking the cross product of two of the sides:

$$\mathcal{A}_T = \frac{1}{2}\sin\alpha\sin\beta\sin\gamma$$

Since $2i\sin\alpha = \zeta - \zeta^{-1}$, and similarly for $2i\sin\beta$ and $2i\sin\gamma$, the area $\mathcal{A}_T$ is i times an element of the field $\mathbb{Q}(\zeta)$, and if $\mathcal{A}_T$ were rational, or even in $\mathbb{Q}(\zeta)$, then i would belong to $\mathbb{Q}(\zeta)$. When $\alpha = \pi/9$, i does not belong to $\mathbb{Q}(\zeta)$ (see, for example, p. 145 of [5]), so we are done when $\alpha = \pi/9$.

Therefore we assume $\alpha = \pi/12$. If $\mathcal{A}_T$ were rational, then $\mathcal{A}_T$ would be fixed by the Galois group of $\mathbb{Q}(\zeta)$. The elements of that Galois group are the σ_j, for j relatively prime to 24, where $\zeta = e^{2\pi i/24}$ is a primitive 24-th root of unity, and σ_j takes ζ onto ζ^j. Choose $j = 5$, which is relatively prime to 24. Consider σ_5. Since $i = \zeta^6$, σ_5 takes i to $\zeta^{30} = \zeta^6 = i$; in other words, σ_5 fixes i. It takes $i\sin\alpha = \zeta - \zeta^{-1}$ to $i\sin(5\alpha)$ and, since $\beta = 3\alpha = \pi/4$, it take $i\sin\beta$ to $i\sin(5\beta) = -i\sin(\beta)$. It takes $i\sin(\gamma)$ to $i\sin(5\gamma) = -i\sin(\gamma)$. Thus σ_5 changes the signs of

$\sin\beta$ and of $\sin\gamma$; hence it fixes $\sin\beta\sin\gamma$. Assume, for proof by contradiction, that the area of the tile is rational. Then $\mathcal{A}_T = \sin\alpha\sin\beta\sin\gamma$ is fixed by σ_5; since $\sin\beta\sin\gamma$ is fixed, $\sin\alpha$ must also be fixed. Since i is fixed that means $i\sin\alpha$ is also fixed. But as we have already observed, $i\sin\alpha$ goes to $i\sin(5\alpha) = i\cos(\alpha)$, which is not equal to $i\sin\alpha$. Hence the area of the tile is not rational, also in case $\alpha = \pi/12$.

Next we will show more: the area of the tile is not a rational multiple of $\sqrt{3}$. We have proved above that σ_5 changes the sign of $i\sqrt{3}$ and fixes i; so it changes the sign of $\sqrt{3}$. Assume, for proof by contradiction, that $\mathcal{A}_T$ is a rational multiple of $\sqrt{3}$. Then σ_5 changes the sign of $A_T = \sin\alpha\sin\beta\sin\gamma$. But since it fixes $\sin\beta\sin\gamma$, it changes the sign of $\sin\alpha$. But on the contrary, it takes $\sin\alpha$ to $\cos\alpha \neq -\sin\alpha$. That completes the proof of the lemma.

Lemma 8 *Suppose ABC is N-tiled by a triangle T whose largest angle $\gamma = 2\pi/3$, and ABC is not similar to T. Then $\alpha \neq \pi/9$, where α is the smallest angle of T.*

Proof. Note that $\zeta = e^{2\pi i/18}$ has degree $\varphi(18) = 6$ over $\mathbb{Q}$. Since $\mathbb{Q}(\zeta)$ does not contain i (because 4 does not divide 18), the field $\mathbb{Q}(\sin\alpha, \cos\alpha, i)$ has degree 12 over $\mathbb{Q}$. Hence its real subfield $\mathbb{Q}(\sin\alpha, \cos\alpha)$ has degree 6. Hence $\sin\alpha$ and $\cos\alpha$ are both irrational, as if either were rational then the degree would be 4. The automorphism σ_j of $\mathbb{Q}(\zeta)$ takes ζ to ζ^j, where j is relatively prime to 18, namely $j = 1, 5, 7, 11, 13, 17$. Let $a = \sin\alpha$, $b = \sin\beta$, $c = \sin\gamma$, and $d = \sin 4\alpha = \sin 5\alpha$. Note that i does not belong to $\mathbb{Q}(\zeta)$, but $2i\sin m\alpha = \zeta^m - \zeta^{-m}$ does. There is no automorphism that fixes exactly two of a, b, and c. This complicates the algebra. We will work with σ_5, which takes ib to $-ia$, and ic to $-ic$, and ia to id.

Suppose, for proof by contradiction, that angle $A = \alpha$. Let U and V be the lengths of the sides AB and AC. Then the area equation is

$$N\sin\alpha\sin\beta\sin\gamma = UV\sin\alpha$$

Canceling $\sin\alpha$ we have

$$Nbc = UV \tag{11}$$

Let $U = pa + qb + rc$ and $V = ma + nb + \ell c$. Then

$$\begin{aligned} (iU)\sigma &= ipd - iqa - irc \\ (iV)\sigma &= imd - ina - i\ell c \\ (UV)\sigma &= (pd - qa - rc)(md - na - \ell c) \end{aligned}$$

Applying σ to (11) (after inserting two factors of i on each side) we have

$$Nac = (pd - qa - rc)(md - na - \ell c)$$

Dividing this equation by (11) we have

$$\frac{b}{a} = \frac{(pa + qb + rc)(ma + nb + \ell c)}{(pd - qa - rc)(md - na - \ell c)}$$

This equation is solvable when V is composed entirely of c sides (so $n = m = 0$) and U is composed entirely of b sides (so $p = r = 0$), as in the quadratic tiling; and vice-versa when V is composed entirely of c sides and U entirely of b sides. We will show that these are the only possible solutions. Cross multiplying we have

$$a(pa + qb + rc)(ma + nb + \ell c) - b(pd - qa - rc)(md - na - \ell c) = 0$$

Then ζ^{14} times this expression is a polynomial in ζ. Taking its remainder under division by the minimal polynomial of ζ, which is $\zeta^6 - \zeta^3 + 1$, we find (using the *PolynomialRemainder* function in Mathematica)

$$\begin{aligned}
0 \quad = \quad & -2mp + np + mq - 2nq - 3\ell r \\
& +\zeta(2mp - np - mq + 2nq + 3\ell r) \\
& +\zeta^2(-6\ell p + 4mp - 2np - 2mq - nq - 6mr) \\
& +\zeta^3(4mp - 2np - 3q + 4nq + 6\ell r) \\
& +\zeta^4(-4m + 2np + 3mq - 4nq - 6r) \\
& +\zeta^5(3\ell p - 2mp + np + mq + nq + 3mr)
\end{aligned}$$

Each of the five coefficients is zero; but up to constant multiples there are really only two different coefficients, so we have the following two equations:

$$\begin{aligned}
2mp + 2nq + 3\ell r \quad &= \quad np + mq \\
4mp \quad &= \quad 2np + 2mq + 6mr
\end{aligned}$$

Adding twice the first equation to the second we find

$$6mr + 6\ell p + 2nq + 3\ell r = 0.$$

Since all these quantities are nonnegative, this implies $mr = \ell p = nq = \ell r = 0$. Eliminating these terms, our two equations both become

$$2mp = np + mq \tag{12}$$

Suppose $\ell \neq 0$. Then since $\ell p = 0$ and $\ell r = 0$ we have $p = r = 0$. Then $q \neq 0$ because $p + q + r$ is the number of tiles on side U. Since $nq = 0$ we have $n = 0$. Then by (12) we have $0 = mq$; and since $q \neq 0$ we have $m = 0$. This is the first solution mentioned above, in which only ℓ and q are nonzero; we have proved that this is the only solution with $\ell \neq 0$. Now suppose $\ell = 0$. Then the c side of the tile at vertex A lies on side V, since it cannot lie on side U and cannot lie opposite angle A. Hence $r \neq 0$. Since $mr = 0$ we have $m = 0$. Then by (12), we have $np = 0$. But $n \neq 0$ since ℓ and m are both zero. Hence $p = 0$. Since $nq = 0$ and $n \neq 0$, we have $q = 0$. Thus if $\ell = 0$, it follows that only n and r are nonzero; that is the second solution mentioned above. We have thus proved our claim that one of these two solutions must be the case.

Then one of the sides U, V is composed entirely of b sides of tiles. Let T_1 be the tile sharing vertex $A = V_1$. Let V_2 on side U be the vertex at the other end of the b side of T_1. Then the γ angle of T_1 is at V_2. Hence the γ angle of the next tile T_2 on U cannot be at V_2; and since the b side of T_2 is on U, the β angle is opposite that side, and not on U. Hence the β angle of T_2 is at V_2, and the γ angle of T_2 is at the next vertex V_3. Continuing in this way down side U, we find that the last tile T_k on U has its γ angle at the vertex of ABC, which must be vertex C. Therefore angle C is at least γ. Suppose, for proof by contradiction, that it exceeds γ. Then angle C must be $\gamma + \alpha$ and the third angle must be α, making ABC isosceles. But that contradictions Lemma 3. This contradiction shows that angle C is equal to γ. But then ABC is similar to T, which is contrary to hypothesis. That completes the proof by contradiction that angle A cannot be equal to α.

The smallest angle A of ABC cannot be 3α, since then ABC would be equilateral, contradicting Lemma 3. Hence angle A must be $2\alpha = \beta$. Then angle B is at least 3α, since ABC cannot be isosceles. That means angle C is at most 4α. But since we cannot have angle B equal to angle C, the only remaining possibility is that angle $B = 3\alpha$ and angle $C = 4\alpha$. Let U and V be the sides AB and AC (in some order). Then the area equation is

$$N \sin\alpha \sin\beta \sin\gamma = \sin\beta UV$$

or in terms of a, b, and c, it is

$$Nabc = bUV$$

Dividing sides by b and setting $U = pa + qb + rc$ and $V = ma + nb + rc$, we have

$$Nac = (pa + qb + rc)(ma + nb + \ell c)$$

Applying σ we have

$$-Ndc = (pd - qa - rc)(md - na - \ell c)$$

Dividing these two equations we find

$$-\frac{a}{d} = \frac{pa + qb + rc}{pd - qa - rc}\frac{ma + nb + \ell c}{md - na - \ell c}$$

Cross multiplying, we have

$$a(pd - qa - rc)(md - na - \ell c) + d(pa + qb + rc)(ma + nb + \ell c) = 0.$$

After multiplying by ζ^{16}, this becomes a polynomial in ζ. Reducing that polynomial modulo the cyclotomic polynomial $1 - \zeta^3 + \zeta^6$ (using Mathematica) we find

$$\begin{aligned} 0 \quad = \quad & 2mp - np - mq + 2nq + 3\ell r \\ & +\zeta(mp - 2np - 3\ell q - 2mq + nq - 3nr) \\ & +\zeta^2(-4mp + 2np + 2mq - 4nq - 6\ell r) \\ & +\zeta^3(-4mp + 2np + 2mq - 4nq - 6\ell r) \\ & +\zeta^4(mp - 2np - 3\ell q - 2mq + nq - 3nr) \\ & +\zeta^5(2mp - np - mq + 2nq + 3\ell r) \end{aligned}$$

Equating the coefficients to zero gives us only two different equations:

$$\begin{aligned} 2mp - np - mq + 2nq + 3\ell r &= 0 \\ mp - 2np - 3\ell q - 2mq + nq - 3nr &= 0 \end{aligned}$$

Subtracting the second equation from the first we have

$$3\ell q + mp + np + mq + nq + 3\ell r + 3nr = 0$$

Since all these quantities are non-negative this implies that each of ℓq, mp, np, mq, nq, ℓr, and nr is zero. Suppose, for proof by contradiction, that $\ell \neq 0$. Then $q = r = 0$, so $p \neq 0$. Hence $n = m = 0$. Hence U is composed only of a sides of tiles, and V is composed only of c sides. The tile T_1 at A that shares side U therefore has its a side on U and its β angle at A. Hence its γ angle is at vertex V_2, the other vertex of T_1 on U. Hence the next tile T_2 on side U has its β angle at V_2 (since its a side is on U and its γ angle is too large to fit). Hence it has its γ angle at the next vertex V_3. Continuing down side U in this way we find that the last tile on side U has its γ angle at the other vertex of side U of ABC. But this is impossible, as the largest angle of ABC is 4α, which is less than γ. This completes the proof by contradiction that $\ell = 0$. Similarly, if $r \neq 0$ then $n = \ell = 0$, so $m \neq 0$. Hence $p = q = 0$, and by considering tiles on side V we find that the angle at the other end of side V must be at least γ, contradiction. Hence $r = \ell = 0$. Then no tile at A has a c side on the boundary of ABC, so the vertex at A must be split, and there are two tiles T_1 and T_2 with their α angles at A, sharing their c sides and having their b sides along U and V respectively. Hence $q \neq 0$ and $n \neq 0$. But this contradicts $nq = 0$. That contradiction completes the proof of the lemma.

Theorem 1 *Suppose $\gamma = 2\pi/3$ and there is an N-tiling of ABC by the tile with angles α, β, and γ with $\alpha \neq \beta$, and ABC is not similar to the tile. Then the vertex splitting is given by $(3, 3, 0)$, i.e., there are three α angles and three β angles and no γ angles at the vertices of ABC, in total.*

Proof. Assume that the vertex splitting is not $(3, 3, 0)$. Then by Lemma 3, α is $\pi/9$, or $\pi/12$, or $2\pi/15$. First suppose that $\alpha = 2\pi/15$. By Lemma 3, ABC is isosceles with base angle β or 2β; hence the vertex angle is 3β or β. But these cases have been ruled out in Lemma 6. That disposes of the possibility $\alpha = 2\pi/15$. The case $\alpha = \pi/9$ is ruled out in Lemma 8, and the case $\alpha = \pi/12$ is ruled out in Lemma 5. That completes the proof of the theorem.

7 Existence of an edge relation

Definition 1 *An **edge relation** in a tiling of ABC by the tile with sides a, b, and c, is a integer linear relation between a, b, and c, that is determined by an internal line segment of the tiling (composed of tile boundaries) that has different numbers of tile edges of lengths a, b, or c on its two sides.*

For example, there might be 5 b edges on one side of a segment EF and on the other side, 3 a edges and 2 c edges, giving rise to the relation $5b = 3a + 2c$. Of course, there could have been one more b edge on each side, giving rise to the same relation. We emphasize that the concept requires not just that a numerical equation be satisfied, but also that it actually be realized in the tiling by some internal segment composed of tile boundaries.

Lemma 9 *Suppose $\gamma = 2\pi/3$ and there is an N-tiling of ABC by the tile with angles α, β, and γ with $\alpha \neq \beta$. Then there are no vertices in that tiling at which an angle equal to β is made up only of α angles. More generally, if $\beta = k\alpha$ then there is no vertex at which k tiles have an α angle.*

Remarks. In this section generally we do not assume $\alpha < \beta$, but of course if $\alpha > \beta$ an angle of β cannot be made up of α angles, so for this lemma we may assume $\alpha < \beta$. Note that the lemma does not say that β is not a multiple of α; only that if it is, at least the tiling makes no use of that fact by using several α angles to fill a β angle.

Proof. Suppose that $\beta = k\alpha$, for some integer k, and that there is a vertex V where k tiles each have an α angle. Since $\alpha + \beta = \pi/3$, we have $(k + 1)\alpha = \pi/3$, or $\alpha = \pi/(3(k + 1))$. We claim that at vertex V, there are more α angles than β angles. Indeed, at every vertex, either there are only three γ angles, or the number of α angles equals the number of β angles, or one or more β angles are filled by k α angles. Hence, if the latter occurs (as it supposedly does at V) then there are more α angles than β angles.

Since there are, by hypothesis, equal numbers of α and β angles at the vertices of ABC, and there are of course N of each in total (since each tile contributes one α and one β), then there must exist a vertex W with more β than α angles. Then there are zero, one, or two γ angles at W. If there are exactly two, then the remaining angle is $\alpha + \beta$; but since $\alpha < \beta$, two β angles cannot be place at W, contradiction. If there is exactly one γ angle at W, then the angle remaining is $2\gamma = 2\alpha + 2\beta$. If three β angles occur at W then $\beta = 2\alpha$, since $\beta \neq \alpha$ by hypothesis. Then since $\alpha + \beta = \pi/3$, we have $\alpha = \pi/9$. But that contradicts Lemma 8, so it is not the case that there is exactly one γ angle at W. Therefore there are no γ angles at W. We therefore have $2\pi = u\beta + v\alpha$, where $u > v$. Since $2\pi = 6(\pi/3) = 6\beta + 6\alpha$, we must have $u > 6$ and $v < 6$. Subtracting $v\pi/3 = v(\alpha + \beta)$ from both sides of $u\beta + v\alpha = 2\pi$, we have

$$(u - v)\beta = 2\pi - \frac{v\pi}{3} = (6 - v)\frac{\pi}{3}.$$

Then

$$\beta = \left(\frac{6 - v}{u - v}\right)\frac{\pi}{3}$$

Since $\alpha + \beta = \pi/3$ we have

$$\alpha = \left(\frac{u - 6}{u - v}\right)\frac{\pi}{3}.$$

Since $\alpha < \beta$ we have $u - 6 < 6 - v$. Hence $u + v < 12$. But we showed above that α has the form $\pi/(3(k+1))$ for some positive integer k. Therefore

$$\frac{u-6}{u-v} = \frac{1}{k+1}$$

Cross multiplying, we have

$$\begin{aligned} u - v &= (u-6)(k+1) \\ &= ku - 6k + u - 6 \\ 6 &= ku + v \end{aligned}$$

But this is impossible, since $u > 6$ and $k \geq 1$ and $v > 0$. That contradiction completes the proof that there are no vertices at which an angle equal to β is made up only of α angles.

The following terminology will shorten some statements:

Definition 2 *A **suspicious edge** is a tile boundary PQ in (the opposite of) Direction A or Direction C between two tiles that share a vertex P, and one of the tiles has a b edge along PQ and the other tile has an a or c edge along PQ.*

Remark. Such an edge is "suspicious" because it looks as if it might lead to an edge relation; it can only fail to do so if it turns out that there are an equal number of edges of each length on each side of the maximal segment containing the suspicious edge.

When we say a line segment has "direction AB" we mean that it is parallel to AB, and similarly with AC or BC or other specified directions. In addition to the directions given by the sides of ABC, we will also need two other directions.

Definition 3 **"Direction C"** *is given by a line making angle of α with BC and having a negative slope with smaller magnitude than the slope of BC.* **"Direction A"** *is given by a line making an angle of α with AB and having a positive slope of smaller magnitude than the slope of AB.*

Thus the bottom edge of the top tile (the one at B) is either in Direction A or Direction C, depending on its orientation.

Definition 4 *Suppose triangle ABC has angle $B = \beta$. Tiles with their c edge parallel to BC and their a edge parallel to AB are said to be "**of Type I**". Tiles with their a edge parallel to BC and their c edge parallel to AB are "**of Type II**."*

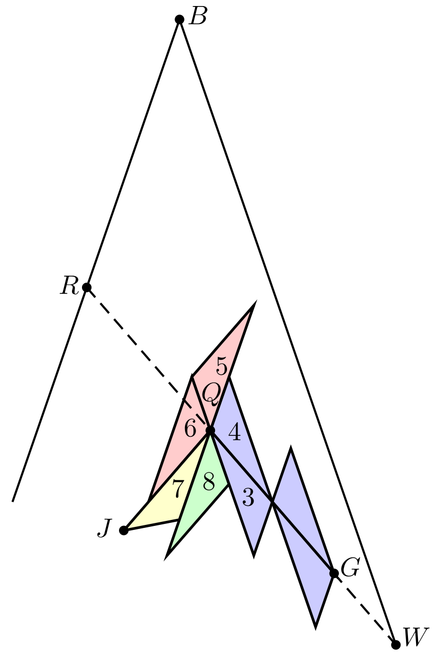
Type I tiles have their b edges in Direction C, and Type II tiles have their b edges in Direction A; Type I tiles have no edges at all in Direction A, and Type II tiles have no edges in Direction C.

Lemma 10 *Suppose $\gamma = 2\pi/3$ and $\alpha \neq \beta$, and there is an N-tiling of triangle ABC by the tile with angles α and β, and angle $ABC = \beta$, and ABC is not similar to the tile. Suppose there are no edge relations $jb = ua + vc$ with integers $j > 0$ and $v > 0$ and $u \geq 0$. Let RW be a line segment in direction C (and contained in the closed triangle ABC) with R northwest of W, and such that all tiles wholly or partially above RW are of Type I or Type II, but there is a point Q on RW that is on the boundary of a tile (below RW) that is neither of Type I nor of Type II. Suppose that Q is the northwest end of a maximal segment lying on RW. Then there is a suspicious edge in direction A ending at Q.*

Remark. Note that the hypothesis allows the case when a/b is rational, i.e., a relation $jb = ua + vb$ with $v = 0$ is allowed.

Proof. Let G be a point on QW such that QG is a maximal segment; by hypothesis, such a point G exists. Let Tile 4 be the tile above QG, with their b edges on QG, and a vertex at Q.

Figure 1: A suspicious edge QJ must exist southwest of Q.



Then Tile 4 has its γ angle at Q and its b edge on QG. Suppose for the moment that Q doe not lie on AB. Then there is a tile west of Tile 4; call it Tile 5. Since Tile 5 lies above RW it is of Type I or Type II. Either way, it has its β angle at Q. Let Tile 6 be the tile southwest of Tile 5; then Tile 6 is partly above RW, so it is of Type I or Type II. Tile 6 is not of Type I, since then its α angle would be at Q and it would have an edge on RW northwest of Q, contradicting the fact that QG is a maximal segment. Hence Tile 6 is of Type II. See Fig. 1 for an illustration. Tile 6 has its γ angle at Q, and its b edge on the southeast. Let J be the southern vertex of Tile 6.

Now (whether or not Q lies on AB), let Tile 3 be the tile below QG with an edge on QG and a vertex at Q. We have assumed there is no relation $jb = ua + vc$ with j and v both positive; that still allows for a relation $jb = ua$, i.e. v could be zero. Since there is no relation $jb = ua + vc$, all the tiles below QG with an edge on QG have their b edges or their a edges on QG, so Tile 3 has its b or its a edge on QG. We claim that Tile 3 does not have its γ angle at Q. If Q lies on AB this is immediate, since angle ARG is less than γ. In case Q does not lie on AB, then Tiles 5 and 6 exist, and angle QJG is equal to $2\alpha + \beta$, which is less than γ, so Tile 3 cannot have its γ angle at Q. Therefore Tile 3 has its α angle at Q. The remaining angle at Q is $\alpha + \beta$. By Lemma 9, this angle must be filled by two tiles, say Tile 7 and Tile 8, in counterclockwise order from Tile 6 back to Tile 3, one of which has an α angle at Q and the other of which has a β angle at Q.

Suppose, for proof by contradiction, that Tile 7 has its b edge along QJ. Then Tile 7 cannot have its β angle at Q, since the β angle must be opposite the b edge. It cannot have its γ angle at Q since angle JQG is $2\alpha + \beta < \gamma$. Hence Tile 7 has its α angle at Q. But then, with its b edge on JQ, it is of Type I. Then Tile 8 has its β angle at Q and the two adjacent sides parallel to AB and BC, so it is of Type I or Type II, contradicting the fact that Q must lie on the boundary of some tile that is neither of Type I nor of Type II. That contradiction shows that Tile 7 does not have its b edge along QJ. Therefore it has its a or its c edge along QJ. See Fig. 1. That completes the proof of the lemma.

We need to state a more general lemma that is proved by the same proof as the previous lemma. The previous lemma assumed that the top angle B of ABC does not split but has just one tile with a vertex at B. The next lemma allows more than one angle at B, but ignores what happens to the east of the tile boundary between the two tiles at B.

Lemma 11 *Suppose $\gamma = 2\pi/3$ and $\alpha \neq \beta$, and there is an N-tiling of triangle ABC by the tile with angles α and β. Suppose there are two tiles at B, and the tile boundary between those two tiles lies on the maximal segment BH, and angle $ABH = \beta$. Suppose there are no edge relations $jb = ua + vc$ with integers $u, v \geq 0$ and $j > 0$. Let RW be a line segment in direction C, with W on BH and R northwest of W, and such that all tiles wholly or partially above RW are of Type I or Type II, but there is a point Q on RW that is on the boundary of a tile (below RW) that is neither of Type I nor of Type II. Suppose that Q is the northwest end of a maximal segment lying on RW. Then there is a suspicious edge in direction A ending at Q.*

Similarly, if ST is a line segment in direction A, with S on AB and T on BH, such that all tiles wholly or partly above ST are of Type I or Type II, but there is a point Q on ST that is on the boundary of a tile that is neither of Type I nor Type II, and Q is the northeast end of a maximal segment lying on ST, then there is a suspicious edge in direction C ending at Q.

Remark. Nothing to the east of BH is relevant, but because the symmetry of the angle ABC is broken by the existence of another tile to the east and the fact that BH may not reach to AC, we need to state the second conclusion about ST separately.

Proof. Same as the previous lemma.

Lemma 12 *Suppose $\gamma = 2\pi/3$ and there is an N-tiling of ABC by the tile with angles α, β, and γ with $\alpha \neq \beta$, and suppose ABC is not similar to the tile. We do not assume $\alpha < \beta$ in this lemma. Suppose there is exactly one tile at vertex B, having angle β at V. Then for some integers $j > 0$ and $u, v \geq 0$, the tiling gives rise to an edge relation $jb = ua + vc$.*

Remark. Definition 1 specifies that an "edge relation" actually arises from some internal segment in the tiling; more is being asserted than the arithmetical relationship between the side lengths.

Proof. Assume, for proof by contradiction, that there are no edge relations $jb = ua + vc$ of the type mentioned in the lemma. For purposes of description, we suppose ABC is drawn with vertex B at the top of the picture, which we call "north", and its angle bisector vertical ("north-south"). The orientation of side AC is not known, because we do not know the other angles of ABC. The tile at B, say Tile 1, has its β angle at B, by hypothesis; and relabeling A and C if necessary we can assume without loss of generality that Tile 1 has its c side on BC, the eastern side of ABC, and its a side on AB (at the west). In other words, Tile 1 is of Type I. We remind the reader that the (unoriented) direction of the bottom edge of Tile 1 is "Direction C". Lines in Direction C make an angle of α with BC.

We consider lines RW in Direction C with R on AB and W on BC, and maximal segments FH (composed of tile boundaries) lying on RW, such that all the tiles wholly or partly above RW that have an edge or vertex on FH are of Type I or Type II. There is at least one such line RW, namely the southern border of the tile at vertex B. Let FH be a maximal segment on such a line RW, with F northwest of H. Then all the tiles above FH with an edge on FH have their b edges on FH. Since FH is a maximal segment, there is a tile below FH with an edge on FH and a vertex at F, and there is a tile below FH with an edge on FH and a vertex at H.

Since there are no edge relations $jb = ua + vc$, all the tiles below FH with an edge on FH have their b edges on FH. We claim that all these tiles are of Type I. To show that, we must show that they are all oriented with their γ angles to the southeast and their α angles to the northwest. There cannot be two γ angles belonging to tiles below FH at any vertex on FH, since $\gamma > \pi/2$. Let Tile 3 be the tile below FH with an edge on FH and a vertex at F. By Lemma 10, there is a suspicious edge QJ extending southwest in Direction A from Q. Since angle JQW is equal to $2\alpha + \beta$, Tile 3 cannot have its γ angle at Q. Then by the pigeonhole principle, all the tiles below FH with an edge on FH have their γ angles to the southeast. Since there are no edge relations $jb = ua + vc$, those tiles all have their b edges on FH; hence they are all of Type I.

Similarly, we consider lines S in Direction A with S on AB and T on BC, and maximal segments FH (composed of tile boundaries) lying on ST, with F southwest of H, such that all the tiles wholly or partly above ST that have an edge or vertex on FH are of Type I or Type II. The entire argument we gave for lines in direction RW did not involve the base AC of the triangle at all, so it is symmetric under reflection in the angle bisector of angle B. (Or we could appeal to Lemma 11). Hence the conclusions we reached about maximal segments on RW also apply to line segments ST in Direction A.

Now we fix RW as the lowest such line (segment) in direction C such that any tile wholly or partially above RW is of Type I or Type II. There must be some such lowest segment, even if only because $W = C$ makes RW the lowest segment in direction C with W on BC. Similarly, we let ST be the lowest line (segment) in direction A, with S on BC and T on AC, such that any tile wholly or partially above ST is of Type I or Type II. For proof by contradiction, we now assume that $W \neq C$ and $S \neq A$. The segments RW and ST contain zero or more maximal segments FH lying on tile boundaries, and may also contain some segments that pass through Type II tiles (which have no tile boundaries in Direction C). By definition of RW and ST, unless $W = C$ or $S = A$, there exists a point Q on RW such that Q is a vertex of a tile, say Tile 2, neither of Type I nor of Type II. We now show that several *a priori* possible locations of Q are actually not possible.

Case 1, Q lies in the interior of a maximal segment FH on RW (so FH lies on tile boundaries) with F northwest of H. Then as shown above, all the tiles below FH with an edge on FH are of Type I, and have their b edges on FH. These tiles form "notches" between them. Each such notch has sides parallel to AB and BC, and the angle remaining to be filled at each notch is β. By Lemma 9, an angle of β cannot be filled by a number of α angles. Hence each notch must be filled by a single tile. That tile is necessarily of Type I or Type II, since it has its a and its c edge parallel to AB and BC, or vice-versa. Hence Q does not occur at a vertex on the interior of segment FH, i.e. Case 2 is impossible. (One such notch can be seen in Fig. 1 along the maximal segment QG in that figure.)

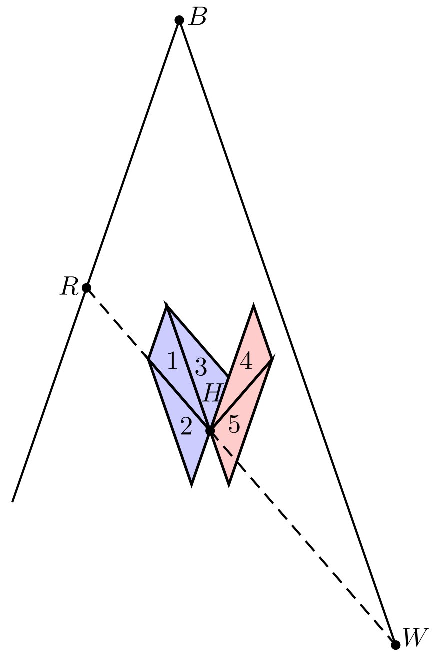
Case 2, $Q = F$ lies on the boundary AB, that is $F = R$, where FH is a maximal segment on RW. Then there is a tile below FH with a vertex at F and its α angle at F, so the angle between that tile and FC is β, and that angle must be filled by a Type I or Type II tile. Hence, if $Q = F$, then Q does not lie on AB, i.e. Case 2 is impossible.

Case 3, Q lies on the boundary BC. Then by Case 1, $Q = H$ for some maximal segment FH on RW. Let Tile 1 be the tile above FH with a vertex at H; since angle $BHF = \alpha$, there is only one such tile, and it is of Type I. As shown above, all the tiles below FH with an edge on FH have their b edge on FH and are of Type I; let Tile 2 be the tile below FH with a vertex at H and an edge on FH. Then Tile 2 is of Type I, so it has its γ angle at H. There is then an unfilled angle of β at H, whose sides are parallel to AB and BC, respectively. By Lemma 9, that angle is filled by a single tile, Tile 3, whose β angle is at H. Then Tile 3 is of Type I or Type II. But that is a contradiction, since Q is supposed to lie on the boundary of some tile that is neither of Type I nor Type II. That rules out Case 3.

Case 4, Q is the southeastern end H of a maximal segment FH on RW, and Q does not lie on BC. Let Tile 1 be the tile above FH with an edge on FH and a vertex at H; let Tile 2 be the tile below FH with an edge on FH and a vertex at H. Then Tile 1 and Tile 2 both have their b edges on FH; Tile 1 has its α angle at H and Tile 2 has its γ angle at H. Let Tile 3 be the tile northeast of Tile 1; then it has an edge in Direction C, so it is not of Type II, but it is above RW, so it must be of Type I. Then it shares its c edge with Tile 1 and has its β angle at H. Let Tile 4 be east of Tile 3. If H should not be a vertex of Tile 4, but instead lie on the interior of an edge of Tile 4, then H would not be the northern vertex of some tile not of Type

I or Type II, contrary to the hypothesis of Case 4. Hence Tile 4 has a vertex at H. It cannot be of Type I, since then it would have a boundary extending FH past H, but FH is a maximal segment. Hence Tile 4 is of Type II. Hence it has its α angle at H and its c edge along the boundary with Tile 3. Let Tile 5 be south of Tile 4, with a vertex at H. Then Tile 5 is at least partly above RW, so it is of Type I or Type II. Since it has an edge in Direction A (shared with Tile 4), it is not of Type I. Therefore Tile 5 is of Type II. Then it has its γ angle at H. Now the gap between Tile 2 and Tile 5 leaves an unfilled angle at H of β. By Lemma 9, that gap must be filled by a single tile, say Tile 6. But the sides of Tile 6 that meet at H are parallel to AB and BC, so Tile 6 must be of Type I or Type II, contradiction, since H is supposed to be the northern vertex of some tile that is neither of Type I nor of Type II. That disposes of Case 4. See Fig. 2 for an illustration.

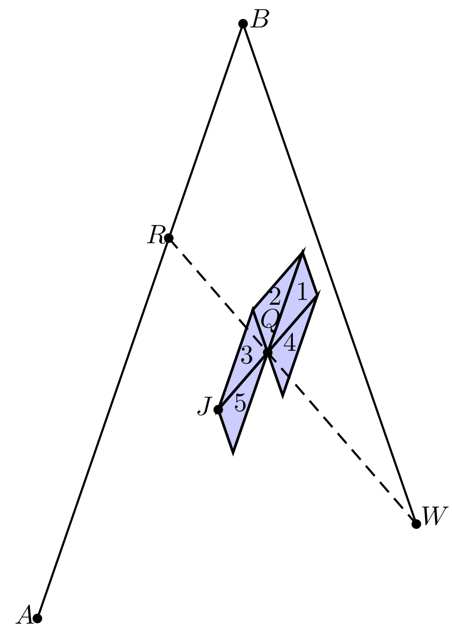
Figure 2: Case 4, Q does not occur at the southeast end of a maximal segment FH.



There are only two remaining possibilities: Q could occur at the northwest end F of a maximal segment FH on RW, or it could occur in a region of Type II tiles, i.e. Q might not lie on any maximal segment FH on RW. By Lemma 10, in the former case there is a suspicious edge southwest of Q in Direction A. We will show that this is also true in the latter case. Suppose that Q (which by definition lies on RW) does not lie on any tile boundary on RW. Since the tiles partly above RW with a vertex at Q are of Type I or Type II, they must be of Type II (or they would have their b edge on RW). Then there is a tile, say Tile 1, with its b edge in Direction A northeast of Q, and its α angle at Q. West of Tile 1 there is Tile 2, also of Type II, with its β angle at Q. Southwest of Tile 2 is Tile 3, which still lies partly above RW, so it too is of Type II, and it has its γ angle at Q. Southeast of Tile 1 is Tile 4, also of Type II, with its γ angle at Q. Let J be the southwest vertex of Tile 3. Then JQ is in Direction A. We are trying to prove that JQ is a suspicious edge. Let Tile 5 be the tile east of Tile 4; if it has its a or c edge on JQ, we are finished. Assume, for proof by contradiction, that it has its b edge on QJ. Since there is not for its γ angle between Tiles 3 and 4, Tile 5 must have its α angle at Q. Then Tile 5 is of Type II, and the angle remaining between Tile 5 and Tile 4 is β. By Lemma 9, it is filled by a single tile. The edges of that tile that meet at Q are parallel to AB

and BC, so it is of Type I or Type II, contradiction. That proves that there is a suspicious edge in direction A ending at Q. See Fig. 3 for an illustration. The figure shows the contradictory situation in which Tile 5 has its b edge on QJ, so the remaining angle can only be filled by a Type I or Type II tile.

Figure 3: There is a suspicious edge QJ ending at Q when Q lies in a Type II region.



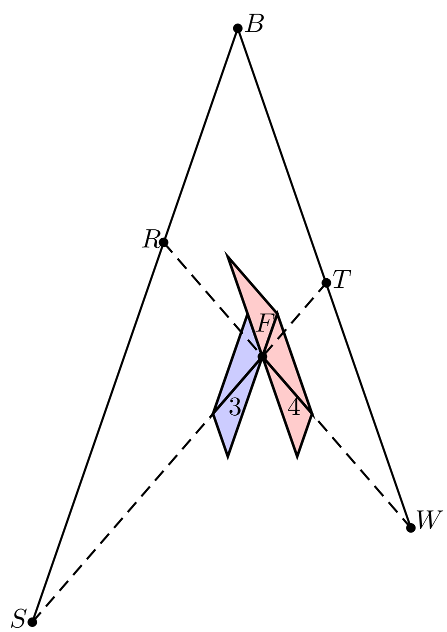
We claim that RW and ST intersect. If not then one lies entirely below the other; by symmmetry we may assume ST lies below RW. Then every tile below RW with an edge on RW is above ST, and hence is of Type I or Type II. But then RW can be lowered slightly and still have every tile wholly or partially above it of Type I or Type II, contradicting the definition of RW. Hence RW and ST have an intersection point. Since their slopes are different they have exactly one intersection point. Call that point F. We claim that F does not lie on the boundary of ABC. Suppose, for proof by contradiction, that it does; by symmetry we may assume $F = W$ lies on BC and ST lies below RW except at F. By definition of RW, there is a point Q on RW that lies on the boundary of a tile, say Tile 2, that is not of Type I or Type II; and we proved above that Q does not lie on BC, and hence, since ST is now assumed to lie below RW, Q lies above ST. But then Tile 2 is at least partially above ST, and hence, by definition of ST, Tile 2 is of Type I or Type II, contradiction. That contradiction completes the proof that RW and ST do intersect in an interior point F.

We now fix Q on RW as the northernmost point on RW that lies on the boundary of a tile not of Type I or Type II, and similarly we fix P as the northernmost point on ST that bounds a tile not of Type I or Type II. Then P is southwest of F (or equal to F), since otherwise P would lie above RW, and hence all tiles on whose boundary P lies would be of Type I or Type II. Similarly, Q is southeast of F (or equal to F).

Suppose, for proof by contradiction, that $Q = F$. Then also $P = F$, by definition of P. As shown above, either F lies at the northwest end of a maximal segment FH on RW, or T lies in a region of Type II tiles. If F lies in a region of Type II tiles, then F is the northeast end of a maximal segment GF lying on STE. But then, the tile northeast of F is not of Type II (since F is an endpoint of a maximal segment), and since it lies at least partly above RW, it is of Type I. Hence it has a tile boundary on RW. Hence F does not lie in a region of Type II tiles, after all. Then F must lie at the northwest end of a maximal segment FH on RW. Then the tile northwest of F cannot be of Type I, but since it lies at least partly above ST, it must be of Type II. Then it has a boundary on ST. Hence F lies at the northeast end of a maximal segment GF on ST, as well as at the northwest end of a maximal segment FH on RW. Then there are Type I and Type II tiles (only) above GFH.

Angle SFW is equal to $\beta + 2\alpha$. Let Tile 3 be the tile below EF with its b edge on EF and a vertex at F, and let Tile 4 be the tile below FG with its b edge on FG and a vertex at F. Then as shown above (or simply because γ is larger than angle SFW), Tile 3 and Tile 4 have their α angles at F. The unfilled angle between Tile 3 and Tile 4 is thus equal to β, and its two sides meeting at F are parallel to AB and BC. By Lemma 9, it is filled by the β angle of a single tile, which must therefore be of Type I or Type II. But that is a contradiction, since now every tile with a vertex at Q is of Type I or Type II. That completes the proof that $Q \neq F$. See Fig. 4.

Figure 4: Q cannot occur at the intersection of RW and ST.



So far, P and Q are just any points on ST and RW that lie on the boundary of tiles not of Type I or type II. Now we fix P and Q to be the northernmost such points. Thus every tile below FP with an edge or vertex on PF or PQ (except at P or Q) is of Type I or Type II. We now define J to be the point such that $PFQJ$ is a parallelogram; that is, QJ is parallel to AB and PJ is parallel to BC. Since P and Q each lie at the northern vertex of a tile not of Type I or Type II, neither P nor Q lies on the boundary of ABC (but J might lie on the boundary or even outside ABC).

We consider the configuration of tiles at Q. As proved above, whether Q is on tile boundaries in Direction C or not, there is a suspicious edge southwest of Q in Direction A, with a b edge above it and an a or c edge below. Applying reflection in the angle bisector of angle ABC, we obtain a similar result for vertex P. Under reflection, Direction C changes to Direction A, Type I changes to Type II and vice-versa, and Q changes to P. We find that the tile above PJ at P has its b edge on PJ, and the tile below PJ at P has its a edge or its c edge on PJ.

Using the terminology introduced in Definition 2, what we have proved is that there is a suspicious edge southwest of Q in Direction A, and a suspicious edge southeast of P in Direction C.

Now consider other parallelograms $PF'HG$, where $F'G$ is in Direction C (hence parallel to FQ), and F' lies between P and F, and HG is in direction A (hence parallel to QJ and PF), and H lies inside the open parallelogram $PFQJ$, or $H' = Q$. such that, with Q' the intersection point of lines QJ and $F'H$, we have

(i) All the tiles wholly or partially above $F'Q'$ (not just $F'H$) are of Type I or Type II, and

(ii) There is a suspicious edge in direction A ending at H (i.e., extending southwest from H).

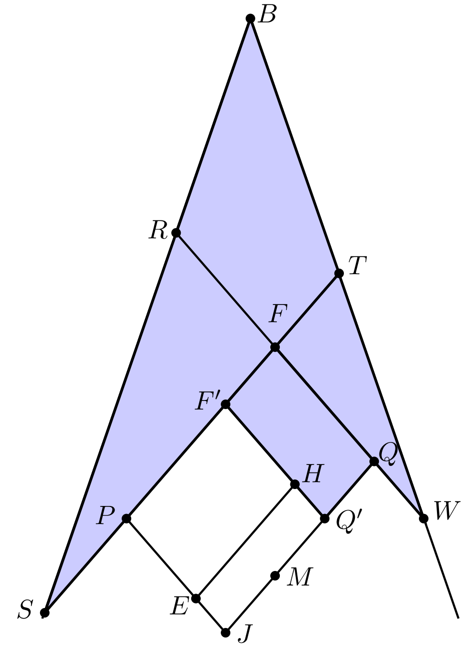
We have shown that there is at least one such parallelogram, namely the one with $F' = F$ and $H = Q$. For notational simplicity, we drop the primes, and assume that $FQJP$ is the smallest such parallelogram, that is, the one with F as close to P as possible. That means, in effect, that while we keep the assumption that tiles even partially above ST are of Type I or Type II, now we only assume that tiles (even partially) above RQ are of Type I or Type II, where we formerly had RW in place of RQ. We will derive a contradiction by constructing a smaller such parallelogram.

We claim that there is some tile in the parallelogram $FQJP$ that is neither of Type I nor Type II. Suppose, for proof by contradiction, that there is no such tile. Since there are no tile boundaries in Direction A among Type I tiles, all the tiles in $FQJP$ with edges on JQ are of Type II, and since Type II tiles all have their b edges in Direction A, all the tiles in $FQJP$ with an edge on JQ have their b edges on JQ. But because a suspicious edge lies on JQ, ending at Q, there is at least one a or c edge on JQ belonging to a tile below JQ. (Note that this argument is valid even if J lies outside ABC.) Let M be a point on JQ southwest of Q, chosen so that MQ is the longest segment composed of tile boundaries in direction A containing point Q extending to the southwest from Q. At Q there are tiles both above and below MQ with a vertex at Q, by the definition of "suspicious edge". Hence all the tiles above MQ with an edge on MQ have their b edge on MP. But since there is at least one tile with an a or c edge on MQ below MQ, there is an edge relation $jb = ua + vc$, contrary to hypothesis. That completes the proof that there is some tile in the parallelogram $FQJP$ that is neither of Type I nor of Type II.

Now let us lower FQ. Specifically, let $F'Q'$ be the lowest line in Direction C (thus parallel to FQ), with F' on PF and Q' on JQ, such that all the tiles above $F'Q'$ with an edge (or part of an edge, but more than a point) on $F'Q'$ are of Type I or Type II. Because of the existence of a tile in $FQJP$ that is neither of Type I nor of Type II, $F'Q'$ is still above PJ. But because we chose Q as far to the north as possible, Q' does not lie on FQ, i.e., $F'Q'$ definitely lies below FQ. See Fig. 5.

Then all the tiles in the (nonempty) parallelogram $F'FQQ'$ are of Type I or Type II, since otherwise $F'Q'$ would have been higher. Then $Q'Q$, being in Direction A, has only Type II tiles with edges on its upper side, since Type I tiles do not have any tile boundaries in Direction A. These edges must all be b edges, since Type II tiles have their b edges in Direction A. Since there is at least one a or c edge below $Q'Q$ (at Q), and since by hypothesis there are no edge relations $jb = ua + vc$, we see that Q' is not a vertex of a tile below QQ' with an edge on QQ', since if it were, there would be an edge relation $jb = ua + vc$, contrary to hypothesis.

Figure 5: Lowering FQ to $F'Q'$. Then H is the new Q.



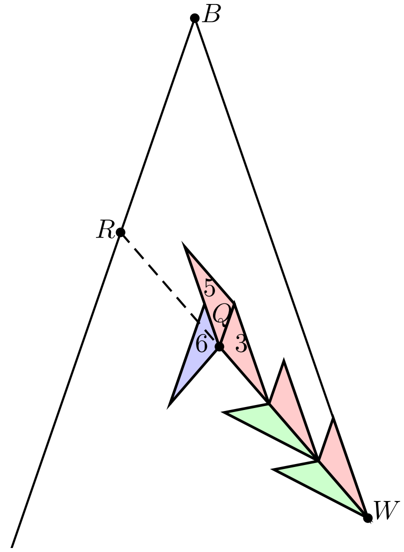
Let MP be the longest segment composed of tile boundaries in direction A southwest of Q. Then M lies (strictly) southwest of Q', since otherwise MP has only b edges on its northwest side, but at least one a or c edge below it (at P), contradicting the hypothesis that there are no edge relations $jb = ua + vc$. Hence segment $Q'Q$ is composed of tile edges (and one part of a tile edge near Q', which is not a vertex of a tile below QQ'). Since $Q'Q$ is in Direction A, and the tiles above it are of Type I or Type II, the tiles above $Q'Q$ with an edge on $Q'Q$ are of Type II and have their b edges on $Q'Q$. There is a point H on $F'Q'$ such that H is a vertex of a tile that is neither of Type I nor of Type II, since otherwise $F'Q'$ would have been lower. Let E be a point on PJ such that EH is in direction A. We claim that $F'HEP$ is a smaller parallelogram than $FQJP$, but still satisfies conditions (i) and (ii).

Condition (i) says that all the tiles wholly or partially above $F'Q'$ are of Type I or Type II. That is true by the definition of $F'Q'$.

The segment $F'Q'$ fulfills the hypotheses of Lemma 10, with (R, W, Q) in the lemma instantiated to (F', Q', H). Hence there is a suspicious edge in direction A extending southwest from H. That is condition (ii). Hence our claim is proved: we have indeed constructed a smaller parallelogram than $FQJP$ that satisfies conditions (i) and (ii). But that contradicts the definition of $FQJP$ as the smallest such parallelogram.

This contradiction was obtained under the assumptions that $S \neq A$ and $W \neq C$. Therefore, we have now proved that either $S = A$ or $W = C$. Suppose, for proof by contradiction, that $W = C$. Let Tile 1 be the tile with an edge on BC and vertex at C. Then Tile 1 is of Type I or Type II, since it is partly above RW. It cannot be of Type II, since then it would have its γ angle at C, but by assumption there are only α and β angles at the vertices of ABC. Hence Tile 1 is of Type I. Hence there is a maximal segment QC on RW. All the tiles above QC are of Type I and have their b edges on QC. Hence all the tiles below QC also have their b edges on QC. Let Tile 6 be southwest of Tile 5 with a vertex at Q. Then Tile 6 lies partly above RW, since besides Tile 6 there are only two tiles, Tile 5 and Tile 3, above RW with vertices and Q, and those tiles have one γ and one β angle at Q. Hence Tile 6 is of Type I or Type II. It cannot

Figure 6: $C = W$ is impossible: another tile can't fit below QW.



be of Type I, since then it would have its α angle at Q and an edge on RW, so QC would not be a maximal segment. Hence Tile 6 is of Type II. The western boundary of Tile 5 is parallel to BC, so Tile 6 has its a side against Tile 5 and its γ angle at Q. Now Tiles 3, 5, and 6 have together $2\gamma + \alpha$ at Q. Let Tile 4 be the tile below QC with its b edge on QC and a vertex at Q. Then Tile 4 cannot have its γ angle at Q. Since $\gamma > \pi/2$, no vertex on CQ has two tiles below CQ with their γ angles there. Since every tile below CQ with an edge on CQ has its b edge on CQ, each such tile has a γ angle at a vertex on CQ. But neither endpoint (C or Q) has a γ angle. That contradicts the pigeonhole principle. This contradiction proves $W \neq C$. See Fig. 6. The figure does not show Tile 4, because it is impossible to fit it in, which is the point.

By symmetry about the angle bisector of angle ABC, it is also impossible that $S = A$. This final contradiction completes the proof of the lemma.

We now consider the case when, instead of an unsplit vertex at B, we have a vertex where two tiles meet, one with angle α and the other with angle α or angle β.

Lemma 13 *Suppose $\gamma = 2\pi/3$ and there is an N-tiling of ABC by the tile with angles α, β, and γ with $\alpha \neq \beta$, and suppose ABC is not similar to the tile. We do not assume $\alpha < \beta$ in this lemma. Suppose there are exactly two tiles at B, one of which has an edge on AB and has angle α at B, and suppose there is no edge relation $ja = ub + vc$ with $j > 0$ and $u, v \geq 0$. Let BH be the maximal segment extending the boundary between the two tiles at B. Then there is a point S on AB such that angle BSH is β, and only Type I and Type II tiles lie above SH, where Type I and Type II tiles are the two types of tiles with their b and c edges parallel to AB and BH.*

Proof. The proof is similar to the proof of the previous lemma, but it has some differences. Let Tile 1 and Tile 2 be the two tiles at B, with Tile 1 having a side on AB and angle α at B. We orient ABC so that the boundary between Tile 1 and Tile 2 is exactly north-south, and Tile 1 is on the west. Let E be the point on AC such that angle $ABE = \alpha$; then AE contains the tile boundary between the two tiles at B. The point H mentioned in the lemma lies on AE. If $H = E$ then the tiling induces a tiling of triangle EAC, which has a single tile with angle

α at B, and hence Lemma 12 (with α and β switched) implies that there is an edge relation $ja = ub + vc$, contradicting our present hypothesis. Therefore $H \neq E$.

In Lemma 12, we did not assume $\alpha < \beta$, and the orientation of side AC and the exact angles at B and C were important only at the end of the proof. Therefore we can use the technique of that lemma on ABE, which has vertex angle α. In applying the proof, the "new vertical" is the angle bisector of angle ABE, and so "Type I" and "Type II" have different meanings; the new Type I tiles have their c edges parallel to BE and their b edges parallel to AB. The new Type II tiles have their b edges parallel to BE and their c edges parallel to AB. (Note that since β and α have been switched, b and a also are switched. But the conclusion of the lemma has been altered to allow for that, by mentioning edge relations $ja = ub + vc$ instead of $jb = ua + vc$.) This time R and S lie on AB, and W and T lie on BE. We modify the conditions defining RW and ST to include the new condition that W and T should lie on the closed segment BH. Then, the situation is as in Lemma 11, so the suspicious edges from P and Q still exist. Then, as in the proof of Lemma 12, unless $W = H$ and $S = A$, we are finished. If $S = A$ then all the tiles below ST have their a edges on ST and since angle A is less then γ, all the tiles below ST with an edge on ST have their β angles towards A and their γ angles towards T. In particular the tile below SH at H has its γ angle at H, so $T = H$, since that tile will be partly east of BE. But since T lies above W, and W lies above H, this is impossible. Hence $S \neq A$. Hence we may assume without loss of generality that $W = H$. But we still can lower ST until $T = W$; T will hit H before S hits C, since $S = C$ is contradictory. Then the situation in triangle BAE is this: All the tiles above SH are of Type I or Type II. That completes the proof of the lemma.

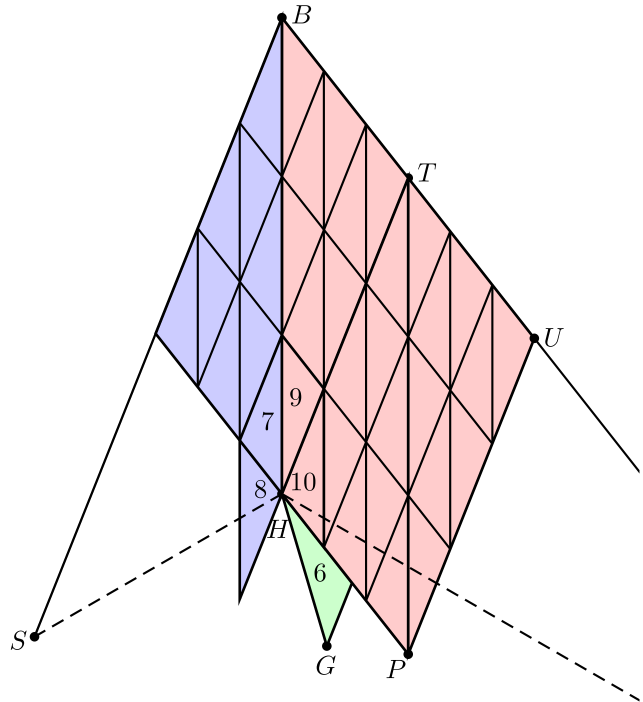
Lemma 14 *Suppose $\gamma = 2\pi/3$ and there is an N-tiling of ABC by the tile with angles α, β, and γ with $\alpha \neq \beta$, and suppose ABC is not similar to the tile. We do not assume $\alpha < \beta$ in this lemma. Suppose there are exactly two tiles at B, one of which has angle α at B, and the other has angle β at B. Then there is an edge relation $ja = ub + vc$, or an edge relation $jb = ua + vc$.*

Proof. Suppose, for proof by contradiction, that there is no edge relation of either type mentioned in the lemma. Let E be the point on AC such that the boundary between the two tiles at B lies on BE. Arrange triangle ABC as in the proof of the previous lemma, with B at the north and BE exactly north-south. Let H be the southern end of the maximal segment on BE starting at B. By Lemma 12, $H \neq E$. By Lemma 13, there is a point S on AB such that SH is in Direction A, and above SH all tiles are Type I or Type II.

Now we consider what happens east of BE. Lemma 13 will apply here, if we interchange α and β. (To make the pictures match we would also need to switch east and west.) Since we have supposed there is no edge relation of either type, the hypothesis of the lemma is fulfilled. Types I and II in the lemma become new types, say Type III and IV, where Type III tiles have their c edge parallel to BH and their a edge parallel to BC, and Type IV tiles have their a edge parallel to BH and their c edge parallel to BC. Surprise, Type III is the same as Type I, but that is not important now. Now Lemma 13 gives us a point W on BC such that angle BWH is α, and above HW all tiles are Type III or Type IV.

All tiles west of BH with an edge on BH have their b or c edges on BH, and all tiles east of BH with an edge on BH have their a or c edges on BH. We thus have some relation $ub + vc = Ua + Vc$, with nonnegative coefficients. If $U > 0$ then $Ua = ub + (v - V)c$, which is a forbidden relation if $v \geq V$ and if $v < V$ then $ub = Ua + (V - v)c$, and we have $u > 0$ since $V - v > 0$, so this is also forbidden. If $u > 0$, then $ub = Ua + (V - v)c$, which is forbidden if $V \geq v$; and if $V < v$ then $Ua = ub + (v - V)c$, also forbidden. Hence the only possibility is that the entire segment BH is composed of c edges on both the east and the west. These tiles are Type I west of BH, and Type III east of BH; on the west they have their α angle to the north and their β angle to the south, and on the east they have their β angle to the north and their α angle to the south. (Again we note that Type III and Type I are actually the same type.) Let Tile 7 be the tile west of BH with its c edge on BH and a vertex at H. Then Tile 7 is of Type II and has its β angle at H. Let Tile 8 be the tile southwest of Tile 7. It lies partly above SH, so it must be of Type II (as Type I tiles have no edges in the direction of southwest boundary of Tile 7). These two tiles share an a edge. Tile 8 has its γ angle at H and SH passes through its interior. See Fig. 7. In the figure, the dashed line on the left is ST for ABE; the other dashed line is $R'W$. Above SHW, that is, above the two dashed lines, are only Type I and Type II tiles (Type II shown).

Figure 7: Part of the proof of Lemma 14



We argue similarly on the eastern side of BH. Let Tile 9 be the tile east of BH with an edge on BH and a vertex at H; then Tile 9 is of Type III, has its α angle at H, and its southwest edge on HT. Let Tile 10 be the tile below HT, sharing an edge with Tile 9. Then Tile 10 has its γ angle at H, and is also of Type III. Now Tiles 7, 8, 9, and 10 together account for an angle of $2\gamma + \alpha + \beta$ at H, leaving $\alpha + \beta$ unfilled. This must be filled by two tiles, one with angle α at H and the other with angle β at H, by Lemma 9. The boundary between these two tiles cannot extend line BH, since BH is a maximal segment. Let Tile 6 be the tile south of Tile 10 with a vertex at H; then Tile 6 has its α angle at H, not its β angle. Therefore, it does not have its a side along Tile 10. Let P be the point farthest southeast of H on RH extended such HP lies on tile boundaries.

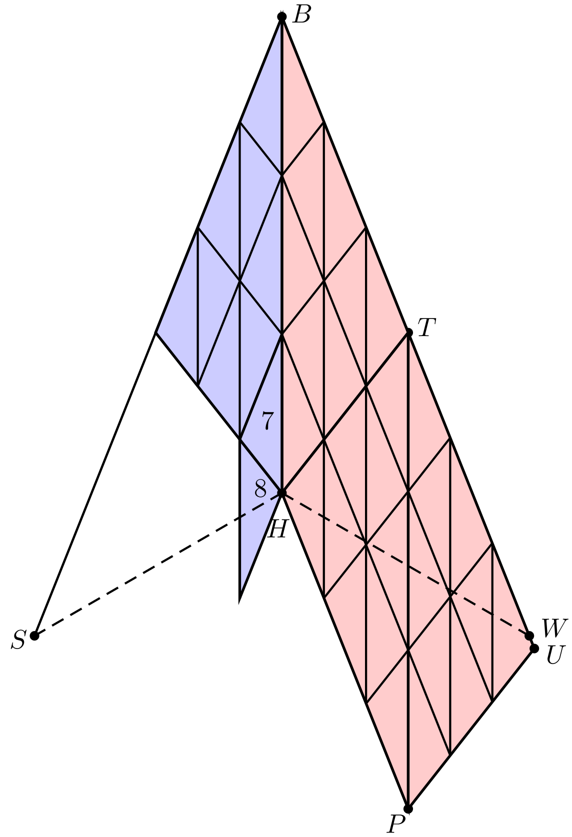
Now we "lower HT". That is, we consider lines VU parallel to HT, with V on HP and F on TW, and let VU be the lowest such line such that all tiles in the quadrilateral $HTUV$ are of Type III or Type IV. Then all tiles above VU with an edge on VU have their b edges on VU, and as in the proof of Lemma 12, at the endpoints of segments on VU composed of tile boundaries, there are tiles above and below VU with vertices at the endpoints (since beyond those endpoints are Type II tiles, which have no edges in the direction of VU). Since there are no edge relations, all the tiles below VU with an edge on VU have their b edges on VU. As in the proof of Lemma 12, there cannot be two γ angles at the northeast end of a maximal segment on VU, so the tiles below VU with an edge on VU are oriented with their γ angles at the southwest, so they are of Type III. But since VU is as low as possible, unless $V = P$ there must be a "notch" between two of the tiles below VU that is filled by a tile not of Type III. Since its two sides are parallel to BE and BC respectively, and the notch angle is β, that tile must be of Type IV. Hence VU can be lowered further, unless $V = P$. But since V is as low as possible, by definition, we must have $V = P$. Then all the tiles above VP with an edge on VP have their a edges on VP, and since VP is composed of tile boundaries, VP is composed (on its northeast side) entirely of a edges. But Tile 11, southwest of VP at H, does not have its a edge on VP. Hence there is an edge relation of the form $ja = ub + vc$, contradiction. That completes the proof of the lemma.

Lemma 15 *Suppose $\gamma = 2\pi/3$ and there is an N-tiling of ABC by the tile with angles α, β, and γ with $\alpha \neq \beta$, and suppose ABC is not similar to the tile. We do not assume $\alpha < \beta$ in this lemma. Suppose there are exactly two tiles at B, each of which has angle α at B. Then there is an edge relation $ja = ub + vc$.*

Remark. If we only wanted a relation $ja = ub+vc$ or $jb = ua+vc$, we would not need a separate argument, since by Theorem 1, there are only six tiles at the vertices of ABC, so there is either a vertex that does not split, or a vertex with an $\alpha + \beta$ angle, so one of the previous lemmas applies. But the conclusion of the lemma specifies a relation $ja = ub + vc$, so we do need a direct proof.

Proof. The proof is similar to the proof of Lemma 14, but easier. Again we let E be the point on BC such that the boundary between the two tiles at B lies on BE, and we let H be the southern end of the maximal segment on BE starting at H. Again we apply Lemma 13 to ABH and BCH; this time we do not have to switch α and β in the application to BCH, so this lemma can assert the existence of a relation $ja = ub + vc$ instead of allowing also the possibility $jb = ua + vc$. The points S and W are constructed as in the previous proof, and as before, all the tiles on the east or west of BH have their c edges on BH. But now, since there are two α angles at B, all those tiles have their α angles to the north and their β angles to the south. Hence Tiles 7, 8, 9, and 10 have vertices at H leaving only an angle of 2α below H. That angle can only be filled by two more tiles with their α angles at H; but the boundary between those tiles will lie on BE, contradicting the definition of H as the southernmost point on BE such that BH lies on tile boundaries. That completes the proof of the lemma. See Fig. 8. The last part of the proof of Lemma 14 (about lowering HT) is not necessary in this proof.

Figure 8: The gap at H must be filled by two α angles, so BH is not maximal after all.



Theorem 2 *Suppose triangle ABC is N-tiled by a tile with $\gamma = 2\pi/3$ and side lengths (a, b, c), and the tile is not similar to ABC. Then there is a maximal segment in the tiling that gives rise to an edge relation $ja = ub + vc$ or $jb = ua + vc$, where u and v are nonnegative integers and j is a positive integer. Moreover, if there is a double angle 2α (or 2β) then there is a relation $ja = ub + vc$ (or $jb = ua + vc$).*

Proof. This has been proved in Lemma 12 in case ABC has a vertex that does not split, and in Lemma 14 in case ABC has a vertex with angle $\alpha + \beta$. But one of these cases must hold, since by Theorem 1, the vertex splitting is $(3, 3, 0)$. There are thus a total of six α and β angles of tiles at the vertices of ABC. If one vertex does not split then the first case applies. If all vertices split then there are exactly two tiles at each vertex of ABC. Then at least one of the vertices has angle $\alpha + \beta$, so the second case applies. The last claim of the theorem is Lemma 15. That completes the proof of the theorem.

8 Uniqueness of the edge relation when a/c is irrational

In the previous section we proved that if $(P, Q, R) = (3, 3, 0)$ then there must be at least one edge relation. In this section we show there is at most one!

Lemma 16 *Suppose T is a triangle whose largest angle γ is $2\pi/3$; let the sides of T be a, b, and c, with c opposite γ. (We do not assume $a < b$.) Suppose $b = \lambda a + \mu c$ with λ and μ rational. Then a/c is uniquely determined by λ and μ (whether or not a/c is rational). If a/c is irrational, then λ and μ are also uniquely determined by a/c; that is, there is at most one relation $b = \lambda a + \mu c$ possible for a given value of a/c.*

Remarks. We do not assume λ and μ are nonnegative or even nonzero. If a/c and b/c are rational (for example, $(a, b, c) = (3, 5, 7)$), there will of course be many true equations $b = \lambda a + \mu c$; whether they occur as edge relations in tilings is another question. This lemma reflects a property of a tile with an angle of 120°; it does not even mention a tiling.

Proof. We have $a = \sin\alpha$, $b = \sin\beta$, and $c = \sin\gamma = \sqrt{3}/2$. By the law of cosines

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos\gamma \\ &= a^2 + b^2 - 2ab\Big(-\frac{1}{2}\Big) \\ c^2 &= a^2 + b^2 + ab \qquad (13) \end{aligned}$$

(We could replace c^2 by $3/4$, since $c = \sqrt{3}/2$, but we just leave it c^2.) By hypothesis, there exist rational numbers λ and μ such that

$$b = \lambda a + \mu c. \qquad (14)$$

We regard (13) and (14) as two equations in b, the first quadratic and the second linear. Therefore the remainder of $b^2 + ab + (a^2 - c^2)$ on division by $b - (\lambda a + \mu c)$ is zero. Computing that remainder we find

$$0 = (\lambda a + \mu c)(\lambda a + \mu c + a) + a^2 - c^2$$

Writing this as a quadratic equation for a we have

$$0 = a^2(1 + \lambda + \lambda^2) + a(2\lambda + 1)\mu c + (\mu^2 - 1)c^2$$

Setting $x = a/c$ we have

$$0 = x^2(1 + \lambda + \lambda^2) + (2\lambda + 1)\mu x + (\mu^2 - 1) \qquad (15)$$

This is a quadratic equation for x with rational coefficients. This equation shows the remarkable fact that one edge relation (given by λ and μ) completely determines the shape of the tile, since that is given once a/c is known.

Now we investigate when λ and μ are uniquely determined by a/c, that is, by x. Suppose, for proof by contradiction, that there is a second edge relation $b = La + Mc$. Then $\lambda \neq L$ since if $\lambda = L$, we also have $\mu = M$. Then $\lambda a + \mu c = La + Mc$, and since $\lambda \neq L$, we have

$$x = \frac{M - \mu}{\lambda - L}. \qquad (16)$$

Thus x is rational. Hence, if x is irrational, there cannot be a second such relation. That completes the proof of the lemma.

9 Some consequences of the area equation

We take up the case when one angle (say angle B) does not split, i.e., there is just one tile with a vertex at B. The conclusion is that in such a tiling, the sides are commensurate, i.e. a/c and b/c are rational.

Here is a sketch of the proof in this section (the details follow the sketch). An edge relation can be used to express one of a, b, c linearly in terms of the other two; say b in terms of a and c. Then the law of cosines (in the tile) shows that a^2 is linear in ac, and the side lengths X and Z of the triangle ABC are linear in a and c. Their product XZ involves terms in ac, a^2, and c^2, but a^2 is linear in ac, so XZ becomes a linear function of ac. On the other hand, if angle $B = \beta$, then the area equation $N\mathcal{A}_T = A_{ABC}$ gives us $Nabc = XZb$, or $Nac = XZ$, so the two terms of XZ as a linear function of ac must each be zero, unless ac is rational. But these equations turn out to lead to a contradiction, so ac is rational; and since $c^2 = 3/4$, a/c is rational. Then because of the edge relation, b/c is also rational.

That sketch (and the first lemma below) assumes angle $B = \beta$, but we do not assume $\alpha < \beta$, so it covers the case in which any vertex of ABC has only one tile at that vertex. Otherwise, every vertex has exactly two tiles, so one of them must have angle $\alpha + \beta = \pi/3$. Then we have $Nabc = XZc$ so $Nab = XZ$, where X and Z are the two sides of ABC adjacent to the $\pi/3$ angle, and we express X and Z in terms of ab instead of ac.

Recall the notation for the $\mathbf{d}$ matrix:

$$\mathbf{d} = \begin{pmatrix} p & d & e \\ g & m & f \\ h & \ell & r \end{pmatrix}.$$

Recall also that X is the length of side BC, Y the length of side AC, and Z the length of side AB, so that

$$\mathbf{d}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}.$$

Lemma 17 *Suppose ABC is N-tiled by a triangle T whose largest angle γ is $2\pi/3$, and that no angle of ABC is equal to γ. Suppose ABC is not similar to T, and T is not isosceles, i.e. is not the tile used in the equilateral 3-tiling. Then none of the elements e, f, and r of the* $\mathbf{d}$ *matrix are zero.*

Remarks. This means that sides AB and BC must each contain at least one edge of length c.

Proof. Suppose, for proof by contradiction, that $e = 0$. That means that there are no c edges on AB, so every tile with an edge on AB has a γ angle at a vertex on AB. Since $\gamma > \pi/2$, there cannot be two γ angles at the same vertex. Since Tile 1 does not have a γ angle at B, the pigeonhole principle tells us that all these tiles have the γ angle to the south, i.e. nearer to A than to B. In particular, the last tile, the one with a vertex at A, has its γ angle at A. Hence angle A is at least γ. But that is impossible, since ABC is not similar to T. The other three cases are treated in the same way, changing AB to BC or AC. That completes the proof of the lemma.

Lemma 18 *Suppose ABC is N-tiled by a triangle T whose largest angle γ is $2\pi/3$. Suppose ABC is not similar to T, and T is not isosceles, i.e. is not the tile used in the equilateral 3-tiling. Let a, b, and c be the sides of the tile opposite α, β, and γ. Suppose $b = \lambda a + \mu c$ with λ and μ rational and nonnegative. Then*

$$a^2 = \frac{-\mu(2\lambda+1)}{1+\lambda+\lambda^2}ac + \frac{c^2(1-\mu^2)}{1+\lambda+\lambda^2}$$

and

$$\begin{aligned} XZ &= (p+\lambda d)(h+\lambda\ell)\Big(\frac{-\mu(2\lambda+1)}{1+\lambda+\lambda^2}ac + \frac{c^2(1-\mu^2)}{1+\lambda+\lambda^2}\Big) \\ &\quad + ac\Big\{(d\mu+e)(h+\lambda\ell) + (p+\lambda d)(\ell\mu+r)\Big\} + c^2(d\mu+e)(\ell\mu+r) \end{aligned}$$

Remark. If we also assume that the tile is scaled so $c = \sin\gamma$, then of course c^2 can be replaced by 3/4; but the lemma does not assume that hypothesis.

Proof. We have $a = \sin\alpha$, $b = \sin\beta$, and $c = \sin\gamma = \sqrt{3}/2$. By the law of cosines

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos\gamma \\ \Big(\frac{\sqrt{3}}{2}\Big)^2 &= a^2 + b^2 - 2ab\Big(-\frac{1}{2}\Big) \\ c^2 &= a^2 + b^2 + ab \end{aligned} \tag{17}$$

Since $\lambda \ge 0$ and $b < c$ we have

$$\mu < 1$$

Substituting $b = \lambda a + \mu c$ into (17), we have

$$\begin{aligned} c^2 &= a^2 + (\lambda a + \mu c)^2 + a(\lambda a + \mu c) \\ 0 &= a^2(1 + \lambda + \lambda^2) + ac(2\lambda\mu + \mu) + c^2(\mu^2 - 1) \\ a^2 &= \frac{-\mu(2\lambda + 1)}{1 + \lambda + \lambda^2}ac \ + \frac{c^2(1 - \mu^2)}{1 + \lambda + \lambda^2} \end{aligned} \tag{18}$$

Then

$$\begin{aligned} X &= pa + db + ec \\ &= pa + d(\lambda a + \mu c) + ec \\ &= a(p + \lambda d) + c(e + d\mu) \\ Z &= ha + \ell b + rc \\ &= ha + \ell(\lambda a + \mu c) + rc \\ &= a(h + \ell\lambda) + c(r + \ell\mu) \end{aligned}$$

Multiplying these two expressions we have

$$\begin{aligned} XZ &= (a(p + \lambda d) + c(e + d\mu))(a(h + \ell\lambda) + c(r + \ell\mu)) \\ &= (p + \lambda d)(h + \lambda\ell)a^2 + ac\Big\{(d\mu + e)(h + \lambda\ell) + (p + \lambda d)(\ell\mu + r)\Big\} + c^2(d\mu + e)(\ell\mu + r) \end{aligned}$$

Now substitute for a^2 from (18):

$$\begin{aligned} XZ \ &= \ (p + \lambda d)(h + \lambda\ell)\Big(\frac{-\mu(2\lambda + 1)}{1 + \lambda + \lambda^2}ac \ + \frac{c^2(1 - \mu^2)}{1 + \lambda + \lambda^2}\Big) \\ &\quad + ac\Big\{(d\mu + e)(h + \lambda\ell) + (p + \lambda d)(\ell\mu + r)\Big\} + c^2(d\mu + e)(\ell\mu + r) \end{aligned} \tag{19}$$

That completes the proof of the lemma.

Lemma 19 *Suppose ABC is N-tiled by a triangle T whose largest angle γ is $2\pi/3$. Suppose ABC is not similar to T, and T is not isosceles, i.e. is not the tile used in the equilateral 3-tiling, and that angle $ABC = \beta$ (but we do not assume $\alpha < \beta$). Let a, b, and c be the sides of the tile opposite α, β, and γ. Then a/c and b/c are rational.*

Remark. For this lemma and proof, we drop the convention we sometimes use, that the vertices A, B, and C are labeled in order of size of the angle, and instead specify B to be a vertex that does not split, i.e., at which only one tile has a vertex.

Proof. By Lemma 12, there exist integers $j, u, v \ge 0$ such that $jb = ua + vc$. It will be convenient to introduce $\lambda = u/j$ and $\mu = v/j$, so that λ and μ are nonnegative rational numbers and $b = \lambda a + \mu c$. Since $vc \le jb$ and $b < c$ implies $vc < jc$, so $v < j$, and dividing by j we have

$$\mu < 1 \tag{20}$$

By Lemma 18, we have

$$a^2 = \frac{-\mu(2\lambda+1)}{1+\lambda+\lambda^2}ac + \frac{c^2(1-\mu^2)}{1+\lambda+\lambda^2}$$

and

$$\begin{aligned} XZ &= (a(p+\lambda d)+c(e+d\mu))(a(h+\ell\lambda)+c(r+\ell\mu)) \\ &= (p+\lambda d)(h+\lambda\ell)a^2 + ac\Big\{(d\mu+e)(h+\lambda\ell)+(p+\lambda d)(\ell\mu+r)\Big\} + c^2(d\mu+e)(\ell\mu+r) \end{aligned}$$

The area of triangle ABC is given by the formula $2\mathcal{A}_{ABC} = XZ\sin B$. Now we assume that the tile is scaled so that $c = \sin\gamma = \sqrt{3}2$. Since angle B is β and $\sin\beta = b$, we have $2\mathcal{A}_{ABC} = XZb$. On the other hand the area of the tile is given by $2\mathcal{A}_T = ac\sin\beta = abc$. Since there is an N-tiling, we have $\mathcal{A}_{ABC} = N\mathcal{A}_T$. Writing this out we have

$$\begin{aligned} 2N\mathcal{A}_T &= 2\mathcal{A}_{ABC} \\ Nabc &= XZb \end{aligned}$$

Canceling b we have

$$\begin{aligned} Nac &= XZ \\ &= (p+\lambda d)(h+\lambda\ell)\Big(\frac{-\mu(2\lambda+1)}{1+\lambda+\lambda^2}ac + \frac{c^2(1-\mu^2)}{1+\lambda+\lambda^2}\Big) \\ &\quad +ac\Big\{(d\mu+e)(h+\lambda\ell)+(p+\lambda d)(\ell\mu+r)\Big\} + c^2(d\mu+e)(\ell\mu+r) \end{aligned}$$

Suppose, for proof by contradiction, that a/c is not rational. Since $c^2 = 3/4$ is rational, 1 and ac are linearly independent in $\mathbb{Q}(a,c)$, and hence $\{1, ac)\}$ can be completed to a basis. Therefore we can equate the coefficients of ac and the rational parts in the formulas of Lemma 18. Equating the rational parts we have

$$0 = \frac{3}{4}(p+\lambda d)(h+\lambda\ell)\frac{1-\mu^2}{1+\lambda+\lambda^2} + \frac{3}{4}(d\mu+e)(\ell\mu+r) \tag{21}$$

Since $\mu < 1$ by (20), both terms on the right are nonnegative. Therefore both are zero. In particular the second term is zero. That is,

$$(d\mu+e)(\ell\mu+r) = 0.$$

Then $d\mu + e = 0$, so $e = 0$, contradicting Lemma 17. That contradiction proves that a/c is rational.

It remains to prove that b/c is also rational. Since $b = \lambda a + \mu c$, we have $b/c = \lambda(a/c) + \mu$, which is rational since a/c is rational. That completes the proof of the lemma.

Next we take up the case where one angle of ABC is $\alpha + \beta$.

Lemma 20 *Suppose ABC is N-tiled by a triangle T whose largest angle γ is $2\pi/3$. Suppose ABC is not similar to T, and T is not isosceles, i.e. is not the tile used in the equilateral 3-tiling, and that angle $ABC = \alpha + \beta = \pi/3$. Let a, b, and c be the sides of the tile opposite α, β, and γ. Then*

(i) a/c is rational if and only if b/c is rational, and

(ii) a/b is rational (whether or not a/c is irrational), and

Proof. We do not assume $\alpha < \beta$ in the proof. We will prove (i) by contradiction, so we assume that one of a/c is irrational and the other rational. By relabeling a and b if necessary, we can assume without loss of generality that a/c is irrational and b/c is rational.

By Lemma 13, there is an edge relation $jb = ua + vc$ or $ja = ub + vc$ for integers j, u, v with $j > 0$ and $u, v \geq 0$. First assume the relation is $ja = ub + vc$. If $u = 0$ then a/c is a rational multiple of v/c and we are finished, so we may assume $u \neq 0$. Then we have $ub = ja - vc$, and dividing by u we have $b = \lambda a - \mu c$ with $\lambda = u/j \neq 0$ and $\mu \geq 0$, and $b/c = \lambda(a/c) - \mu$. Since $\lambda \neq 0$, b/c is rational if and only if a/c is rational, and we are finished with part (i).

Therefore we may assume, without loss of generality, that there is an edge relation of the form $jb = ua + vc$. Then dividing by j we have $b = \lambda a + \mu c$ for rational λ and μ, with λ and μ nonnegative. Then $b/c = \lambda(a/c) + \mu$, so provided $\lambda \neq 0$, b/c is rational if and only if a/c is rational. Hence, to prove (i), it suffices to prove that $\lambda \neq 0$.

We have the area equation

$$Nabc = XZ \sin B$$

But now, $B = \alpha + \beta = \pi/3$, so $\sin B = \sin(\pi/3) = \sin(2\pi/3) = c$, so

$$Nabc = XZc$$

and canceling c we have

$$Nab = XZ \tag{22}$$

instead of $Nac = XZ$ as in the previous lemma. Then

$$\begin{aligned} XZ &= Nab \\ &= Na(\lambda a + \mu c) \qquad \text{since } b = \lambda a + \mu c \\ &= N\lambda a^2 + N\mu ac \end{aligned}$$

From Lemma 18 we have

$$a^2 = \frac{-\mu(2\lambda+1)}{1+\lambda+\lambda^2}ac + \frac{\frac{3}{4}(1-\mu^2)}{1+\lambda+\lambda^2}$$

Substituting that in the previous equation we have

$$XZ = N\lambda\frac{\frac{3}{4}(1-\mu^2)}{1+\lambda+\lambda^2} + ac\Big\{N\mu - \frac{\mu(2\lambda+1)}{1+\lambda+\lambda^2}\Big\} \tag{23}$$

The coefficients of 1 and ac in XZ have already been calculated in Lemma 18:

$$\begin{aligned} XZ &= (p+\lambda d)(h+\lambda\ell)\Big(\frac{-\mu(2\lambda+1)}{1+\lambda+\lambda^2}ac + \frac{\frac{3}{4}(1-\mu^2)}{1+\lambda+\lambda^2}\Big) \\ &\quad +ac\Big\{(d\mu+e)(h+\lambda\ell) + (p+\lambda d)(\ell\mu+r)\Big\} + \frac{3}{4}(d\mu+e)(\ell\mu+r) \end{aligned} \tag{24}$$

Since a/c is irrational and c^2 is rational, ac is irrational, so we can work in the field $\mathbb{Q}(ac)$. Equating the rational part of XZ as given by (23) and (24), we have

$$N\frac{\frac{3}{4}\lambda(1-\mu^2)}{1+\lambda+\lambda^2} = (p+\lambda d)(h+\lambda\ell)\frac{\frac{3}{4}(1-\mu^2)}{1+\lambda+\lambda^2} + \frac{3}{4}(d\mu+e)(\ell\mu+r)$$

Subtracting the first term on the right from both sides, and canceling $3/4$, we have

$$\Big(N - (p+\lambda d)(h+\lambda\ell)\Big)\frac{\lambda(1-\mu^2)}{1+\lambda+\lambda^2} = (d\mu+e)(\ell\mu+r) \tag{25}$$

Our aim is to prove $\lambda \neq 0$. Assume, for proof by contradiction, that $\lambda = 0$. Then (25) becomes

$$0 \quad = \quad (d\mu + e)(\ell\mu + r)$$

Since μ, d, and ℓ are all nonnegative, we have $0 \geq er$; but by Lemma 17, we have $er > 0$, contradiction. That contradiction shows that $\lambda \neq 0$. That completes the proof of (i).

Now to prove (ii). If a/c is rational, then $a/b = (a/c)/(b/c)$ is also rational, so we may assume without loss of generality that a/c is irrational. By Lemma 13, there is an edge relation of the form $ja = ub + vc$ or $jb = ua + vc$ (and this is actually realized in the tiling along some maximal segment EF). For convenience above, we relabel a and b, if necessary, so we can assume the relation has the form $jb = ua + vc$. By part (i) of Lemma 20, the hypotheses "a/c is irrational" and "b/c irrational" are equivalent, so this relabeling does not affect the hypothesis. Dividing by j we have $b = \lambda a + \mu c$.

Assume, for proof by contradiction, that a/b is irrational. Then $\mu \neq 0$. Now let $x = a/c$ and $y = b/c$. Then $y = \lambda x + \mu$. The law of cosines for the tile is $c^2 = a^2 + b^2 + ab$. Dividing by c we have $1 = x^2 + y^2 + xy$. Substituting $y = \lambda x + \mu$ we find

$$x^2(1 + \lambda + \lambda^2) + x(2\lambda + 1)\mu + (\mu^2 - 1) = 0. \tag{26}$$

Thus x is quadratic over $\mathbb{Q}$. Then $\mathbb{Q}(x) = Q(\sqrt{D})$, where D is a square times the discriminant of the quadratic equation for x (just displayed). So for some constants P, Q, R, and S, we have

$$\begin{aligned}
x &= P + Q\sqrt{D} \\
y &= R + S\sqrt{D} \\
\frac{y}{x} &= \frac{r + S\sqrt{D}}{P + Q\sqrt{D}} \\
&= \frac{(P + Q\sqrt{D})(R - S\sqrt{D})}{R^2 - DS^2} \\
&= \frac{Pr - QSD + \sqrt{D}(QR - PS)}{R^2 - DS^2}
\end{aligned}$$

from which we see that y/x is rational if and only if $QR = PS$. Now $y^2 + xy + x^2 - 1 = 0$ is a quadratic equation for y over $\mathbb{Q}(x)$, with discriminant $x^2 - 4(x^2 - 1) = 4 - 3x^2$. Then y belongs to $\mathbb{Q}(x)$ if and only if $4 - 3x^2$ is a square in $\mathbb{Q}(x)$; that is, if and only if $4 - 3x^2$ is a rational square times D, say $\rho^2 D$. Substituting $x = P + Q\sqrt{D}$, we have

$$\begin{aligned}
\rho^2 D &= 4 - 3x^2 \\
&= 4 - 3(P + Q\sqrt{D})^2 \\
&= 4 - 3P^2 + 6PQ\sqrt{D}) + 3DQ^2
\end{aligned}$$

Since D is not a rational square, this implies $PQ = 0$. If $Q = 0$ then x is rational, contrary to our assumption, so we may assume $Q \neq 0$. Then $P = 0$ and $x = Q\sqrt{D}$. Hence x^2 is rational. Solving (26) for the linear term in x we have

$$x = \frac{1 - \mu^2 - x^2(1 + \lambda + \lambda^2)}{(2\lambda + 1)\mu}.$$

The right side is rational, since x^2 is rational. But that contradicts the fact that x is irrational. That completes the proof of the lemma.

Lemma 21 *Suppose ABC is N-tiled by a triangle T whose largest angle γ is $2\pi/3$. Suppose ABC is not similar to T, and T is not isosceles, i.e. is not the tile used in the equilateral 3-tiling, and that angle $ABC = \alpha + \beta = \pi/3$. Let a, b, and c be the sides of the tile opposite α, β, and γ. Then a/c is rational.*

Proof. Define $\lambda = b/a$, so by Lemma 20, λ is rational. Assume, for proof by contradiction, that a/c is irrational. Then by Lemma 19, every vertex angle of ABC splits; since the vertex splitting must be $(3,3,0)$, at least one angle of ABC must be $\alpha+\beta$. We can assume it is angle B. Then we can equate the coefficients of ac in XZ from Lemma 18 and (23), obtaining

$$\begin{aligned} & (p+\lambda d)(h+\lambda\ell)\Big(\frac{-\mu(2\lambda+1)}{1+\lambda+\lambda^2}\Big) + (d\mu+e)(h+\lambda\ell) + (p+\lambda d)(\ell\mu+r) \\ = \quad & \Big(N\mu - \frac{\mu\lambda(2\lambda+1)}{1+\lambda+\lambda^2}\Big) \end{aligned} \tag{27}$$

We have $\mu = 0$ since $b = \lambda a$. Putting $\mu = 0$ this equation becomes

$$e(h+\lambda\ell) + (p+\lambda d)r \quad = \quad 0$$

Since both terms on the left are nonnegative, they are each zero; in particular $e(h+\lambda\ell) = 0$. By Lemma 17, $e \neq 0$. Hence $h + \lambda\ell = 0$. Since $\lambda > 0$ we have $h = \ell = 0$. That means that side AB, whose length is Z, is made entirely of c edges. With $h = \ell = 0$ we obtain $(p+\lambda d)r = 0$; but by Lemma 17, $r \neq 0$, and since $\lambda > 0$ we have $p = d = 0$. That means that side X is made entirely of c edges. We have $Z = rc$ and $X = ec$. The area equation tells us $Nab = XZ$; then we have

$$\begin{aligned} N\lambda a^2 &= Nab \qquad \text{since } b = \lambda a^2 \\ &= XZ \\ &= (ec)(rc) \\ &= erc^2 \\ \frac{N\lambda a^2}{c^2} &= er \\ \Big(\frac{a}{c}\Big)^2 &= \frac{er}{N\lambda} \end{aligned}$$

Hence a/c has a rational square root. Define $x = a/c$; then as shown in (26), we have

$$x^2(1+\lambda+\lambda^2) + x(2\lambda+1)\mu + (\mu^2-1) = 0.$$

and substituting the rational expression $er/(N\lambda)$ for x^2 we obtain a linear equation for x:

$$\frac{er(1+\lambda+\lambda^2)}{N\lambda} + x(2\lambda+1)\mu + (\mu^2-1) = 0$$

Since $2\lambda + 1 \neq 0$, we can solve this equation for x. Hence $x = a/c$ is rational, contrary to assumption. That completes the proof of the lemma.

10 Conclusions

Theorem 3 *Suppose ABC is N-tiled by a tile non-isosceles tile T with a 120° angle, and T is not similar to ABC. Then T is similar to a triangle with sides a, b, and c, where a, b, and c are integers.*

Proof. By Lemma 20, a/c is rational. Applying Lemma 21, b/c is rational too. Now just multiply a, b, and c by the least common multiple of their denominators. That completes the proof of the theorem.

Theorem 4 *Suppose ABC is N-tiled by the tile with angles α, β, and $\gamma = 2\pi/3$. Then α is not a rational multiple of π.*

Proof. We may assume (by scaling the tile) that $a = \sin\alpha$. Then $c = \frac{1}{2}\sqrt{3}$. By Theorem 3, a/c is rational. Hence a is a rational multiple of $\sqrt{3}$, so a belongs to $\mathbb{Q}(\sqrt{3})$. We will prove that $\cos\alpha$ is a rational multiple of a/c. From Lemma 2, we know that $\cos\beta$ is a rational multiple of $\cos\alpha$; let ξ be a rational number such that $\cos\beta = \xi\cos\alpha$. Let $\lambda = b/a$; then since a/c and b/c are rational, λ is also rational. Then

$$\begin{aligned} c &= \sin(\pi/3) \\ &= \sin(\alpha+\beta) \\ &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ &= a\cos\beta + b\cos\alpha \\ &= a(\xi\cos\alpha) + \lambda a\cos\alpha \\ &= a\cos\alpha(\xi+\lambda) \end{aligned}$$

Solving for $\cos\alpha$ we have

$$\cos\alpha = \left(\frac{c}{a}\right)\frac{1}{\xi+\lambda}$$

The right side is a rational multiple of c/a, since ξ, and λ are rational. But c/a is a rational multiple of a/c, since $(a/c)^2$ is rational. Hence $\cos\alpha$ is a rational multiple of a/c, as claimed.

By Corollary 1, since a/c is rational, if α is a rational multiple of 2π, then it is an integer multiple of $\pi/4$, $\pi/5$, $\pi/6$, or $2\pi/5$. Since $\alpha < \beta$ and $\alpha + \beta = \pi/3$, we have $\alpha < \pi/6$, ruling out all four possibilities. That completes the proof of the theorem.

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