

Triangle Tiling II: Nonexistence theorems

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Abstract

An N -tiling of triangle ABC by triangle T is a way of writing ABC as a union of N triangles congruent to T , overlapping only at their boundaries. The triangle T is the “tile”. The tile may or may not be similar to ABC . We wish to understand possible tilings by completely characterizing the triples (ABC, T, N) such that ABC can be N -tiled by T . In particular, this understanding should enable us to specify for which N there exists a tile T and a triangle ABC that is N -tiled by T ; or given N , determine which tiles and triangles can be used for N -tilings; or given ABC , to determine which tiles and N can be used to N -tile ABC . This is the second of four papers on this subject. In [1], we dealt with the case when ABC is similar to T , and the case when T is a right triangle. In this paper, we assume that ABC is not similar to T , and T is not a right triangle, and furthermore that if T has a 120° angle then T is isosceles.

The main theorem is that under those hypotheses, the only N -tilings are of an equilateral triangle by an isosceles tile with base angles $\pi/6$, and N three times a square.

Under those hypotheses, there are only two families of tilings. There are tilings of an equilateral triangle ABC by an isosceles tile with base angles $\pi/6$; and there are newly-discovered “triquadratic tilings”, which are treated in the third paper [2] in this series. These arise in the special case that $3\alpha + 2\beta = \pi$ or $3\beta + 2\alpha = \pi$ with α not a rational multiple of π , where α and β are the two smallest angles of the tile, and the sides of the tile have rational ratios.

The case when the tile has an angle $2\pi/3$ and is not isosceles is taken up in [3]. At the time of writing, our analysis of that case is incomplete, but there are no known tilings of any ABC in that case.

We use techniques from linear algebra and elementary field theory, and in one case we use some algebraic number theory. We use some counting arguments and some elementary geometry and trigonometry.

1 Introduction

For a general introduction to the problem of triangle tiling, see [1]. This paper is largely concerned with the non-existence of tilings, rather than their existence. In [1] we enumerated the known tilings; it is our aim to prove that these families exhaust all the possible tilings, or at least, exhaust all the triples (ABC, N, T) such that ABC can be N -tiled by tile T . There will be few pictures of beautiful tilings here (though there will be at least *some*); most of the paper is filled with calculations dedicated to showing that other apparently possible tilings actually are not possible.

We begin by studying a number of special cases that we will need later; then we finish with a frontal attack on the general case, in which the tile is neither a right triangle nor similar to the tiled triangle ABC . This methodical approach succeeds in all but two cases. One of those cases is treated in the third paper in this series [2], and the other is taken up in [3].

The following lemma identifies those relatively few rational multiples of π that have rational tangents or whose sine and cosine satisfy a polynomial of low degree over \mathbb{Q} . The lemma and

its proof are of course well-known, but it is short and may help to make the paper more self-contained.

Lemma 1 *Let $\zeta = e^{i\theta}$ be algebraic of degree d over \mathbb{Q} , where θ is a rational multiple of π , say $\theta = 2m\pi/n$, where m and n have no common factor.*

Then $d = \varphi(n)$, where φ is the Euler totient function. In particular if $d = 4$, which is the case when $\tan \theta$ is rational and $\sin \theta$ is not, then n is 5, 8, 10, or 12; and if $d = 8$ then n is 15, 16, 20, 24, or 30.

Remark. For example, if $\theta = \pi/6$, we have $\sin \theta = 1/2$, which is of degree 1 over \mathbb{Q} . Since $\cos \theta = \sqrt{3}/2$, the number $\zeta = e^{i\theta}$ is in $\mathbb{Q}(i, \sqrt{3})$, which is of degree 4 over \mathbb{Q} . The number ζ is a 12-th root of unity, i.e. n in the theorem is 12 in this case; so the minimal polynomial of ζ is of degree $\varphi(12) = 4$. This example shows that the theorem is best possible.

Remark. The hypothesis that θ is a rational multiple of π cannot be dropped. For example, $x^4 - 2x^3 + x^2 - 2x + 1$ has two roots on the unit circle and two off the unit circle.

Proof. Let f be a polynomial with rational coefficients of degree d satisfied by ζ . Since $\zeta = e^{i2m\pi/n}$, ζ is an n -th root of unity, so its minimal polynomial has degree $d = \varphi(n)$, where φ is the Euler totient function. Therefore $\varphi(n) \leq d$. If $\tan \theta$ is rational and $\sin \theta$ is not, then $\sin \theta$ has degree 2 over \mathbb{Q} , so ζ has degree 2 over $\mathbb{Q}(i)$, so ζ has degree 4 over \mathbb{Q} . The stated values of n for the cases $d = 4$ and $d = 8$ follow from the well-known formula for $\varphi(n)$. That completes the proof of (ii) assuming (i).

Corollary 1 *If $\sin \theta$ or $\cos \theta$ is rational, and $\theta < \pi$ is a rational multiple of π , then θ is a multiple of $2\pi/n$ where n is 5, 8, 10, or 12.*

Proof. Let $\zeta = \cos \theta + i \sin \theta = e^{i\theta}$. Under the stated hypotheses, the degree of $\mathbb{Q}(\zeta)$ over \mathbb{Q} is 2 or 4. Hence, by the lemma, θ is a multiple of $2\pi/n$, where $n = 5, 8, 10,$ or 12 (if the degree is 4) or $n = 3$ or 6 (if the degree is 3). But the cases 3 and 6 are superfluous, since then θ is already a multiple of $2\pi/12$.

In [1], we introduced the \mathbf{d} matrix and the \mathbf{d} matrix equation,

$$\mathbf{d} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

where $a, b,$ and c are the sides of the tile, and $X, Y,$ and Z are the lengths of the sides of ABC , in order of size. The angles of ABC are, in order of size, $A, B,$ and C , so $X = \overline{BC}$, $Y = \overline{AC}$, and $Z = \overline{AB}$. We keep this convention even if some the angles are equal. The \mathbf{d} matrix has nonnegative integer entries, describing how the sides of ABC are composed of edges of tiles.

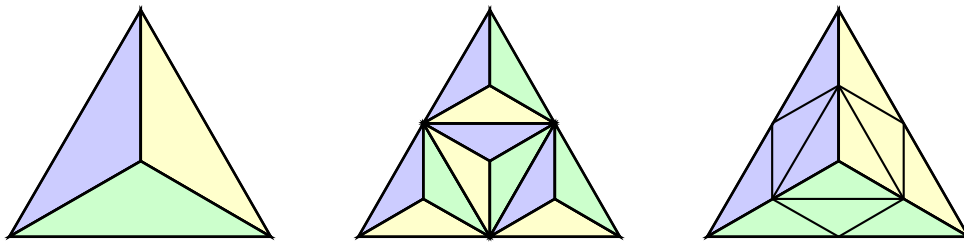
The d matrix is used in almost all our proofs. To avoid having every page filled with cumbersome subscript notation \mathbf{d}_{ij} for the entries of the matrix, we introduce letters for the entries. While this eliminates subscripts, it does require the reader to remember which element is denoted by which letter. Here, for reference, we define

$$\mathbf{d} = \begin{pmatrix} p & d & e \\ g & m & f \\ h & \ell & r \end{pmatrix}$$

2 An isosceles tile T with base angle $\pi/6$

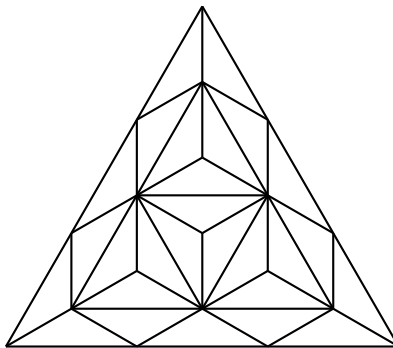
In this section, we investigate the tilings that can be constructed from the tile T with two $\pi/6$ angles and one $2\pi/3$ angle, in which T is not similar to ABC . One can find examples of such tilings by first quadratically tiling an equilateral triangle, and then 3-tiling each tile; the first 12-tiling shown Fig. 1 is of that form.

Figure 1: Tilings by the isosceles tile with base angle $\pi/6$



But this does not exhaust the possibilities, as we can also 3-tile an equilateral triangle ABC and then quadratically tile the three resulting triangles. In this way we can produce, for example, a (different) 12-tiling, illustrated in the second tiling in Fig. 1. These tilings are not prime; that is, they are made by first constructing one tiling and then tiling the tile. We once thought that the equilateral 3-tiling might be the only prime tiling constructed from T . That is incorrect, however, as shown by the prime 27-tiling in Fig. 2.

Figure 2: A prime 27-tiling by the isosceles tile with base angle $\pi/6$



Lemma 2 *Let triangle T be isosceles with a vertex angle of $2\pi/3$. Suppose triangle ABC is N -tiled by T and not similar to T . Then ABC is equilateral, and N has the form $3m^2$ for some integer m .*

Remark. For each integer N of the form mentioned in the lemma, there does exist an N -tiling of an equilateral triangle by the tile T of the lemma, as discussed above.

Proof. Let $\alpha = \pi/6$ and $\gamma = 2\pi/3$ be the angles of T . We note that no tile has its γ angle at a vertex of ABC , since that would force the other two vertices of ABC to have angle α and hence ABC would be similar to T , contrary to hypothesis. Hence the vertex angles of ABC are composed (altogether) of six α angles. The only two ways of writing 6 as a sum of smaller positive integers are $6 = 1 + 2 + 3$ and $6 = 2 + 2 + 2$. Hence, ABC is either equilateral, or it is a 30-60-90 right triangle.

We are free to choose the size of T ; let us suppose that the sides of T are $a = 1$ and $c = \sqrt{3}$, so the altitude of T is $1/2$ and its area is $\sqrt{3}/4$ (because its base is $\sqrt{3}$ and its height is $1/2$). If

X , Y , and Z are the sides of ABC , then the fundamental tiling equation is

$$\mathbf{d} \begin{pmatrix} 1 \\ 1 \\ \sqrt{3} \end{pmatrix} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

Our equations will be slightly simpler if we recall the convention that when the tile is isosceles, as it is in this proof, so that $b = a$, that we take the middle column of the \mathbf{d} matrix to be zero (counting short sides as a rather than b , since it is arbitrary).

We begin by proving that it is not the case that along one side of ABC , all the tiles sharing that side of ABC have an a edge on the boundary of ABC . Assume, for proof by contradiction, that all the tiles on side BC have an a edge on BC . Then each of these tiles has a γ angle at one of its vertices on BC . But there cannot be a γ angle at the endpoints B and C . Hence, there is one more γ angle than vertices on (the interior of) BC , so by the pigeonhole principle, some vertex has more than one γ angle. But this is impossible since $\gamma > \pi/2$. That contradiction shows that it is impossible for all the tiles with an edge on BC to have an a edge on BC . The same argument applies to each side of ABC , since all the angles of ABC are less than γ . To phrase this result in terms of the \mathbf{d} matrix: it is not the case that any entry in the third column is zero. That is, $e \neq 0$, $f \neq 0$, and $r \neq 0$.

We now take up the case when ABC is equilateral. Let X be the side of equilateral triangle ABC . Then the altitude of ABC is $X\sqrt{3}/2$, and its area is $X^2\sqrt{3}/4$. But this area is also N times the area of T . Since the area of T is $\sqrt{3}/4$, we have $N\sqrt{3}/4 = X^2\sqrt{3}/4$. Hence $N = X^2$, and $X = \sqrt{N}$. The tiling equations are

$$\mathbf{d} \begin{pmatrix} 1 \\ 1 \\ \sqrt{3} \end{pmatrix} = \sqrt{N} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

We can assume that the components of this vector equation correspond to edges AB , AC , and BC respectively. Then, remembering that the middle column of \mathbf{d} is zero because the tile is isosceles, we have

$$\begin{aligned} \sqrt{N} &= p + e\sqrt{3} \\ N &= (p + e\sqrt{3})^2 \\ &= p^2 + 3e^2 + 2pe\sqrt{3} \end{aligned}$$

Since $\sqrt{3}$ is irrational, we have $pe = 0$. Since we proved above that $e \neq 0$, we have $p = 0$. That is, no a sides of the tile occur on the side BC of ABC corresponding to the first row of the tiling equation. Then BC is entirely composed of c edges (as it indeed is in the 27-tiling). Since each side of ABC has length X and $c = \sqrt{3}$, the number of tiles on each edge is $X/\sqrt{3}$, which must be an integer, say m . Then $X^2 = 3m^2$, so $N = X^2 = 3m^2$. That completes the proof in case ABC is equilateral.

We may therefore assume that ABC is a 30-60-90 triangle. Let the $\pi/6$ angle of ABC be at A , the right angle at C and the $\pi/3$ angle at B . Let the sides of ABC be X , $X\sqrt{3}$, and $2X$. Then the area of ABC is $X^2\sqrt{3}/2$, which must be equal to the area of N tiles, namely $N\sqrt{3}/4$. Hence $X^2 = \frac{N}{2}$, or $X = \sqrt{N/2}$. Then we have

$$\mathbf{d} \begin{pmatrix} 1 \\ 1 \\ \sqrt{3} \end{pmatrix} = \sqrt{\frac{N}{2}} \begin{pmatrix} 1 \\ \sqrt{3} \\ 2 \end{pmatrix}.$$

From the first row of this equation, we have

$$p + e\sqrt{3} = \sqrt{\frac{N}{2}}.$$

Squaring this equation we have

$$\begin{aligned}\frac{N}{2} &= (p + e\sqrt{3})^2 \\ &= p^2 + 3e^2 + 2pe\sqrt{3}\end{aligned}$$

Since the left side is rational, we have $pe = 0$, and since $e \neq 0$, we have $p = 0$. Then

$$\begin{aligned}\frac{N}{2} &= 3e^2 \\ N &= 6e^2.\end{aligned}\tag{1}$$

From the second row of the \mathbf{d} matrix equation, we have

$$g + f\sqrt{3} = \sqrt{\frac{N}{2}}\sqrt{3}.$$

Squaring this equation, we have

$$g^2 + 3f^2 + 2gf\sqrt{3} = 3N$$

Since $\sqrt{3}$ is irrational, we have $gf = 0$. Since $f \neq 0$ we have $g = 0$. Then

$$\begin{aligned}3f^2 &= 3N \\ N &= f^2\end{aligned}$$

But by (1), we have $N = 6e^2$. Hence $f^2 = 6e^2$, so $\sqrt{6} = f/e$, contradicting the irrationality of $\sqrt{6}$. That completes the proof of the lemma.

3 An isosceles tile T with base angle $\pi/5$

Lemma 3 *Let $\alpha = \pi/5$. Let T be an isosceles triangle with two angles α and one angle 3α . Let ABC be an isosceles triangle with base angles 2α and vertex angle α . Then there is no tiling of ABC by tile T .*

Proof. Let $a = \sin \alpha$. We will work in the field $\mathbb{Q}(a)$ so we begin with some identities in that field. We have

$$\begin{aligned}a &= \sin \alpha = \frac{1}{4}\sqrt{10 - 2\sqrt{5}} \\ a^2 &= \frac{1}{16}(10 - 2\sqrt{5}) \\ 16a^2 - 10 &= -2\sqrt{5} \\ \sqrt{5} &= 5 - 8a^2 \\ (8a^2 - 5)^2 &= 5 \\ 64a^4 - 80a^2 + 25 &= 5 \\ 16a^4 - 20a^2 + 5 &= 0\end{aligned}$$

This is the minimal polynomial of a over \mathbb{Q} . It is irreducible by Eisenstein's criterion, since 20 and 5 are divisible by 5, but 16 is not, and 5 is not divisible by 5^2 . So $\mathbb{Q}(a)$ has degree 4 over \mathbb{Q} and $\{1, a, a^2, a^3\}$ is a basis for $\mathbb{Q}(a)$ over \mathbb{Q} . From the minimal polynomial we see

$$a^4 = 20a^2 - 5\tag{2}$$

We have

$$\begin{aligned}
b &= \cos \alpha \\
&\quad \frac{1 + \sqrt{5}}{4} \\
&= \frac{1 + (5 - 8a^2)}{4} \quad \text{since } \sqrt{5} - 5 = -8a^2 \\
&= \frac{3 - 4a^2}{2}
\end{aligned}$$

Hence

$$b = \frac{3}{2} - 2a^2 \quad (3)$$

We have $\sin 2\alpha = 2ab = 2a(\frac{3}{2} - 2a^2)$. Hence

$$\sin 2\alpha = 3a - 4a^3 \quad (4)$$

Recall (or prove) the trig identity (which is true for any α)

$$\frac{\sin 3\alpha}{\sin \alpha} = 3 - 4 \sin^2 \alpha$$

Expressing this more briefly using a we have

$$\frac{\sin 3\alpha}{\sin \alpha} = 3 - 4a^2 \quad (5)$$

With these algebraic preliminaries in hand, we turn to the proof of the lemma. The sides of T are $\sin \alpha$ and $\sin(3\alpha)$. The altitude of T is $\sin^2 \alpha$, so the area of T is $\frac{1}{2} \sin(3\alpha) \sin^2 \alpha$. On the other hand, for some number λ , the sides of ABC are $\lambda \sin 2\alpha$ and $\lambda \sin \alpha$. The altitude of ABC is $\frac{1}{2} \lambda \sin \alpha \sin(2\alpha)$, so its area is $\frac{1}{2} \lambda^2 \sin \alpha \sin^2 2\alpha$. Since the area of ABC must be N times the area of T , we have

$$\begin{aligned}
N \frac{1}{2} \sin(3\alpha) \sin^2 \alpha &= \frac{1}{2} \lambda^2 \sin \alpha \sin^2 2\alpha \\
N \sin(3\alpha) \sin \alpha &= \lambda^2 \sin^2 2\alpha \\
\lambda &= \frac{\sqrt{N \sin \alpha \sin(3\alpha)}}{\sin(2\alpha)}
\end{aligned}$$

Let \mathbf{d} be the \mathbf{d} matrix of the tiling. Since the tile is isosceles, we only have sides a and c , not b , and the middle column of the \mathbf{d} matrix is zero. Then we have

$$\begin{aligned}
\mathbf{d} \begin{pmatrix} \sin \alpha \\ \sin \alpha \\ \sin 3\alpha \end{pmatrix} &= \lambda \begin{pmatrix} \sin \alpha \\ \sin 2\alpha \\ \sin 2\alpha \end{pmatrix} \\
&= \frac{\sqrt{N \sin \alpha \sin(3\alpha)}}{\sin(2\alpha)} \begin{pmatrix} \sin \alpha \\ \sin 2\alpha \\ \sin 2\alpha \end{pmatrix}
\end{aligned}$$

Writing the second of these three equations out we have

$$g \sin \alpha + f \sin 3\alpha = \sqrt{N \sin \alpha \sin(3\alpha)}$$

Dividing both sides by $\sin \alpha$ we have

$$g + f \frac{\sin 3\alpha}{\sin \alpha} = \sqrt{N \frac{\sin 3\alpha}{\sin \alpha}}$$

Expressing this in terms of $a = \sin \alpha$, we have

$$g + f(3 - 4a) = \sqrt{N(3 - 4a)}$$

Squaring both sides we have

$$\begin{aligned} (g + f(3 - 4a))^2 &= N(3 - 4a) \\ g^2 + 2gf(3 - 4a) + f^2(3 - 4a)^2 &= N(3 - 4a) \\ g^2 + 6gf + 9f^2 + a(-8gf - 24f^2) + a^2(16f^2) &= 3N - 4aN \end{aligned}$$

Since $\{1, a, a^2, a^3\}$ is a basis for $\mathbb{Q}(a)$ over \mathbb{Q} , the coefficients of like powers of a must be equal on both sides. Therefore the coefficient of a^2 on the left is zero; that is, $f = 0$. Then the equation becomes

$$g^2 = 3N - 4aN$$

But now there is a nonzero coefficient of a on the right, and no a term on the left. This contradiction completes the proof of the lemma.

4 A non-isosceles tile T with largest angle $2\pi/5$

Lemma 4 *Suppose ABC is N -tiled by a triangle T . Suppose ABC is not similar to T , and T is not isosceles. Then the largest angle γ of T cannot be $2\pi/5$.*

Proof. Since ABC is not similar to T , we have vertex splitting. As before, let P , Q , and R be the total number of α , β , and γ angles, respectively, at vertices of ABC . Since T is not similar to ABC , we have $P + Q + R \geq 5$. Since T is not isosceles, $\alpha < \beta$. We have $\beta > (\pi - \gamma)/2 = 3\pi/10 = (3/4)\gamma$. The number R of γ angles is at most 2, since $3\gamma = 6\pi/5 > \pi$.

Assume, for proof by contradiction, that $R = 2$. Then $P\alpha + Q\beta = \pi/5$, and since $\beta > 3\pi/10 > \pi/5$, we must have $Q = 0$. Hence $\alpha = \pi/(5P)$ and $\beta = \pi - \gamma - \alpha = 3\pi/5 - \pi/(5P)$. Since $\beta < \gamma$, we have

$$\begin{aligned} \frac{3}{5}\pi - \frac{\pi}{5P} &< \frac{2}{5}\pi \\ \frac{3}{5} - \frac{1}{5P} &< \frac{2}{5} \\ 3P - 1 &< 2P \\ P &< 1 \end{aligned}$$

This contradicts $P + Q \geq 5$. That completes the proof by contradiction that $R \neq 2$. Since $R \leq 2$, that implies $R = 1$ or $R = 0$.

Now assume, for proof by contradiction, that $R = 1$. Then the equation $P\alpha + Q\beta + R\gamma = \pi$ becomes $P\alpha + Q\beta = 3\pi/5$, and since $\beta > 3\pi/10$, we must have $Q \leq 1$. We have two equations for α and β :

$$\begin{aligned} P\alpha + Q\beta &= \frac{3\pi}{5} && \text{since } R = 1 \\ \alpha + \beta &= \frac{3\pi}{5} && \text{since } \alpha + \beta + \gamma = \pi \end{aligned}$$

We cannot have $P = Q = 1$, since $P + Q \geq 5$. Hence $P \neq Q$, and the equations are uniquely solvable for α and β . By Cramer's rule we have

$$\begin{aligned} \alpha &= \frac{3\pi}{5} \frac{1 - Q}{P - Q} \\ \beta &= \frac{3\pi}{5} \frac{P - 1}{P - Q} \end{aligned}$$

Since $Q \leq 1$, the equation for α shows that $Q = 0$, since $\alpha > 0$. Hence

$$\begin{aligned}\alpha &= \frac{3\pi}{5P} \\ \beta &= \frac{3\pi}{5} \frac{P-1}{P}\end{aligned}$$

We have $\beta < \gamma = 2\pi/5$. Hence

$$\frac{3\pi}{5} \frac{P-1}{P} < \frac{2\pi}{5}$$

Dividing by π and multiplying by $5P$, we have

$$\begin{aligned}3(P-1) &< 2P \\ 3P-3 &< 2P \\ P &< 3\end{aligned}$$

But this contradicts $P + Q + R \geq 5$, since $R = 1$ and $Q = 0$. That completes the proof that $R \neq 1$.

Hence $R = 0$. The equation $P\alpha + Q\beta + R\beta = \pi$ then becomes $P\alpha + Q\beta = \pi$, and we have the two equations

$$\begin{aligned}P\alpha + Q\beta &= \pi \\ \alpha + \beta &= \frac{3\pi}{5} \quad \text{since } \alpha + \beta + \gamma = \pi\end{aligned}$$

If $P = Q$, then we have $\alpha + \beta = \pi/P = (3\pi)/5$. Hence $P = 5/3$, which is impossible since P is an integer. Therefore $P \neq Q$, and the equations can be solved uniquely for α and β . By Cramer's rule we have

$$\begin{aligned}\alpha &= \frac{\pi - (3\pi/5)Q}{P - Q} \\ \beta &= \frac{(3\pi/5)P - Q\pi}{P - Q}\end{aligned}$$

Assume, for proof by contradiction, that $Q > 1$. Then the numerator of α is negative, so the denominator $P - Q$ is also negative, so $Q > P$. Then

$$\beta = \frac{Q - (3/5)P}{Q - P}\pi$$

Since $\beta < \gamma = 2\pi/5$, we have

$$\frac{Q - (3/5)P}{Q - P} < \frac{2}{5}$$

Multiplying both sides by $5(Q - P)$, which is positive since $Q > P$, we have

$$\begin{aligned}5Q - 3P &< 2Q - 2P \\ 3Q &< P\end{aligned}$$

But we have $P < Q$, so $3Q < P < Q$, which is a contradiction. Thus our assumption that $Q > 1$ is untenable. Hence $Q \leq 1$.

Now assume $Q = 0$. Then the formula for β above simplifies to $3\pi/5$, which is more than γ , contradiction. Hence $Q = 1$. Then we have

$$\beta = \frac{(3/5)P - 1}{P - 1}\pi$$

The equation $\beta < \gamma$ then implies

$$\frac{(3/5)P - 1}{P - 1} < \frac{2}{5}$$

Multiplying both sides by $5(P - 1)$ we have

$$\begin{aligned} 3P - 5 &< 2(P - 1) \\ P &< 3 \end{aligned}$$

But that contradicts $P + Q + R \geq 5$, since $R = 0$ and $Q = 1$. This contradiction does not depend on any assumption, and hence completes the proof of the lemma.

5 The case $\alpha = \pi/11$ and $\beta = 3\alpha$

In this section and the next, we prove there are no tilings of any triangle ABC in which the tile T is not similar to ABC , for two special tiles T that seem to slip through the cracks of our arguments in subsequent sections. We deal with these two special cases using some field theory computations in the relevant cyclotomic fields, combined with simple geometric arguments. Some of the computations were made using a computer algebra system, but they could have been made by hand with enough patience, and they can be checked (once made) with a calculator, as we shall now explain.

We used the computer algebra systems Sage and Mathematica. The results we obtained are of two kinds. Some are polynomial equations in a root of unity ζ , with integer coefficients. You can check these with a scientific calculator, and verify that the two sides agree to many decimal places. While that is not a proof, it is good evidence that no mistake has been made in preparing input for Sage or typesetting the output, or by Sage itself. On the other hand, some of the results are equations containing not only ζ but also parameters for unknown integers p, q, r , etc. Here a computer algebra system is used to divide one polynomial (with variables in the coefficients) by another and get the remainder. The computations would be very difficult, but not impossible, to perform by hand. To check whether a mistake has been made in the input or the typesetting, you would have to give the problem to Sage or Mathematica or Maple yourself. Mathematica and Maple are also capable of doing these calculations, and we checked some of them in both Mathematica and Sage.

The most curious thing about this part of the work is that the two special cases are surprisingly different. The two arguments have a method in common (field theory), but the fields are different, and the resulting equations are different, and the geometric arguments at the end are different. They do not seem to be special cases of a more general proof.

Lemma 5 *Let ABC be tiled using as a tile the triangle T with $\alpha = \pi/11$, $\beta = 3\pi/11$, and $\gamma = 7\pi/11$, and suppose the angles of triangle ABC are anything but 2α , 4α , and 5α . Then ABC is similar to T .*

Remark. We were not able to deal with the exceptional case mentioned by this method of proof.

Proof. First we note that $\sin \alpha$ and $\cos \alpha$ are not rational, by Corollary 1. Let

$$\begin{aligned} a &:= \sin \alpha \\ \zeta &:= e^{i\alpha} \end{aligned}$$

Then $\zeta^{22} = 1$. Since 22 is not divisible by 4, i does not belong to $\mathbb{Q}(\zeta)$. The degree of $\mathbb{Q}(\zeta)$ over \mathbb{Q} is $\varphi(22) = 10$, and $\mathbb{Q}(\zeta, i)$ then has degree 20. There are no convenient automorphisms of $\mathbb{Q}(\zeta)$ that fix two of ia, ib , and ic and move the third, so we are forced to consider the area equation directly.

The minimal polynomial of ζ is the cyclotomic polynomial

$$x^{10} - x^9 + x^8 - x^7 + x^6 - x^5 + x^4 - x^3 + x^2 - x + 1.$$

We note that $\sin^2 \alpha$ is not rational, since if it were, then $\sin \alpha$ and $\cos \alpha$ would be of degree 2, so $\mathbb{Q}(\sin \alpha, \cos \alpha)$ would be of degree 1, 2, or 4, and $\mathbb{Q}(\zeta, i) = \mathbb{Q}(\sin \alpha, \cos \alpha, i)$ would be of degree 1, 2, 4, or 8, but we have shown that it is 20. Hence $\mathbb{Q}(a^2) \neq \mathbb{Q}$. Since $a^2 = -(\zeta - \zeta^{-1})^2$ belongs to $\mathbb{Q}(\zeta)$, it is of degree dividing 10. Since it is fixed under the automorphism σ_{21} that takes ζ to $\zeta^{21} = \zeta^{-1}$, in fact $\mathbb{Q}(a^2)$ is of degree 5 over \mathbb{Q} .

The minimal polynomial of $a = \sin(\pi/11)$ is

$$x^{10} - \frac{11}{4}x^8 + \frac{11}{4}x^6 - \frac{77}{64}x^4 + \frac{55}{256}x^2 - \frac{11}{1024}$$

according to Sage. We will show that $\cos \alpha$ belongs to $\mathbb{Q}(a)$. We have

$$\sin 6\alpha = \sin 5\alpha$$

Expanding both sides in $\sin \alpha$ and $\cos \alpha$ (using one's favorite computer algebra system, or pencil and paper), subtracting one from the other, and then using $\cos^2 \alpha = 1 - \sin^2 \alpha$, one finds

$$0 = (32 \sin^4 \alpha \cos \alpha - 16 \sin^4 \alpha - 32 \sin^2 \alpha \cos \alpha + 20 \sin^2 \alpha + 6 \cos \alpha - 5) \sin \alpha$$

Dividing by $\sin \alpha$ we have

$$\begin{aligned} 0 &= 32 \sin^4 \alpha \cos \alpha - 16 \sin^4 \alpha - 32 \sin^2 \alpha \cos \alpha + 20 \sin^2 \alpha + 6 \cos \alpha - 5 \\ &= \cos \alpha (32 \sin^4 \alpha - 32 \sin^2 \alpha + 6) - 16 \sin^4 \alpha + 20 \sin^2 \alpha - 5 \end{aligned}$$

Solving for $\cos \alpha$ we find

$$\begin{aligned} \cos \alpha &= \frac{16 \sin^4 \alpha - 20 \sin^2 \alpha + 5}{32 \sin^4 \alpha - 32 \sin^2 \alpha + 6} \\ &= \frac{16a^4 - 20a^2 + 5}{32a^4 - 32a^2 + 6} \end{aligned}$$

Squaring and using $\cos^2 \alpha = 1 - \sin^2 \alpha$, one soon can confirm the minimal polynomial reported by Sage; we stop here, having proved that $\cos \alpha$ belongs to $\mathbb{Q}(a^2)$. It is awkward to have $\cos \alpha$ expressed as a fraction; we want to have it expressed in terms of the basis elements of $\mathbb{Q}(a^2)$. Sage can do that for us, and we can verify the result numerically to check that there was no human or machine error.¹

$$\cos \alpha = 128a^8 - 288a^6 + 216a^4 - 60a^2 + \frac{9}{2} \tag{6}$$

We next wish to express $b = \sin \beta$ and $c = \sin \gamma$ in terms of $a = \sin \alpha$ and $\cos \alpha$. We have

$$\begin{aligned} b &= \sin 3\alpha \\ &= 3a - 4a^3 \end{aligned}$$

¹If we wanted to do it by hand, we would use the extended Euclidean algorithm to find polynomials λ and μ such that $\lambda(32x^4 - 32x^2 + 6) + \mu\psi(x) = 1$, where ψ is the minimal polynomial of a ; then the answer is $(16x^4 - 20x^2 + 5)\lambda \bmod \psi$. This is tedious by hand, easy with Mathematica or Sage, but then you might as well just let Sage do the whole calculation.

and

$$\begin{aligned}
c &= \sin 4\alpha \\
&= \sin(2 \cdot 2\alpha) \\
&= 2 \sin 2\alpha \cos 2\alpha \\
&= 4 \sin \alpha \cos \alpha (1 - 2 \sin^2 \alpha) \\
&= 4a(1 - 2a^2) \cos \alpha \\
&= 4a(1 - 2a^2) \left(128a^8 - 288a^6 + 216a^4 - 60a^2 + \frac{9}{2}\right)
\end{aligned}$$

Now we ask Sage to evaluate this expression in $\mathbb{Q}(a)$. It tells us

$$c = -64a^7 + 112a^5 - 56a^3 + 7a$$

As a check, this value is numerically equal to $\sin(4\alpha)$ to many decimal places.

Let θ be one of the angles of triangle ABC and let U and V be the adjacent sides. Let $p, q, r, m, n,$ and ℓ be the rows of the \mathbf{d} matrix associated with U and V , i.e. the numbers of tiles with $a, b,$ and c sides on U and V , respectively. Explicitly,

$$\begin{aligned}
U &= pa + qb + rc \\
V &= ma + nb + \ell c
\end{aligned}$$

The area equation $2N\mathcal{A}_T = 2\mathcal{A}_{ABC}$ becomes

$$\begin{aligned}
Nbc &= UV \sin \theta \\
&= (pa + qb + rc)(ma + nb + \ell c) \sin \theta \\
&= (pa + q(3a - 4a^3) + r(-64a^7 + 112a^5 - 56a^3 + 7a)) \\
&\quad \left(ma + n(3a - 4a^3) + \ell(-64a^7 + 112a^5 - 56a^3 + 7a)\right) \sin \theta
\end{aligned}$$

Dividing both sides by $a^2 \sin \theta$ we have

$$\begin{aligned}
\frac{Nbc}{a \sin \theta} &= \left(p + q(3 - 4a^2) + r(-64a^6 + 112a^4 - 56a^2 + 7)\right) \\
&\quad \left(m + n(3a - 4a^2) + \ell(-64a^6 + 112a^4 - 56a^2 + 7)\right)
\end{aligned}$$

Multiplying out, and reducing by the minimal polynomial of a , we have (according to Sage)

$$\begin{aligned}
\frac{Nbc}{a \sin \theta} &= (256(q + 14r)\ell + 256(14\ell + n)r - 7168\ell r)a^8 \\
&\quad + (-448(q + 14r)\ell - 448(14\ell + n)r - 64(p + 3q + 7r)\ell - 64(7\ell + m + 3n)r + 13376\ell r)a^6 \\
&\quad + (16(q + 14r)(14\ell + n) + 112(p + 3q + 7r)\ell + 112(7\ell + m + 3n)r - 4576\ell r)a^4 \\
&\quad + (-4(q + 14r)(7\ell + m + 3n) - 4(14\ell + n)(p + 3q + 7r) + 704\ell r)a^2 \\
&\quad + (p + 3q + 7r)(7\ell + m + 3n) - 33\ell r
\end{aligned}$$

Expanding and simplifying the coefficients of a^8 and a^6 we find

$$\begin{aligned}
\frac{Nbc}{a \sin \theta} &= 256(q\ell + rn)a^8 \\
&\quad - (640(q\ell + rn) + 64(\ell p + \ell r + mr))a^6 \\
&\quad + (16(q + 14r)(14\ell + n) + 112(p + 3q + 7r)\ell + 112(7\ell + m + 3n)r - 4576\ell r)a^4 \\
&\quad + (-4(q + 14r)(7\ell + m + 3n) - 4(14\ell + n)(p + 3q + 7r) + 704\ell r)a^2 \\
&\quad + (p + 3q + 7r)(7\ell + m + 3n) - 33\ell r
\end{aligned} \tag{7}$$

Now suppose, for proof by contradiction, that $\theta = \gamma$. Then $\sin \theta = c$ and we can cancel c from numerator and denominator on the left. Substituting $b = 3a - 4a^3$ on the left, a factor of a cancels and we obtain $3N - 4Na^2$ on the left. Comparing coefficients of a^8 we have zero on the left and $ql + rn$ on the right; hence $ql = 0$ and $rn = 0$ (since both quantities are nonnegative). Then the coefficient of a^6 is zero on the left, and on the right it is $64(lp + lr + mr)$. Hence $lp = 0 = lr = mr$. Now we have proved that $nr = mr = lr = 0$, so $r(n + m + \ell) = 0$. But $n + m + \ell > 0$, so $r = 0$. Then the coefficient of a^4 simplifies to $16nq$; hence $nq = 0$. We have also proved $lp = 0 = lq = lr$, so $\ell(p + q + r) = 0$. But $p + q + r > 0$, so $\ell = 0$. The coefficient of a^2 simplifies to $-4(mq + np)$ on the right and $-4N$ on the left. Therefore $mq + np = N$. The constant term is $3N$ on the left and on the right it simplifies to $mp + 3mp + 3np$. Hence $mp + 3mp + 3np = 3N$. Subtracting three times the equation $mq + np = N$ we find $mp = 0$. Assume for the moment that $m \neq 0$. Then $p = 0$; since $\ell = 0$ we must have $q \neq 0$. Then since $nq = 0$ we have $n = 0$, and side V is composed entirely of a sides of tiles, while side U is composed entirely of b sides of tiles. On the other hand if $m = 0$ then side V is composed entirely of b sides of tiles, so $n \neq 0$; then since $nq = 0$ we have $q = 0$ and side U is composed entirely of a sides of tiles. Note that these deductions are consistent with the case when ABC is similar to T and the tiling is quadratic. Interchanging the arbitrary labels U and V if necessary, we may assume that side U is composed entirely of a sides of tiles. The other end of side U is at either A or B . The angle of ABC at A cannot be α , for then ABC would be similar to T , since angle $C = \gamma$; but it must be less than β , since $2\beta + \gamma > \pi$. Also the angle at B must be less than β , since $A + B = \alpha + \beta$, and angle $A > \alpha$. Hence the angle at the other endpoint of U is less than β , whether that endpoint is A or B . Let T_1 be the tile along U at that vertex of ABC . Then T_1 has its α angle in the corner of ABC . But then it cannot have its a side along U , contradiction. That completes the proof that $\theta \neq \gamma$. Since θ was originally any angle of ABC , we can now assume that none of the angles of ABC are equal to γ .

Returning to (7), the last equation before we assumed $\theta = \gamma$, we now assume, for proof by contradiction, that $\theta = \alpha$. Then $\sin \theta = a$ and on the left we have Nbc/a^2 . Substituting the expressions derived above for b/a and c/a on the left, we have

$$\begin{aligned}
& N(3 - 4a^2)(-64a^6 + 112a^3 - 56a^2 + 7) \\
= & 256(ql + rn)a^8 \\
& -(640(ql + rn) + 64(lp + lr + mr))a^6 \\
& +(16(q + 14r)(14\ell + n) + 112(p + 3q + 7r)\ell + 112(7\ell + m + 3n)r - 4576lr)a^4 \\
& +(-4(q + 14r)(7\ell + m + 3n) - 4(14\ell + n)(p + 3q + 7r) + 704lr)a^2 \\
& +(p + 3q + 7r)(7\ell + m + 3n) - 33lr
\end{aligned}$$

Expanding the left hand side we have

$$\begin{aligned}
& N(256a^8 - 640a^6 + 560a^4 - 196a^2 + 21) \\
= & 256(ql + rn)a^8 \\
& -(640(ql + rn) + 64(lp + lr + mr))a^6 \\
& +(16(q + 14r)(14\ell + n) + 112(p + 3q + 7r)\ell + 112(7\ell + m + 3n)r - 4576lr)a^4 \\
& +(-4(q + 14r)(7\ell + m + 3n) - 4(14\ell + n)(p + 3q + 7r) + 704lr)a^2 \\
& +(p + 3q + 7r)(7\ell + m + 3n) - 33lr
\end{aligned}$$

Now we can equate coefficients of like powers of a . From the coefficient of a^8 we see $N = ql + rn$. Then from the coefficient of a^6 we have

$$\begin{aligned}
640N &= 640(ql + rn) + 64(lp + lr + mr) \\
&= 640N + 64(lp + lr + mr)
\end{aligned}$$

Subtracting the left side from the right we see $\ell p = \ell r = mr = 0$. Turning to the coefficient of a^4 , we have

$$\begin{aligned} 560N &= 16 \cdot 14(q\ell + rn) + 16qn + 3 \cdot 112(q\ell + rn) \\ &= 560(q\ell + rn) + 16qn \\ &= 560N + 16qn \end{aligned}$$

Subtracting the left side from the right, we see $qn = 0$. Finally, from the constant term we have

$$\begin{aligned} 21N &= pm + 3pn + 3qm + 21(q\ell + rn) \\ &= pm + 3pn + 3qm + 21N \end{aligned}$$

Subtracting the left side from the right, we see $pm = 0 = pn = qm$. Since we already deduced $p\ell = 0$ we now have $p(m + n + \ell) = 0$, which implies $p = 0$. Similarly, having deduced $mp = mq = rm = 0$, we have $m(p + q + r) = 0$, which implies $m = 0$. Now, if $q \neq 0$ then since $nq = 0$ we would have $n = 0$; and then since $m = 0$ we would have $\ell \neq 0$, and since $\ell r = 0$ we would have $\ell = 0$. Thus if $q \neq 0$ then U is composed entirely of b sides of tiles (since $p = r = 0$) and V is composed entirely of c sides of tiles (since $m = n = 0$). On the other hand, if $q = 0$ then since $p = 0$, we would have $r \neq 0$, so $\ell = 0$, so $n \neq 0$, and then U would be composed entirely of c sides of tiles and V entirely of b sides. Hence in either case, one of U and V is composed entirely of b sides; we may assume without loss of generality that it is U . Let T_1 be the tile at vertex A (there is only one since the angle there is α). Let V_2 be the next vertex on side U , at the other end of the b side of T_1 . Then T_1 has its γ angle at V_2 , since it has its β angle opposite its b side on U , and its α angle at A . Since $\gamma > \pi/2$, there cannot be two γ angles on the same side of U at any one vertex. Then by the pigeonhole principle, the last tile T_k must have its γ angle at the vertex C of ABC , and U is the side AC .

But angle C cannot be exactly γ , since then ABC would be similar to T (and besides, we already showed none of the angles of ABC are equal to γ). Assume, for proof by contradiction, that angle $C = \gamma + \alpha$. Let P be the point on side AB such that angle ACP is γ ; since angle C is more than γ , such a point P exists. Then triangle ACP is similar to T , and the similarity factor is q , since the side opposite the β angle of ACP (which is AC) has length qb . Hence the length of AP is qc and the area of triangle ACP is $q^2\mathcal{A}_T$. Since the length of AP is qc and that is less than the length of AB , which is ℓc , we have $q < \ell$.

Observe that triangle CPB is similar to triangle ABC , since angle PCB is α and the two triangles share angle $B = 2\alpha$. The length of PC is qa , because q is the similarity factor between T and triangle ACP . Side PC is opposite angle B in PCB ; and AC , which has length qb , is opposite angle B in triangle ABC . The similarity factor between CPB and ABC is thus a/b . We now decompose triangle ABC into triangle ABP and triangle CPB (although we do not claim that segment CP belongs to the tiling). We have

$$\begin{aligned} \mathcal{A}_{ABC} &= \mathcal{A}_{ACP} + \mathcal{A}_{CPB} \\ &= q^2\mathcal{A}_T + \frac{a}{b}\mathcal{A}_{ABC} \end{aligned}$$

Solving for \mathcal{A}_{ABC} we find

$$\mathcal{A}_{ABC} = \frac{q^2}{1 - \frac{a}{b}}\mathcal{A}_T$$

On the other hand $\mathcal{A}_{ABC} = N\mathcal{A}_T$ because ABC is N -tilled by T . Hence

$$N\mathcal{A}_T = \frac{q^2}{1 - \frac{a}{b}}\mathcal{A}_T$$

Canceling \mathcal{A}_T and simplifying, we find

$$\begin{aligned} q^2 &= N\left(1 - \frac{a}{b}\right) \\ q^2 b &= N(b - a) \end{aligned}$$

Substituting $b = 3a - 4a^3$ and canceling a from both sides we have

$$\begin{aligned} q^2(3 - 4a^2) &= N(2 - 4a^2) \\ -4q^2 a^2 + 3q^2 &= -4Na^2 + 2N \end{aligned}$$

Comparing the coefficients of a^2 we have $N = q^2$. Comparing the constant coefficients we have $q^2 = 2N/3$. Since $N > 0$ this is a contradiction. Hence, if angle $A = \alpha$, we do not have angle $C = \gamma + \alpha$.

Now suppose, for proof by contradiction, that angle $A = \alpha$ and angle $C = \gamma + 2\alpha$. Then angle $B = \alpha$, so ABC is isosceles. Let P and Q be points on AB such that angle $ACP = \gamma$ and angle $ACQ = \gamma + \alpha$. Then angle $QCB = \alpha$. Since angle $B = \alpha$, triangle ABC is similar to triangle CAB , as both have two α angles. Angle $PCQ = \alpha$ and angle $CPQ = \pi - \beta = \alpha + \gamma$. Therefore triangle PCQ is similar to triangle CQA . Triangle ACP is similar to the tile, with similarity factor q , so $PC = qa$. In the similarity between PCQ and CQA , side PC corresponds to AC , so the similarity factor from CQA to PCQ is $PC/AC = a/b$. Then

$$\begin{aligned} AC &= qb \\ AP &= qc \\ PC &= qa \\ QC &= (a/b)AQ \\ &= (a/b)(AP + PQ) \\ &= (a/b)qc + (a/b)PQ \\ PQ &= (a/b)QC \end{aligned}$$

Substituting this value of PQ into $QC = (a/b)qc + (a/b)PQ$ we have

$$QC = (a/b)qc + (a/b)^2 QC$$

Solving for QC we have

$$QC = \frac{abqc}{b^2 - a^2}$$

Since $PQ = (a/b)QC$ this give us

$$PQ = \frac{a^2 qc}{b^2 - a^2}$$

Since triangle ABC is isosceles we have $BC = AC = qb$. Since triangle QBC is isosceles we have $QB = QC$. We compute

$$\begin{aligned} AB &= AP + PQ + QB \\ &= qc + PQ + QB \\ &= qc + \frac{a^2 qc}{b^2 - a^2} + QB \\ &= qc + \frac{a^2 qc}{b^2 - a^2} + QC \text{ since } QB = QC \\ &= qc + \frac{a^2 qc}{b^2 - a^2} + \frac{abqc}{b^2 - a^2} \end{aligned}$$

$$\begin{aligned}
&= qc\left(1 + \frac{a^2 + ab}{b^2 - a^2}\right) \\
&= qc\left(1 + \frac{a(b+a)}{b^2 - a^2}\right) \\
&= gc\left(1 + \frac{a}{b-a}\right) \\
&= gc\left(\frac{b-a+a}{b-a}\right) \\
&= gc\left(\frac{b}{b-a}\right)
\end{aligned}$$

But AB , which was the side V , is composed entirely of c edges. Hence $gb(b-a)$ is an integer, namely ℓ where $V = \ell c$. In particular $b(b-a) = \ell/g$ is rational. But $b = 3a - 4a^3$, so we have

$$\begin{aligned}
\frac{\ell}{g} &= (3a - 4a^3)((3a - 4a^3) - a) \\
0 &= (3a - 4a^3)(2a - 4a^3) - \frac{\ell}{g}
\end{aligned}$$

But then a satisfies a rational polynomial of degree 6, contradicting the fact that its minimal polynomial has degree 10. This contradiction shows that ABC cannot have an α angle and a $\gamma + 2\alpha$ angle. But we have now checked all the possibilities for angle C if angle $A = \alpha$. Hence, angle A cannot be α .

Now assume, for proof by contradiction, that triangle ABC is isosceles, with two angles equal to 2α . We may suppose the two angles are angles A and B . Then angle $C = \gamma$, since $\gamma + 4\alpha = \pi$. But we have already shown that triangle ABC cannot have a γ angle. Hence triangle ABC cannot be isosceles with two 2α angles.

The smallest angle of triangle ABC cannot exceed 2α , since if it is 3α or more then the second smallest angle is at least 4α , and the largest angle would exceed 5α , making the total at least 12α , while it must be only $11\alpha = \pi$.

The smallest angle of triangle ABC must therefore be exactly 2α . Then the second smallest angle is at least $3\alpha = \beta$. If the next smallest angle is $\beta = 3\alpha$ then the largest angle is γ , which we have shown to be impossible. Hence the second smallest angle must be at least 4α . If it is 4α then the third angle is 5α . The second smallest cannot be 5α or more because then the sum of all three angles would exceed $\pi = 11\alpha$. Hence there is only one possible shape of ABC remaining when angle $A = 2\alpha$, angle $B = 4\alpha$, and angle $C = 5\alpha$. But that shape is ruled out by hypothesis. That completes the proof of the lemma.

6 The case $\alpha = \pi/14$ and $\beta = 4\alpha$

Here is a another special case that requires a separate treatment, which we provide by using a cyclotomic field.

Lemma 6 *Let ABC be tiled using as a tile the triangle T with $\alpha = \pi/14$, $\beta = 4\pi/14$, and $\gamma = 9\pi/14$. Suppose angle C is equal to γ or $\gamma + \alpha$. Then ABC is similar to T .*

Proof. Let $\zeta = e^{2\pi i/28}$. The degree of $\mathbb{Q}(\zeta)$ over \mathbb{Q} is $\varphi(28) = 12$. Since 28 is divisible by 4, i belongs to $\mathbb{Q}(\zeta)$. Let $\sigma = \sigma_{15}$ be the automorphism of $\mathbb{Q}(\zeta)$ that takes ζ to ζ^{15} . Then σ takes i to $-i$. Since $2\sin(j\alpha) = -i(\zeta^j - \zeta^{-j})$, σ takes $\sin(j\alpha)$ to $-\sin(15j\alpha)$, so σ fixes $a = \sin \alpha$ and also fixes c since

$$\begin{aligned}
c\sigma &= -\sin(15 \cdot 9\alpha) \\
&= -\sin 23\alpha
\end{aligned}$$

$$\begin{aligned}
&= \sin 5\alpha \\
&= \sin 9\alpha \\
&= \sin \gamma = c
\end{aligned}$$

On the other hand $b\sigma = -\sin 60\alpha = -\sin 4\alpha = -b$.

Let U and V be (the lengths of) two sides of ABC , and let θ be the angle between sides U and V . Then let the number of a , b , and c sides of tiles on side U be p , q , and r respectively, and let m , n , and ℓ be the number of a , b , and c sides on V . Then we have

$$\begin{aligned}
U &= pa + qb + rc \\
V &= ma + nb + \ell c \\
U\sigma &= pa - qb + rc \\
V\sigma &= ma - nb + \ell c
\end{aligned}$$

All the angles of ABC are multiples of α . For which J do we have $\sin J\alpha$ fixed by σ ? Exactly for J even, because

$$\begin{aligned}
(2i \sin J\alpha)\sigma &= (\zeta^J - \zeta^{-J})\sigma \\
&= (\zeta\sigma)^J - (\zeta\sigma)^{-J} \\
&= \zeta^{15J} - \zeta^{-15J} \\
&= (-\zeta)^J - (-\zeta)^{-J} \\
&= (-1)^J(\zeta^J - \zeta^{-J}) \\
&= (-1)^J 2i \sin J\alpha
\end{aligned}$$

But σ changes the sign of i , since $i\sigma = \zeta^7\sigma = \zeta^{7-15} = -i$. Hence the left side is equal to $-2i(\sin J\alpha)\sigma$, and we can divide by $-2i$, obtaining

$$(\sin J\alpha)\sigma = (-1)^{J+1} \sin J\alpha$$

So for J odd, σ fixes $\sin J\alpha$, and for J even, σ changes the sign of $\sin J\alpha$.

For some integer J , angle $\theta = J\alpha$. The area equation is

$$\begin{aligned}
2N\mathcal{A}_T &= 2\mathcal{A}_{ABC} \\
Nabc &= UV \sin \theta \\
&= (pa + qb + rc)(ma + nb + \ell c) \sin \theta
\end{aligned} \tag{8}$$

Applying σ we find

$$-Nabc = (-1)^{J+1}(pa - qb + rc)(ma - nb + \ell c)(-1)^{J+1} \sin \theta$$

since σ fixes a and c but changes the sign of b . Dividing the last two equations, and multiplying by -1 , we have

$$1 = (-1)^J \frac{(pa + qb + rc)(ma + nb + \ell c)}{(pa - qb + rc)(ma - nb + \ell c)}$$

The fraction on the right can be 1 only if the b terms do not appear, i.e. $q = n = 0$, and it can be -1 only if the a and c terms do not appear, i.e. $p = r = m = \ell = 0$. That is, it can be 1 only if U and V both have no b edges, and it can be -1 only if U and V are composed *only* of b edges. (If that conclusion is not obvious, it can be reached by multiplying by the denominator and simplifying, remember that all the letters denote nonnegative quantities.) Hence: if J is even, U and V have no b edges, and if J is odd, they have only b edges.

Consider the case in which J is odd, and U and V both are composed of only b sides of tiles. In that case the area equation becomes $Nabc = qnb^2 \sin \theta$. Canceling b we have

$$Nac = qnb \sin \theta.$$

Now the plan is to express a , b , and $\sin \theta$ in terms of ζ . The minimal polynomial $\psi(x)$ of ζ is the 28-th cyclotomic polynomial, so it is known. The degree of ψ is $\varphi(28) = 12$, where φ is the Euler totient function. We will get a polynomial $f(x)$ that is satisfied by ζ , so it must be zero mod ψ . But it will turn out not to be zero mod ψ . Here are the details. First, the minimal polynomial of ζ is

$$\psi(x) = x^{12} - x^{10} + x^8 - x^6 + x^4 - x^2 + 1 \quad (9)$$

as can be found using the `cyclotomic_polynomial` or the `monopoly` function of Sage, or by factoring $x^{28} - 1$ in Mathematica or Maple, or by using techniques found in any number theory book.

Next, we express a , b , and $\sin \theta$ in terms of ζ :

$$\begin{aligned} a &= \sin \frac{\pi}{14} = -\frac{i}{2}(\zeta - \zeta^{-1}) \\ b &= \sin \frac{4\pi}{14} = -\frac{i}{2}(\zeta^4 - \zeta^{-4}) \\ c &= \sin \frac{9\pi}{14} = -\frac{i}{2}(\zeta^9 - \zeta^{-9}) \\ \theta &= \sin \frac{J\pi}{14} = -\frac{i}{2}(\zeta^J - \zeta^{-J}) \\ 4Nac &= -N(\zeta - \zeta^{-1})(\zeta^9 - \zeta^{-9}) \\ 4qnb \sin \theta &= -qn(\zeta^4 - \zeta^{-4})(\zeta^J - \zeta^{-J}) \end{aligned}$$

Since $Nac = qnb \sin C$ we can equate the last two right hand sides:

$$\begin{aligned} N(\zeta - \zeta^{-1})(\zeta^9 - \zeta^{-9}) &= qn(\zeta^4 - \zeta^{-4})(\zeta^J - \zeta^{-J}) \\ N(\zeta^{10} - \zeta^8 - \zeta^{-8} + \zeta^{-10}) &= qn(\zeta^{J+4} - \zeta^{J-4} - \zeta^{4-J} + \zeta^{-J-4}) \end{aligned}$$

Now we want to multiply by a sufficient power of ζ to kill the negative exponents; that power has to be at least 10 and at least $J + 4$. Since $14\alpha = \pi$, if ABC has any angles that are odd multiples of α , then it has at least two such, and the smallest one is less than $\pi/2 = 7\alpha$, so we only need to consider $J = 1, 3$, and 5 among odd J . Hence ζ^{10} is a sufficient power to kill the negative exponents. Multiplying by ζ^{10} we have

$$N(\zeta^{20} - \zeta^{18} - \zeta^2 + 1) = qn(\zeta^{J+14} - \zeta^{J+6} - \zeta^{14-J} + \zeta^{6-J})$$

Since $\zeta^{14} = -1$ we can simplify this a bit:

$$N(-\zeta^6 + \zeta^4 - \zeta^2 + 1) = qn(-\zeta^J - \zeta^{J+6} - \zeta^{14-J} + \zeta^{6-J})$$

Define

$$f(x) := Nx^6 - Nx^4 - qnx^{J+6} - qnx^J - qnx^{14-J} + qnx^{6-J} + Nx^2 - N$$

Then f is a polynomial satisfied by ζ . Therefore $f(x)$ must be zero mod the cyclotomic polynomial ψ exhibited in (9), which has degree 12. The degree of f can only exceed 12 if $J+6 \geq 12$ or $14-J \geq 12$. We only have to consider $J = 1, 3$, or 5 , and when $J = 3$ or $J = 5$ we are done already, because the degree of f is less than 12 and the constant term $-N$ is not zero, so f is not the zero polynomial. That leaves only the case $N = 1$, in which we must compute f mod ψ . We did this computation in Sage as follows for $J = 1$:

```

sage: R.<x,N,q,n> = PolynomialRing(QQ,4)
sage: f = N*x^6 - N*x^4 - q*n* x^7 - q*n*x^13 + q*n*x^5 + N*x^2 - N
sage: psi = x^12 - x^10 + x^8 - x^6 + x^4 - x^2 + 1
sage: f.quo_rem(psi)
(-q*n*x, -q*n*x^11 + q*n*x^9 - 2*q*n*x^7 +
  2*q*n*x^5 + x^6*N - q*n*x^3 - x^4*N + q*n*x + x^2*N - N)

```

This tells us that $f(x) \bmod \psi(x)$ is

$$-qnx^{11} + qnx^9 - 2qnx^7 + Nx^6 + 2qnx^5 - Nx^4 - qnx^3 + Nx^2 + qnx - N$$

This is not the zero polynomial since its constant term N is not zero. Hence ABC has no angle that is an odd multiple of α .

Now suppose θ is an even multiple of α . Since ABC has no angles that are odd multiples of α , all its angles are even multiples of α , so the smallest one is less than or equal to 4α ; hence we only have to rule out angles $\theta = 2\alpha$ and 4α . If θ is an even multiple of α , then (as shown above) U and V have no b edges, and the area equation (8) becomes

$$Nabc = (pa + rc)(ma + lc) \sin \theta \quad (10)$$

First we take up the case $J = 4$. Then $\theta = 4\alpha = \beta$, so $\sin \theta = b$. Then we can divide the equation by b , obtaining

$$Nac = (pa + rc)(ma + lc)$$

The plan is to express everything in terms of ζ and then take the equation mod ψ . We have

$$\begin{aligned} N(\zeta - \zeta^{-1})(\zeta^9 - \zeta^{-9}) &= (p(\zeta - \zeta^{-1}) + r(\zeta^9 - \zeta^{-9}))(m(\zeta - \zeta^{-1}) + \ell(\zeta^9 - \zeta^{-9})) \\ N\zeta^{10} - N\zeta^8 - N\zeta^{-8} + N\zeta^{-10} &= \zeta^{18}r\ell + \zeta^{10}(rm + \ell p) + \zeta^2pm + \zeta^{-2}pm \\ &\quad + \zeta^{-10}(rm + \ell p) + \zeta^{-18}r\ell \end{aligned}$$

Since $\zeta^{14} = -1$ we have

$$\begin{aligned} N\zeta^{10} - N\zeta^8 - N\zeta^{-8} + N\zeta^{-10} &= \zeta^{10}(rm + \ell p) - \zeta^4r\ell + \zeta^2pm \\ &\quad + \zeta^{-2}pm - \zeta^{-4}r\ell + \zeta^{-10}(rm + \ell p) \end{aligned}$$

Bringing everything to one side of the equation we have

$$\begin{aligned} 0 &= \zeta^{10}(rm + \ell p - N) + N\zeta^8 - \zeta^4r\ell + \zeta^2pm \\ &\quad + \zeta^{-2}pm - \zeta^{-4}r\ell + N\zeta^{-8} + \zeta^{-10}(rm + \ell p - N) \end{aligned}$$

Multiplying by ζ^{10} we have

$$0 = \zeta^{20}(rm + \ell p - N) + N\zeta^{18} - \zeta^{14}r\ell + \zeta^{12}pm + \zeta^8pm - \zeta^6r\ell + N\zeta^2 + (rm + \ell p - N)$$

Again using $\zeta^{14} = -1$ we have

$$\begin{aligned} 0 &= -\zeta^6(rm + \ell p - N) - N\zeta^4 + r\ell + \zeta^{12}pm + \zeta^8pm - \zeta^6r\ell + N\zeta^2 + (rm + \ell p - N) \\ &= pm\zeta^{12} + pm\zeta^8 - \zeta^6(rm + \ell p - N + r\ell) - N\zeta^4 + N\zeta^2 + (rm + \ell p - N) \end{aligned}$$

This polynomial in ζ has the same degree as the minimal polynomial ψ , so it must be either identically zero, or a constant multiple of ψ . It is not a constant multiple, since it lacks a term in ζ^{10} , while ψ has such a term. It is not identically zero, since the coefficient of ζ^2 is $N \neq 0$. This contradiction shows that ABC has no angle $\theta = 4\alpha$.

Therefore ABC must have an angle $\theta = 2\alpha$. There is no longer a canceling factor in the area equation (10), so the algebra is more tedious. We reach for computer support:

```

sage: R.<x,N,p,r,m,l> = PolynomialRing(QQ,6)
sage: a = x - x^-1
sage: b = x^4-x^-4
sage: c = x^9 - x^-9
sage: f = N*a*b*c - (p*a+r*c)*(m*a + l *c)
sage: t = x^2 - x^-2
sage: f = x^20*(N*a*b*c - (p*a+r*c)*(m*a + l *c)*t)
sage: f
-x^40*r*1 + x^36*r*1 + x^34*N - x^32*r*m - x^32*p*1 - x^32*N
+ x^30*r*m + x^30*p*1 + x^28*r*m + x^28*p*1 - x^26*r*m - x^26*p*1
- x^26*N - x^24*p*m + x^24*N + 2*x^22*p*m + 2*x^22*r*1 - 2*x^18*p*m
- 2*x^18*r*1 + x^16*p*m - x^16*N + x^14*r*m + x^14*p*1 + x^14*N
- x^12*r*m - x^12*p*1 - x^10*r*m - x^10*p*1 + x^8*r*m + x^8*p*1
+ x^8*N - x^6*N - x^4*r*1 + r*1

```

That is,

$$\begin{aligned}
f(x) = & -rlx^{40} + rlx^{36} + x^{34}N - x^{32}(rm - pl - N) + x^{30}(rm + pl) + x^{28}(rm + pl) \\
& + x^{26}(rm - pl - N) + x^{24}(N - pm) + 2x^{22}(pm + rl) - 2x^{18}(pm + rl) + x^{16}(pm - N) \\
& + x^{14}(rm + pl + N) - x^{12}(rm + pl) - x^{10}(rm + pl) + x^8(rm + pl + N) \\
& - Nx^6 - rlx^4 + rl
\end{aligned}$$

Since $x^{14} = -1$ and $x^{28} = 1$ we have

$$\begin{aligned}
f(x) = & rlx^{12} + rlx^8 + x^6N - x^4(rm - pl - N) + x^2(rm + pl) + (rm + pl) \\
& - x^{12}(rm - pl - N) - x^{10}(N - pm) - 2x^8(pm + rl) + 2x^4(pm + rl) - x^2(pm - N) \\
& + (rm + pl - N) - x^{12}(rm + pl) - x^{10}(rm + pl) + x^8(rm + pl + N) \\
& - Nx^6 - rlx^4 + rl
\end{aligned}$$

Collecting like terms we have

$$\begin{aligned}
f(x) = & x^{12}(rl - 2rm + N) + x^{10}(pm - N - rm - pl) + x^8(-2pm - rl + rm + pl + N) \\
& x^4(-rm + pl + N + 2pm + rl) + x^2(rm + pl - pm + N) + (rm + pl - N + rl)
\end{aligned}$$

This polynomial must vanish at ζ , and it has degree at most 12, so either it is identically zero or it is a multiple of ψ . Since f has a zero coefficient of x^6 , it is not a multiple of ψ ; hence it must be identically zero. From the constant term we have $N = rm + pl + rl$; substituting that value for N into the coefficient of x^4 we have

$$\begin{aligned}
0 &= -rm + pl + N + 2pm + rl \\
&= -rm + pl + (rm + pl + rl) + 2pm + rl \\
&= 2pl + rl + 2pm + rl \\
&= 2\ell(p + r) + 2pm
\end{aligned}$$

Hence $pm = 0$ and $\ell(p + r) = 0$. Hence either $\ell = 0$ or $p = r = 0$. If $p = r = 0$ then the constant term is $-N \neq 0$, so we are finished. Hence we can assume $\ell = 0$. Then the constant term is $rm - N$, so $N = rm$. Hence $m \neq 0$; but since $pm = 0$ we have $p = 0$. Then the coefficient of x^2 is $rm + pl - \pm + N = rm + N$, but since $rm = N$ that is $2N$, which is not zero. That contradiction completes the proof of the lemma.

7 The case when $3\alpha + 2\beta = \pi$ or $2\alpha + 3\beta = \pi$

For this section only, we drop the assumption that $\alpha \leq \beta$, assuming only that $\alpha \leq \gamma$ and $\beta \leq \gamma$. We consider tilings of a triangle ABC that is not isosceles, by a tile with angles α , β , and γ , where $3\alpha + 2\beta = \pi$ and α is not a rational multiple of π . Note that we must have $\gamma = \beta + 2\alpha$, since $\gamma = \pi - \alpha - \beta = 3\alpha + 2\beta - \alpha - \beta = 2\alpha + \beta$. The possible shapes of ABC are quite limited:

Lemma 7 *Let ABC be N -tiled by a tile with angles α , β , and γ , and suppose $3\alpha + 2\beta = \pi$. Suppose ABC is not similar to the tile and not isosceles, and suppose that α is not a rational multiple of π . Then either ABC has angles 2α , β , and $\beta + \alpha$, or ABC has angles α , 2α , and 2β .*

Proof. Note that β is not a rational multiple of α , since then the relation $3\alpha + 2\beta = \pi$ would make α a rational multiple of π ; similarly β is not a rational multiple of π . Suppose first that ABC does not have an α angle. If one angle (say A) is 2α , then the tiling must split that angle into two α angles, since $\beta \neq 2\alpha$. Suppose, for proof by contradiction, that the tiling split one of the other two vertex angles of ABC into several α angles. It must be at least 3 of them since ABC is not isosceles. Hence at least five α angles are involved at the vertices of ABC . Hence the equation that says the sum of the vertex angles is π is an integral relation of the form $n\alpha + m\beta + r\gamma = \pi$, with $n \geq 5$. Since $\gamma = \beta + 2\alpha$ we have $(n + 2r)\alpha + (m + r)\beta = \pi$. Since $n + 2r \geq 5$ this relation is not a multiple of $3\alpha + 2\beta = \pi$, and hence the two equations can be solved for α and β , which will be rational multiples of π . This contradicts the hypothesis, and the contradiction shows that ABC does not have an angle that splits into α angles. As a consequence of this observation, ABC does not contain an angle larger than $\gamma - \alpha = \beta + \alpha$, since if it did, then the third angle would be less than β and so would have to split into α angles. Hence the other two angles of ABC must be at least β and at most $\beta + \alpha$. There is only one possibility, namely the first triangle mentioned in the lemma.

Now suppose that ABC does have an α angle. If the second smallest angle is 2α , then the remaining angle must be 2β ; that is the second possibility mentioned in the lemma. If the second smallest angle is greater than 2α , it must be at least β , since if not it will split into at least three α angles, giving rise to an integral relation that is not a multiple of $3\alpha + 2\beta = \pi$. It cannot be exactly β as that would make ABC similar to the tile. Therefore the second angle is more than β , and the third angle is therefore less than γ . If neither of these two angles splits into α angles, then each one contains a β angle plus one or more α angles. But since ABC is not isosceles, they cannot both be $\beta + \alpha$; hence at least four α angles are involved in the tiling at the vertices of ABC . That is contradictory, since $4\alpha + 2\beta > \pi$. Hence one of the second two angles does split into α angles. Since the second smallest angle is more than 2α , at least 3 α angles are required, and that gives rise to an integer relation that is not a multiple of $3\alpha + 2\beta$, contradiction. That completes the proof of the lemma.

We next introduce notation and definitions that will be used throughout this section, i.e., regardless of the shape of ABC . As always, a , b , and c are the sides of the tile, and we assume the tile and the triangle ABC are scaled so that

$$\begin{aligned} a &= \sin \alpha \\ b &= \sin \beta \\ c &= \sin \gamma \end{aligned}$$

For use in this entire section, we introduce the following notation:

$$\begin{aligned} z &= e^{i\pi\alpha/2} \\ \zeta &= e^{i\pi\beta} \end{aligned}$$

We note that

$$z^6 \zeta^2 = -1$$

since $3\alpha + 2\beta = \pi$. Hence $\zeta = iz^{-3}$. We have

$$\begin{aligned}
2ia &= z^2 - z^{-2} \\
2ib &= \zeta - \zeta^{-1} \\
&= i(z^3 + z^{-3}) \\
2b &= z^3 + z^{-3} \\
2 \sin 2\alpha &= -i(z^4 - z^{-4}) \\
2ic &= e^{i(2\alpha+\beta)} - e^{-i(2\alpha+\beta)} \\
&= z^4 \zeta - z^{-4} \zeta^{-1} \\
&= iz^4 z^{-3} + iz^{-4} z^3 \quad \text{since } \zeta = iz^{-3} \\
&= i(z + z^{-1}) \\
2c &= z + z^{-1}
\end{aligned}$$

In other words, $\sin \gamma = \cos(\alpha/2)$. We have

$$\begin{aligned}
\sin(\beta + \alpha) &= \sin(\pi - \gamma) \\
&= \sin \gamma \\
&= \frac{1}{2}(z + z^{-1})
\end{aligned}$$

We note the factorizations

$$\begin{aligned}
z^2 - z^{-2} &= (z + z^{-1})(z - z^{-1}) \\
z^3 + z^{-3} &= (z + z^{-1})(z^2 - 1 + z^{-2})
\end{aligned}$$

In view of these factorizations we have

$$\begin{aligned}
\frac{a}{c} &= -i(z - z^{-1}) \\
\frac{b}{c} &= z^2 - 1 + z^{-2}
\end{aligned}$$

We introduce

$$\begin{aligned}
t &:= 2 \cos \alpha \\
&= z^2 + z^{-2} \\
s &:= 2 \sin(\alpha/2) \\
&= -i(z - z^{-1})
\end{aligned}$$

Then $t = 2 - s^2$, and

$$\begin{aligned}
\frac{a}{c} &= i(z - z^{-1}) \\
&= s \\
\frac{b}{c} &= z^2 - 1 + z^{-2} \\
&= t - 1 \\
&= 1 - s^2 \\
\sin 2\alpha &= 2 \sin \alpha \cos \alpha \\
&= at \\
&= a(2 - s^2)
\end{aligned}$$

The \mathbf{d} matrix equation is

$$\mathbf{d} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

We introduce individual names for the components of the \mathbf{d} matrix to avoid so many subscripts:

$$\mathbf{d} = \begin{pmatrix} p & d & e \\ g & m & f \\ h & \ell & r \end{pmatrix}$$

Therefore we have for the three sides X , Y , and Z of triangle ABC

$$\begin{aligned} X &= pa + db + ec \\ Y &= ga + mb + fc \\ Z &= ha + \ell b + rc \end{aligned}$$

Generally our convention is that X , Y , and Z are in order of size, so X is opposite the smallest angle A , and Y is opposite the middle angle B , and Z is opposite the largest angle C . In this section, however, we are not assuming $\alpha < \beta$, and that means that we may not have $Y < Z$ either. In case triangle ABC has angles α , 2α , and 2β , we will assume that X is opposite the α angle at A , and the 2α angle is at B , opposite Y , and that Z is opposite the 2β angle. In case ABC has angles 2α , β , and $\beta + \alpha$, we will assume X is opposite the 2α angle at A , and Y is opposite the β angle at B , and Z is opposite the $\beta + \alpha$ angle at C . Thus in case $\beta < \alpha$, we may not have X, Y, Z in order of size.

Lemma 8 *Suppose triangle ABC is N -tiled by a tile in which $\gamma > \pi/2$ (as is the case when $3\alpha + 2\beta = \pi$). Suppose all the tiles along one side of ABC do not have their c sides along that side of ABC . Then there is a tile with a γ angle at one of the endpoints of that side of ABC .*

Proof. Let PQ be the side of ABC with no c sides of tiles along it. Then the γ angle of each of those tiles occurs at a vertex on PQ , since the angle opposite the side of the tile on PQ must be α or β . Let n be the number of tiles along PQ ; then there are $n - 1$ vertices of these tiles on the interior of PQ . Since $\gamma > \pi/2$, no one vertex has more than one γ angle. By the pigeonhole principle, there is at least one tile whose γ angle is not at one of those $n - 1$ interior vertices; that angle must be at P or Q . That completes the proof of the lemma.

Lemma 9 *Suppose $3\alpha + 2\beta = \pi$, and suppose triangle ABC has angles 2α , β , and $\beta + \alpha$. Let T be a triangle with angles α , β , and γ and suppose that ABC is N -tiled by T in such a way that just five tiles share vertices of ABC . Then the following restrictions on the elements of the \mathbf{d} matrix apply: We have $e \neq 0$, $f \neq 0$, and $r \neq 0$. In other words, the third element in each row is nonzero.*

Proof. None of the angles of ABC is large enough to accommodate the γ vertex of a tile, since those angles are 2α , β , and $\beta + \alpha < \beta + 2\alpha = \gamma$. The lemma then follows from Lemma 8, since the three numbers e , f , and r are the numbers of c sides of tiles on the three sides of ABC . That completes the proof of the lemma.

Lemma 10 *Let $3\alpha + 2\beta = \pi$, and let ABC have angles α , 2α , and 2β , and suppose there is an N -tiling of ABC by a tile with angles α , β , and γ such that exactly two tiles meet at the 2β vertex of ABC . Then we cannot have $m = \ell = 0$, i.e. the bottom two entries in the second column cannot both be zero. Also $e \neq 0$ and $f \neq 0$ and $r \neq 0$.*

Proof. At vertex A , where the angle is α , there is just one tile. It must have its b side on AB and its c side on AC , or vice-versa. The number of b sides on AC is m and the number of b sides on AB is ℓ , so they cannot both be zero.

In triangle ABC , no γ angles of tiles can occur at any vertex, since the α and 2α angles are too small and the 2β angle splits into two β angles. Hence the third column of the \mathbf{d} matrix cannot contain any zero entries, by Lemma 8. Hence e , f , and r are each nonzero. That completes the proof of the lemma.

7.1 The case when ABC has angles α , 2α , and 2β

Lemma 11 *Let $3\alpha + 2\beta = \pi$, and let ABC have angles α , 2α , and 2β , and let N be arbitrary. Then there is no N -tiling of ABC by a tile with angles α , β , and γ in which exactly two tiles meet at the 2β angle of ABC . (Here we do not assume $\alpha < \beta$.)*

Proof. Let X , Y , and Z be the sides of triangle ABC opposite A , B , and C respectively. Let λ be the ratio of the sides to the opposite angle, which (by the law of sines) is the same for all three sides:

$$\begin{aligned} X &= \lambda \sin \alpha \\ Y &= \lambda \sin 2\alpha \\ Z &= \lambda \sin 2\beta \end{aligned}$$

We note that

$$\begin{aligned} \sin \alpha &= a \\ \sin 2\alpha &= 2 \sin \alpha \cos \alpha \\ &= ta \\ &= (2 - s^2)a \\ \sin 2\beta &= \sin 3\alpha \\ &= 4 \sin^3 \alpha - 3 \sin \alpha \\ &= 4a^3 - 3a^2 \end{aligned}$$

The \mathbf{d} matrix equation is

$$\begin{pmatrix} p & d & e \\ g & m & f \\ h & \ell & r \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \lambda \begin{pmatrix} a \\ \sin 2\alpha \\ \sin 2\beta \end{pmatrix} \quad (11)$$

As noted above, we have $\sin 2\beta = 4a^3 - 3a^2$ and $\sin 2\alpha = a(2 - s^2)$ we have

$$\begin{pmatrix} p & d & e \\ g & m & f \\ h & \ell & r \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \lambda \begin{pmatrix} a \\ a(2 - s^2) \\ 4a^3 - 3a^2 \end{pmatrix}$$

Taking the ratio of the second row to the first row, we have

$$\frac{ga + mb + fc}{pa + db + ec} = 2 - s^2$$

Multiplying by the denominator, we have

$$ga + mb + fc = (2 - s^2)(pa + db + ec)$$

Dividing both sides by c and expressing a/c and b/c in terms of s we have

$$\begin{aligned} g\frac{a}{c} + m\frac{b}{c} + f &= (2-s^2)(p\frac{a}{c} + d\frac{b}{c} + e) \\ gs + m(1-s^2) + f &= (2-s^2)(ps + d(1-s^2) + e) \end{aligned}$$

This is a fourth-degree polynomial equation for s over \mathbb{Q} . Bringing the equation to polynomial form we have

$$0 = \psi(s) := ds^4 - ps^3 + (m - 3d - e)s^2 + (2p - g)s + (2d + 2e - f - m) \quad (12)$$

If ψ is identically zero, then from the first two coefficients of ψ in (12) we have $d = p = 0$. From the coefficient of s^2 we have $m - 3d - e = m - e = 0$, so $m = e$. From the coefficient of s we have $2p - g = 0$, and since $p = 0$ we have $g = 0$. From the constant coefficient we have $2d + 2e - f - m = 2e - f - m = e - f$, so $f = e$. The \mathbf{d} matrix then has the form

$$\begin{pmatrix} 0 & 0 & e \\ 0 & e & e \\ h & \ell & r \end{pmatrix}$$

This is not immediately contradictory, and we shall return to proving that ψ is not identically zero below.

From the first row of the \mathbf{d} matrix equation we have

$$pa + db + ec = \lambda a$$

Dividing by c and using $a/c = s$ and $b/c = 1 - s^2$ we have

$$\lambda s = ps + d(1 - s^2) + e$$

Solving for λ we have

$$\lambda = p - ds + \frac{e + d}{s} \quad (13)$$

The d matrix equation (11) can be written as an eigenvalue problem this way:

$$\begin{pmatrix} p & d & e \\ \frac{gb}{\sin 2\alpha} & \frac{mb}{\sin 2\alpha} & \frac{fb}{\sin 2\alpha} \\ \frac{hc}{\sin 3\alpha} & \frac{lc}{\sin 3\alpha} & \frac{rc}{\sin 3\alpha} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

To find an eigenvector by the cofactor method, we need to compute the cofactors of the matrix

$$\begin{pmatrix} p - \lambda & d & e \\ \frac{gb}{\sin 2\alpha} & \frac{mb}{\sin 2\alpha} - \lambda & \frac{fb}{\sin 2\alpha} \\ \frac{hc}{\sin 3\alpha} & \frac{lc}{\sin 3\alpha} & \frac{rc}{\sin 3\alpha} - \lambda \end{pmatrix}$$

Taking the cofactors of the third row, and multiplying by $\sin 2\alpha$, we find a candidate for an eigenvector (it is only a candidate until we prove that its components are all nonzero):

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} (df - em)b + e\lambda \sin 2\alpha \\ (eg - pf)b + f\lambda b \\ (pm - dg)b - (mb + p \sin 2\alpha)\lambda + \lambda^2 \sin 2\alpha \end{pmatrix} \quad (14)$$

Now assume, for proof by contradiction, that ψ is identically zero. As shown above, the \mathbf{d} matrix then has the form

$$\begin{pmatrix} 0 & 0 & e \\ 0 & e & e \\ h & \ell & r \end{pmatrix}$$

Then we have

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -be^2 + e\lambda \sin 2\alpha \\ eb\lambda \\ eb\lambda + \lambda^2 \sin 2\alpha \end{pmatrix}$$

Since $e \neq 0$ (otherwise the whole first row of the \mathbf{d} matrix is zero), the second two components are nonzero. Suppose, for proof by contradiction, that $u = 0$. Then since $e \neq 0$ we have

$$eb = \lambda \sin 2\alpha$$

Since $\lambda \sin 2\alpha = ga + mb + fc = eb + ec$, we have $eb = eb + ec$, and hence $ec = 0$. Since $e \neq 0$ we have $c = 0$, a contradiction. This contradiction shows that $u \neq 0$ (still under the assumption that ψ is identically zero). Having proved $u \neq 0$, we conclude that the eigenspace has dimension 1, and (u, v, w) is a multiple of (a, b, c) . Therefore $v/w = b/c$. Cross multiplying, we have $vc = bw$. Putting in $v = eb\lambda$ and $w = eb\lambda + \lambda^2 \sin 2\alpha$ we obtain

$$\begin{aligned} eb\lambda c &= eb^2\lambda + b\lambda^2 \sin 2\alpha \\ &= eb^2\lambda + b\lambda^2 at \quad \text{since } \sin 2\alpha = at \end{aligned}$$

Dividing by $bc\lambda$ we have

$$e = e\frac{b}{c} + \lambda t\frac{a}{c}$$

Since $p = d = 0$, the first row of the \mathbf{d} matrix equation is $ec = \lambda a$. Hence $\lambda = ec/a$. Putting this value in for λ we have

$$\begin{aligned} e &= e\frac{b}{c} + \frac{ec}{a}t\frac{a}{c} \\ &= e\frac{b}{c} + et \\ &= e(t-1) + et \quad \text{since } b/c = t-1 \end{aligned}$$

Since $e \neq 0$ we can divide by e :

$$\begin{aligned} 1 &= t-1+t \\ &= 2t-1 \end{aligned}$$

Solving for t we find $t = 1$. Since $t = 2\cos\alpha$ we have $\cos\alpha = 1/2$, so $\alpha = \pi/3$. Then $\beta = \pi/2 - 3\alpha/2 = 0$, contradiction. Note that this is a contradiction even if we do not assume $\alpha \leq \beta$. This contradiction shows that ψ is not identically zero.

We next want to prove that the components of (u, v, w) are not zero, without the assumption that ψ is identically zero. First assume, for proof by contradiction, that $v = 0$. Then

$$(eg - pf)b + f\lambda b = 0$$

Dividing both sides by b we have

$$eg - pf + f\lambda = 0$$

Putting in the value of λ from (13) we have

$$\begin{aligned} 0 &= eg - pf + f\left(p - ds + \frac{e+d}{s}\right) \\ &= (eg - pf)s + f(ps - ds^2 + e + d) \end{aligned}$$

Collecting terms and changing the sign, we have

$$0 = dfs^2 - egs - (e+d)f$$

Let $H(s) = dfs^2 - egs - (e+d)f$. Then $H(0) = (e+d)f \leq 0$, and $H(1) = df - eg - (e+d)f = -eg - ef \leq 0$. Since H is a quadratic polynomial with positive leading coefficient, it cannot have a zero between 0 and 1. But s is such a zero, contradiction. Hence $v \neq 0$.

Next assume, for proof by contradiction, that $u = 0$. Then

$$\begin{aligned} 0 &= (df - em)b + e\lambda \sin 2\alpha \\ &= (df - em)b + e\lambda at \quad \text{since } \sin 2\alpha = at \end{aligned}$$

Dividing both sides by c and using (13) we have

$$0 = (df - em)\frac{b}{c} + e\left(p - ds + \frac{e+d}{s}\right)t\frac{a}{c}$$

Using $b/c = 1 - s^2$ and $t = 2 - s^2$ and $a/c = s$, we have

$$\begin{aligned} 0 &= (df - em)(1 - s^2) + e\left(p - ds + \frac{e+d}{s}\right)s(2 - s^2) \\ &= (df - em)(1 - s^2) + e(ps - ds^2 + e + d)(2 - s^2) \\ &= eds^4 - eps^3 + s^2(em - df - 3ed - e^2) + 2eps + (df - em + 2e^2 + 2de) \end{aligned}$$

By Lemma 8, we have $e \neq 0$, since there cannot be a tile with a γ angle at B (where the 2α angle of ABC is), and there cannot be a tile with a γ angle at C , where the 2β angle of ABC is, since by hypothesis, there are two tiles each with a β angle at that vertex.

Define

$$H(s) := ds^4 - ps^3 + s^2\left(m - \frac{df}{e} - 3d - e\right) + 2ps + \left(\frac{df}{e} - m + 2e + 2d\right)$$

Since $e \neq 0$, we can divide the previous equation by e , obtaining $H(s) = 0$:

$$0 = ds^4 - ps^3 + s^2\left(m - \frac{df}{e} - 3d - e\right) + 2ps + \left(\frac{df}{e} - m + 2e + 2d\right) \quad (15)$$

The first two terms of H are equal to the first two terms of ψ , so $H - \psi$ is a quadratic in s :

$$\begin{aligned} 0 &= H(s) - \psi(s) \\ &= s^2\left(m - \frac{df}{e} - 3d - e - (m - 3d - e)\right) + s(2p - (2p - g)) \\ &\quad + \left(\frac{df}{e} - m + 2e + 2d\right) - (2d + 2e - f - m) \\ &= -s^2\left(\frac{df}{e}\right) + gs + \frac{df}{e} \end{aligned} \quad (16)$$

We have $f \neq 0$, by Lemma 10. Assume, for proof by contradiction, that $d \neq 0$. Then we can divide by $-df/e$, obtaining

$$0 = s^2 - \frac{eg}{df}s - 1$$

This quadratic function is negative when $s = 0$, and negative when $s = 1$, and its leading coefficient is positive. Therefore it has no zero between 0 and 1, contradiction. This contradiction shows that $d = 0$.

Since $d = 0$, equation (16) becomes $gs = 0$. Hence $g = 0$, since $s \neq 0$. The equation (12) ($\psi(s) = 0$) then becomes

$$0 = -ps^3 + (m - e)s^2 + 2ps + (2e - f - m)$$

On the other hand, the equation (15) ($H(s) = 0$) now becomes (with $d = g = 0$)

$$-ps^3 + (m - e)s^2 + 2ps + (2e - m)$$

Subtracting that from the previous equation we find $f = 0$; but we already proved $f \neq 0$ using Lemma 8. That contradiction depended on the assumption $u = 0$, and that completes the proof by contradiction that $u \neq 0$.

Now assume, for proof by contradiction, that $w = 0$. That is,

$$0 = (pm - dg)b - (mb + p \sin 2\alpha)\lambda + \lambda^2 \sin 2\alpha$$

To write this as a function of s , use $\sin 2\alpha = at = cs(2 - s^2)$ and $b = c(1 - s^2)$ and $\lambda = p - ds + (e + d)/s$, and then multiply by s/c . We find (with the aid of a computer algebra system) the following polynomial equation:

$$\begin{aligned} 0 = & -d^2s^7 + dps^5 + (4d^2 + 2de - dm)s^4 + (dg - 3dp - ep)s^3 \\ & (-5d^2 - 6de - e^2 + 2dm + em)s^2 \\ & + (-dg + 2dp + 2ep)s + (2d^2 + 4de + 2e^2 - dm - em) \end{aligned}$$

Computing the polynomial remainder of this on division by ψ , and changing the sign, we find

$$0 = G(s) := dfs^2 - egs - f(d + e)$$

Assume, for proof by contradiction, that $d = 0$. Then $0 = -egs - fe$. By Lemma 10, $e \neq 0$ and $f \neq 0$, so $fe > 0$; since $egs \geq 0$ this is a contradiction. Hence $d \neq 0$. Since $f \neq 0$ as just noted, the leading coefficient df is not zero. Thus G is a quadratic function such that $G(0) = -f(d + e) < 0$. We have $G(1) = df - eg - f(d + e) = -eg - ef < 0$. Since $G(0)$ and $G(1)$ are both negative, and G'' is positive, G has no zero between 0 and 1. This is a contradiction, since $G(s) = 0$ for $s = 2 \sin(\alpha/2)$, which is between 0 and 1 since $\alpha < \pi/3$. That completes the proof by contradiction that $w \neq 0$. Hence, the eigenspace is one-dimensional, and (u, v, w) is a multiple of the eigenvector (a, b, c) .

Returning to equation (14), now that we know (u, v, w) is a multiple of (a, b, c) , we have we have $a/c = u/w$. That is,

$$\frac{a}{c} = \frac{(df - em)g + e\lambda \sin 2\alpha}{(pm - dg)b - (mb + p \sin 2\alpha)\lambda + \lambda^2 \sin 2\alpha}$$

We have $\sin 2\alpha = at = a(2 - s^2)$. Putting that in, and dividing numerator and denominator on the right by c , and then using $b/c = 1 - s^2$ and $a/c = s$, we have

$$\begin{aligned} s &= \frac{(df - em)(1 - s^2) + es\lambda(2 - s^2)}{(pm - dg)(1 - s^2) - (m(1 - s^2) + ps(1 - s^2))\lambda + \lambda^2s(1 - s^2)} \\ &= \frac{(df - em)(1 - s^2) + es\lambda(2 - s^2)}{(1 - s^2)(pm - dg - (m + ps)\lambda) + \lambda^2s} \end{aligned}$$

Multiplying by the denominator we have

$$s(1 - s^2)(pm - dg - (m + ps)\lambda + \lambda^2s) = (df - em)(1 - s^2) + es\lambda(2 - s^2)$$

Putting in $\lambda = p - es + (e + d)/s$ and bringing this to polynomial form (using Mathematica, for example, or by hand if you wish) we have

$$0 = d^2 s^6 - dps^5 + (dm - de - 3d^2)s^4 + (2dp - dg)s^3 \\ + (3d^2 + de - df - 2dm)s^2 + (dg - dp + ep)s + (d^2 - e^2 - df - dm)$$

Now taking this polynomial mod ψ , using the *PolynomialRemainder* command of Mathematica (you won't want to do that polynomial division by hand) we find a simple quadratic equation:

$$0 = d^2 - e^2 - df - dm - (dg - dp + ep)s + (dm + de - d^2)s^2 \quad (17)$$

By Lemma 10, $e \neq 0$. Suppose, for proof by contradiction, that $d = 0$. Then (17) becomes $eps + e^2 = 0$. Since $e \neq 0$ then $ps + e = 0$. Since e, p , and s are all ≥ 0 and $e > 0$, this is a contradiction. Hence $d \neq 0$.

Since $s = -i(z + z^{-1}) = 2 \sin(\alpha/2)$, we have

$$\sin \frac{\alpha}{2} < \frac{1}{2} \quad \text{since } \alpha < \pi/3 \\ s < 1 \quad \text{since } s = 2 \sin(\alpha/2)$$

Then, an N -tiling of ABC as in the lemma gives rise to a solution $s = 2 \sin(\alpha/2)$ of the following two equations (which are (12) and (17)):

$$0 = \psi(s) = ds^4 - ps^3 + (m - 3d - e)s^2 + (2p - g)s + (2d + 2e - f - m) \\ 0 = \chi(s) := (dm + de - d^2)s^2 - (dg - dp + ep)s + d^2 - e^2 - df - dm$$

We will finish the proof by showing that these two equations have no simultaneous solution s between 0 and 1.

We compute a somewhat simpler quartic than ψ satisfied by s , as follows:

$$F(s) := d\psi - \chi = d^2 s^4 - dps^3 - 2d(d + e)s^2 + p(d + e)s + (d + e)^2$$

F has a positive constant coefficient (as well as positive leading coefficient) since $d > 0$. Assume, for proof by contradiction, that $p = 0$. Then

$$F(s) = d^2 s^4 - 2d(d + e)s^2 + (d + e)2 \\ = (ds - (d + e))^2$$

Hence the only zero of F is $s = (d + e)/d \geq 1$. But (if there is an N -tiling), F has a zero between 0 and 1, contradiction. Hence $p \neq 0$.

Now we compute the value of $F(1)$.

$$F(1) = d^2 - dp - 2d(d + e) + p(d + e) + (d + e)^2 \\ = d^2 - dp - 2d^2 - 2de + pd + pe + d^2 + 2de + e^2 \\ = pe + e^2 \\ F(1) > 0 \quad \text{since } p > 0 \text{ and } e > 0$$

We have $F(0) = (d + e)^2$. This is positive since $d > 0$. Now F is a quartic that is positive at 0 and positive at 1, and its derivative at 0 is $F'(0) = p(d + e)$, which is positive since $p \neq 0$ and $d \neq 0$. The derivative $F'(s)$ is a cubic; one of its zeroes is negative, since for large negative s we have $F'(s) < 0$. It therefore has at most two zeroes between 0 and 1. The only way that $F(s)$ can have a zero strictly between 0 and 1 is if F first increases (as s increases from 0), then reaches a maximum and decreases, crossing the s -axis (or possibly just touching the s -axis, if F has a double zero), then reaches a minimum, then increases (crossing the s -axis again, unless it

has a double zero) and reaches its positive value at $s = 1$. Then F' has two zeroes between 0 and 1, and in particular $F'(1)$ must be positive. We compute

$$F'(s) = 4d^2s^3 - 3dps^2 - 4d(d+e)s + p(d+e)$$

The product of its three roots is thus $-p(d+e)/4d^2$, which is negative. Because $F'(0)$ is positive, and $F'(s)$ is negative for large negative s , one of the three roots is negative. But (if there is an N -tiling), the other two are between 0 and 1, so the product of the three roots is negative.

The sum of the three roots of F' is $4d(d+e)/4d^2 = (d+e)/d$. Since one of these roots is negative and the other two are between 0 and 1, the sum is less than 2. Hence $(d+e)/d < 2$. Hence $e < d$. We proved above that $F'(1) \geq 0$. Therefore

$$\begin{aligned} 0 &\leq F'(1) \\ &= 4d^2 - 3dp - 4d(d+e) + p(d+e) \\ &= -2dp - 4de + pe \\ &= p(e-2d) - 4de \end{aligned}$$

But since $e < d$, and $e > 0$ and $d > 0$, the right side is negative, contradiction. That completes the proof of the lemma.

7.2 The case when ABC has angles 2α , β , and $\beta + \alpha$ and $\sin(\alpha/2)$ is irrational

Recall that $\sin(\beta + \alpha) = \sin \gamma = c$ in this case, as follows from $3\alpha + 2\beta = \pi$. The 2α angle is at vertex A , the β angle at vertex B , and the $\beta + \alpha$ angle at vertex C . We do not assume $\alpha < \beta$. The \mathbf{d} matrix equation is

$$\begin{pmatrix} p & d & e \\ g & m & f \\ h & \ell & r \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}.$$

We suppose the tile is scaled so that each side is equal to the sine of the opposite angle. Then

$$\begin{aligned} a &= \sin \alpha \\ b &= \sin \beta \\ c &= \sin(\alpha + \beta) = \sin \gamma \quad \text{since } \alpha + \beta = \pi - \gamma \end{aligned}$$

Let $\lambda = X/\sin(2\alpha)$. By the law of sines for triangle ABC we then have

$$\begin{aligned} X &= \lambda \sin 2\alpha \\ Y &= \lambda \sin \beta \\ Z &= \lambda \sin(\beta + \alpha) \end{aligned}$$

Define $t = (\sin 2\alpha)/\sin \alpha = 2 \cos \alpha$. Then $\lambda at = \lambda(\sin \alpha)t = \lambda \sin 2\alpha = X$. Hence the \mathbf{d} matrix equation can be written

$$\begin{pmatrix} p & d & e \\ g & m & f \\ h & \ell & r \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \lambda \begin{pmatrix} at \\ b \\ c \end{pmatrix} \tag{18}$$

Dividing the top row by t we write it as an eigenvalue equation:

$$\begin{pmatrix} \frac{p}{t} & \frac{d}{t} & \frac{e}{t} \\ g & m & f \\ h & \ell & r \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Our next goal is to prove that the eigenspace of the eigenvalue λ is one-dimensional. To do that, we use the method of computing a candidate eigenvector by the cofactor method, and showing that its three components are each nonzero.

The matrix whose cofactors we need is

$$\begin{pmatrix} \frac{p}{t} - \lambda & \frac{d}{t} & \frac{e}{t} \\ g & m - \lambda & f \\ h & \ell & r - \lambda \end{pmatrix}$$

We expand in cofactors of the elements on the first row. We find a candidate eigenvector (u, v, w) where

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} (m - \lambda)(r - \lambda) - f\ell \\ hf - g(r - \lambda) \\ g\ell - h(m - \lambda) \end{pmatrix}$$

Lemma 12 *Suppose that ABC is N -tiled by tile T with $3\alpha + 2\beta = \pi$ and $\sin(\alpha/2)$ irrational, and triangle ABC has angles 2α , β , and $\beta + \alpha$. Suppose also that not both g and h are zero. Then the eigenspace of λ is one-dimensional and (u, v, w) defined above is a multiple of (a, b, c) .*

Proof. Our first aim is to prove $v \neq 0$. Since $f \neq 0$, if $g = 0$ then also $h = 0$, which contradicts the hypothesis that not both g and h are zero, and similarly, if $h = 0$ then $g = 0$. Hence $g \neq 0$ and $h \neq 0$. Then $hf > 0$. But we have $grc \leq \lambda c$ from the third row of the \mathbf{d} matrix equation; hence $gr \leq \lambda$. Then

$$\begin{aligned} v &= hf - g(r - \lambda) \\ &= hf + g(\lambda - r) \\ &> 0 \quad \text{since not both } g = 0 \text{ and } h = 0 \end{aligned}$$

Next we will prove $w \neq 0$. From the second row of the \mathbf{d} matrix equation, we have $ga + mb \leq \lambda b$. Since $g > 0$, we have $m < \lambda$, not just $m \leq \lambda$. Then

$$\begin{aligned} w &= g\ell - h(m - \lambda) \\ &= g\ell + h(\lambda - m) \\ &> 0 \quad \text{since not both } g = 0 \text{ and } h = 0 \end{aligned}$$

Finally we will prove $u \neq 0$. We have from the second and third rows of the \mathbf{d} matrix equation

$$\begin{aligned} mb + fc &\leq \lambda b && \text{with equality only if } g = 0 \\ \ell b + rc &\leq \lambda c && \text{with equality only if } h = 0 \end{aligned}$$

Rearranging the terms of these equations we have

$$\begin{aligned} fc &\leq (\lambda - m)b \\ \ell b &\leq (\lambda - r)c \end{aligned}$$

Multiplying these two equations and dividing by bc we have

$$\ell f \leq (\lambda - m)(\lambda - r)$$

The left hand side is u . We have proved $u \geq 0$ with equality only if $g = 0$ and $h = 0$. But at present we have assumed that not both g and h are zero; so we have $u > 0$.

We have proved all three components of the candidate eigenvector are nonzero. It follows that the eigenspace is one-dimensional; hence (u, v, w) is a multiple of (a, b, c) . That completes the proof of the lemma.

Lemma 13 Suppose $3\alpha + 2\beta = \pi$, and $s = 2 \sin \alpha/2$. Then we have (whether or not s is rational)

$$\begin{aligned}\sin \gamma &= \cos \frac{\alpha}{2} \\ \frac{a}{c} &= s \\ \frac{b}{c} &= 1 - s^2\end{aligned}$$

Proof. Since $\gamma = \pi - (\alpha + \beta)$, we have

$$\begin{aligned}\sin \gamma &= \sin(\pi - (\alpha + \beta)) \\ &= \sin(\alpha + \beta) \\ &= \cos(\pi/2 - (\alpha + \beta)) \\ &= \cos \frac{\alpha}{2} \quad \text{since } \pi/2 - \beta = 3\alpha/2\end{aligned}$$

Then $c = \sin \gamma = \cos \alpha/2$, and $a = \sin \alpha = 2 \sin(\alpha/2) \cos(\alpha/2)$. Hence

$$\frac{a}{c} = 2 \sin \alpha/2.$$

Since $3\alpha + 2\beta = \pi$, we have

$$\begin{aligned}\sin \beta &= \sin(\pi/2 - 3\alpha/2) \\ &= \cos(3\alpha/2) \\ &= 4 \cos^3 \frac{\alpha}{2} - 3 \cos \frac{\alpha}{2}\end{aligned}$$

Hence

$$\begin{aligned}b/c &= 4 \cos^2(\alpha/2) - 3 \\ &= 4(1 - \sin^2 \alpha/2) - 3 \\ &= 1 - 4 \sin^2 \alpha/2\end{aligned}$$

Then we have

$$\begin{aligned}\frac{a}{c} &= s \\ \frac{b}{c} &= 1 - s^2\end{aligned}$$

That completes the proof of the lemma.

Lemma 14 Assume ABC has angles 2α , β , and $\beta + \alpha$, and $\sin(\alpha/2)$ is irrational. Then a , b , and c are linearly independent over \mathbb{Q} .

Proof. Suppose, for proof by contradiction, that a , b , and c are linearly dependent over \mathbb{Q} . Then there are (positive or negative) integers n , j , and k such that $na + jb + kc = 0$. Dividing by c and using $a/c = s$ and $b/c = 1 - s^2$ we find

$$0 = ns + j(1 - s^2) + k.$$

If $j = 0$ then $s = -k/n = 2 \sin(\alpha/2)$ so $\sin(\alpha/2)$ is rational, contradicting the hypothesis that it is irrational. Hence $j \neq 0$. Solving for $1 - s^2$ we have

$$1 - s^2 = -\frac{ns}{j} - \frac{k}{j} \tag{19}$$

There is an angle $\beta + \alpha$ at vertex C ; the adjacent sides are X and Y and the area equation can be written as

$$Nab = XY = (pa + db + ec)(ga + mb + fc)$$

Dividing by c^2 we have

$$N\frac{a}{c}\frac{b}{c} = (p\frac{a}{c} + d\frac{b}{c} + e)(g\frac{a}{c} + m\frac{b}{c} + f)$$

Putting everything in terms of s we have

$$\begin{aligned} Ns(1 - s^2) &= ps + d(1 - s^2) + e)(gs + m(1 - s^2) + f \\ Ns\left(-\frac{ns}{j} - \frac{k}{j}\right) &= ps + d\left(-\frac{ns}{j} - \frac{k}{j}\right) + e)(gs + m\left(-\frac{ns}{j} - \frac{k}{j}\right)f) \end{aligned}$$

This is quadratic in s . We can then use (19) to replace s^2 by a term linear in s . Hence s satisfies a linear equation over \mathbb{Q} . Hence s is rational. But $s = 2\sin(\alpha/2)$, so $\sin(\alpha/2)$ is rational. It is not necessary to write the equation out explicitly. That completes the proof of the linear independence of a , b , and c .

Lemma 15 *Suppose ABC has angles 2α , β , and $\beta + \alpha$ and $\sin(\alpha/2)$ is irrational. Then $s = 2\sin(\alpha/2)$ satisfies a cubic equation over \mathbb{Q} ; hence its degree over \mathbb{Q} is 3.*

Proof. Taking the ratio of the first row of (18) to the third, we have

$$\frac{pa + db + ec}{ha + lb + rc} = \frac{at}{c}$$

Multiplying by the denominator and then dividing both sides by c^2 we have

$$p\frac{a}{c} + d\frac{b}{c} + e = \frac{at}{c}\left(h\frac{a}{c} + \ell\frac{b}{c} + r\right)$$

We now express this equation in terms of s . We have already derived $a/c = s$ and $b/c = 1 - s^2$; we also have $t = 2 - s^2$, since t is by definition $2\cos\alpha$, so we have $t = 2(1 - 2\sin^2(\alpha/2)) = 2 - 4\sin^2(\alpha/2) = 2 - s^2$. Putting these values into the previous equation, we have

$$ps + d(1 - s^2) + e = s(2 - s^2)(hs + \ell(1 - s^2) + r)$$

This is a polynomial equation of degree 5. Bringing it to polynomial form we have

$$0 = F(s) := \ell s^5 - hs^4 - (3\ell + r)s^3 + (d + 2h)s^2 + (2\ell + 2r - p)s - (e + d)$$

If F were identically zero, then we would have $\ell = 0$ from the leading coefficient, and then $r = 0$ from the coefficient of s^3 , contradicting Lemma 9. Therefore F is not identically zero.

Taking the ratio of the second row to the third, we have

$$\begin{aligned} \frac{ga + mb + fc}{ha + lb + rc} &= \frac{b}{c} \\ (ga + mb + fc) &= (ha + lb + rc)\frac{b}{c} \\ \left(g\frac{a}{c} + m\frac{b}{c} + f\right) &= \left(h\frac{a}{c} + \ell\frac{b}{c} + r\right)\frac{b}{c} \\ (gs + m(1 - s^2) + f) &= (hs + \ell(1 - s^2) + r)(1 - s^2) \end{aligned}$$

Subtracting the left side from the right and bringing the equation to polynomial form, we have

$$0 = H(s) := \ell s^4 - hs^3 + (m - 2\ell - r)s^2 + (h - g)s + \ell + r - m - f$$

If H were identically zero, we would have $\ell = 0$ from the leading coefficient, and $h = 0$ from the coefficient of s^3 , then $r = m$ from the coefficient of s^2 , then $g = 0$ from the coefficient of s , then

$f = 0$ from the constant coefficient; but by Lemma 9, we do not have $g = f = 0$. Hence H is not identically zero. We compute $K = -(F \bmod H)$ (using the *PolynomialRemainder* command in Mathematica, for example), and we find

$$0 = K(s) = (\ell + m)s^3 - (d + g + h)s^2 - (f + m + \ell + r - p)s + (d + e) \quad (20)$$

If K were identically zero, then we would have $\ell = m = 0$ from the coefficient of s^3 , and $d = g = h = 0$ from the coefficient of s^2 , and then $e = 0$ from the constant term, but $e = 0$ contradicts Lemma 8. Hence K is not identically zero. Assume, for proof by contradiction, that the leading term of K is zero. Then $m = \ell = 0$ and $H(s) = -hs^3 - rs^2 + (h - g)s - (m + f) = 0$. That completes the proof of the lemma.

Lemma 16 *Under the hypotheses in the section title, 1 , $\sin(\alpha/2)$, and $\cos(\alpha)$ are linearly independent.*

Proof. We have $\cos \alpha = 1 - 2\sin^2(\alpha/2) = 1 - (1/2)s^2$, and $\sin(\alpha/2) = s/2$, so the claim is equivalent to the claim that 1 , s , and s^2 are linearly independent over \mathbb{Q} ; that is, that s does not have degree 2 over \mathbb{Q} . By Lemma 15, s satisfies a cubic equation; hence its degree over \mathbb{Q} is either 1 or 3, not 2.

Lemma 17 *Let $3\alpha + 2\beta = \pi$ and assume triangle ABC is N -tiled by a tile with angles α and β . Suppose ABC has angles 2α , β , and $\beta + \alpha$, and suppose $s = 2\sin(\alpha/2)$ is not rational. Then the \mathbf{d} matrix entries g and h are not both zero.*

Proof. Suppose, for proof by contradiction, that $g = h = 0$. The \mathbf{d} matrix equation is, by (18),

$$\begin{pmatrix} p & d & e \\ 0 & m & f \\ 0 & \ell & r \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \lambda \begin{pmatrix} at \\ b \\ c \end{pmatrix}$$

Writing this in eigenvalue form we have

$$\begin{pmatrix} \frac{p}{t} - \lambda & \frac{d}{t} & \frac{e}{t} \\ 0 & m - \lambda & f \\ 0 & \ell & r - \lambda \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

Hence the determinant of the matrix on the left is zero. Expanding it by cofactors we have

$$\left(\frac{p}{t} - \lambda\right)\left((m - \lambda)(r - \lambda) - f\ell\right) = 0$$

There are two cases: either $p = \lambda t$ or not. In case $p = \lambda t$ then the last row of the \mathbf{d} matrix equation is

$$\ell b + rc = \lambda c$$

Dividing by c and using $b/c = 1 - s^2$ and $\lambda = p/t$ we have

$$\begin{aligned} \ell(1 - s^2) + r &= \frac{p}{t} \\ &= \frac{p}{2 - s^2} \\ \ell(1 - s^2)(2 - s^2) &= p \end{aligned}$$

Thus s^2 satisfies a quadratic equation over \mathbb{Q} . Hence the degree of $\mathbb{Q}(s)$ over \mathbb{Q} is either 2 or 4, since s is not rational by hypothesis. But this contradicts Lemma 15, which says s satisfies a cubic equation and hence has degree 3 or 1. That completes the case $p = \lambda t$.

Now assume $p \neq \lambda t$. Then the second factor in the characteristic equation is zero:

$$(m - \lambda)(r - \lambda) - f\ell = 0$$

This is a quadratic equation for λ ; hence $\mathbb{Q}(\lambda)$ has degree 2 or 1 over \mathbb{Q} . But from the third row of the \mathbf{d} matrix equation we have

$$\lambda = \ell(1 - s^2) + r$$

which implies that the degree of $\mathbb{Q}(s)$ over $\mathbb{Q}(\lambda)$ is 1 or 2. Hence the degree of $\mathbb{Q}(s)$ over \mathbb{Q} is 1, 2, or 4. Degree 1 contradicts the hypothesis that s is irrational; degrees 2 or 4 contradict (20). That completes the proof of the lemma.

Lemma 18 *Let $3\alpha + 2\beta = \pi$ and assume triangle ABC has angles 2α and β and is N -tiled by a tile with angles α and β . Suppose $\sin(\alpha/2)$ is irrational. Then $\ell \neq 0$, $p \neq 0$, $h \neq 0$, $g\ell - hm \neq 0$, and we have the following equations between the elements of the \mathbf{d} matrix, where Δ is the determinant of the \mathbf{d} matrix:*

$$\begin{aligned} \frac{p}{g+h} &= \frac{mr - \ell f}{g\ell - hm} \\ \frac{p}{g+h} &= \frac{-p(m+r) + eh + dg - N}{g\ell - hm + hf - gr} \\ \frac{p}{g+h} &= \frac{\Delta + N(m+r)}{-Nh} \end{aligned}$$

and in the second equation, if the denominator on the right is zero, so is the numerator.

Proof. Since $t = N/\lambda^2$, t belongs to $\mathbb{Q}(\lambda)$. The third row of the \mathbf{d} matrix equation, $ha + \ell b + rc = \lambda c$, shows that λ belongs to $\mathbb{Q}(a, b, c) = \mathbb{Q}(t)$. Since t belongs to $\mathbb{Q}(\lambda)$ and λ belongs to $\mathbb{Q}(t)$, we have $\mathbb{Q}(\lambda) = \mathbb{Q}(t) = \mathbb{Q}(a, b, c)$. This field has degree at least 3 over \mathbb{Q} , by Lemma 14. We will soon see that the degree is exactly 3.

We are in a position to find three different cubic equations for λ . The first equation come from the characteristic equation for λ :

$$\begin{pmatrix} \frac{p}{t} - \lambda & \frac{d}{t} & \frac{e}{t} \\ g & m - \lambda & f \\ h & \ell & r - \lambda \end{pmatrix} = 0$$

Multiplying the top row by t , the determinant is still zero:

$$\begin{pmatrix} p - \lambda t & d & e \\ g & m - \lambda & f \\ h & \ell & r - \lambda \end{pmatrix} = 0$$

Since $\lambda^2 = N/t$, we have $\lambda t = N/\lambda$. Multiplying by λ we have

$$\lambda \begin{pmatrix} p - \frac{N}{\lambda} & d & e \\ g & m - \lambda & f \\ h & \ell & r - \lambda \end{pmatrix} = 0$$

Expanding and collecting like powers of λ we find the following equation, in which Δ is the determinant of the \mathbf{d} matrix:

$$p\lambda^3 + (-p(m+r) + eh + dg - N)\lambda^2 + (\Delta + N(m+r))\lambda - N(mr - \ell f) = 0 \quad (21)$$

This is a cubic polynomial equation for λ . Hence $\mathbb{Q}(\lambda)$ has degree 3 over \mathbb{Q} , and this is the minimal polynomial of λ . Since we have already proved that the eigenspace of λ is one-dimensional,

it is not the case that all the coefficients are zero. Since the degree of $\mathbb{Q}(\lambda)$ is 3, the highest and lowest coefficients are not zero: $p \neq 0$ and $mr \neq 0$.

Next we compute b/c in terms of λ :

$$\begin{aligned} \frac{b}{c} &= (z^2 - 1 + z^2) \\ &= t - 1 \quad \text{since } t = z^2 + z^{-2} \end{aligned}$$

Since $\lambda^2 = N/t$, we have $t = N/\lambda^2$, which gives us

$$\frac{b}{c} = \frac{N}{\lambda^2} - 1 \quad (22)$$

Since the eigenvector (u, v, w) is a multiple of (a, b, c) , we have

$$\begin{aligned} \frac{b}{c} &= \frac{v}{w} \\ \frac{N}{\lambda^2} - 1 &= \frac{hf - g(r - \lambda)}{gl - h(m - \lambda)} \end{aligned}$$

Cross multiplying we have

$$(N - \lambda^2)(gl - hm + h\lambda) = (hf - gr)\lambda^2 + g\lambda^3$$

Subtracting the left side from the right and expanding and collecting, we have

$$0 = (g + h)\lambda^3 + (gl - hm + hf - gr)\lambda^2 - Nh\lambda - N(gl - hm) \quad (23)$$

By Lemma 17, not both g and h are zero. Hence (since both are nonnegative) $g + h > 0$, and this is a non-trivial equation. Since $\mathbb{Q}(\lambda) = \mathbb{Q}(a, b, c)$ (from the third row of the \mathbf{d} matrix equation), and this field has degree 3, the constant term $N(gl - hm) \neq 0$ (or else the equation could be divided by λ to yield a quadratic equation for λ). Hence $gl - hm \neq 0$. Similarly, the constant term of (21), namely $N(mr - \ell f)$, is also not zero, so $mr - \ell f \neq 0$.

Now (23) and (21) are two cubic equations for λ . Therefore (23) is a multiple of (21). The ratio of the coefficients of λ^3 is $p/(g + h)$. Therefore the other nonzero coefficients are also in that ratio. From the constant coefficients (neither of which is zero) we have

$$\frac{p}{g + h} = \frac{mr - \ell f}{gl - hm} \quad (24)$$

It follows that

$$\ell \neq 0 \quad (25)$$

since otherwise the numerator and denominator on the right of (24) have opposite signs, but the left side is positive. From the ratio of the quadratic terms we have (unless the numerator and denominator are both zero)

$$\frac{p}{g + h} = \frac{-p(m + r) + eh + dg - N}{gl - hm + hf - gr} \quad (26)$$

From the linear terms we have (unless both numerator and denominator are zero)

$$\frac{p}{g + h} = \frac{\Delta + N(m + r)}{-Nh} \quad (27)$$

These are the three equations mentioned in the lemma. It remains to prove $h \neq 0$. Assume, for proof by contradiction, that $h = 0$. Then $g \neq 0$, since the constant term $g + h$ of (23) is not zero. Then from (24) we have $p\ell = mr$, or $m = p\ell/r$. When $h = 0$ we have

$$\begin{aligned} \Delta &= \begin{vmatrix} p & d & e \\ g & m & f \\ 0 & \ell & r \end{vmatrix} \\ &= pmr + egl - gdr - pfl \end{aligned}$$

Since both numerator and denominator on the right of (27) are zero we have

$$\begin{aligned} 0 &= \Delta + N(m+r) \\ &= pmr + egl - gdr - pfl + N(m+r) \end{aligned}$$

Substituting $m = p\ell/r$ and multiplying by r we have

$$0 = p^2\ell r + eglr - gdr^2 - pflr + N(p\ell + r^2)$$

Bringing the last term to the left side and changing signs we have

$$\begin{aligned} N(p\ell + r^2) &= pflr + gdr^2 - eglr - p^2\ell r \\ &\leq pflr + gdr^2 \end{aligned}$$

Note that $gd \leq \lambda^2$, since $ga \leq \lambda b$ and $db \leq \lambda a$, from the \mathbf{d} matrix. We also have $r \leq \lambda$ and $fc \leq \lambda b < \lambda c$, so $fr < \lambda^2$. Then

$$\begin{aligned} N(p\ell + r^2) &< p\ell\lambda^2 + r^2\lambda^2 \\ &= (p\ell + r^2)\lambda^2 \end{aligned}$$

Since $p\ell + r^2 \neq 0$ we have

$$N < \lambda^2$$

which is a contradiction, since $\lambda^2 = N/t$ and $t = 2 \cos \alpha > 1$ since $\alpha > \pi/5$. That contradiction completes the proof that $h \neq 0$. Hence equation (27) is valid, i.e. its denominator is not zero. That completes the proof of the lemma.²

Lemma 19 *Suppose $3\alpha + 2\beta = \pi$, and triangle ABC has angles 2α and β and is not isosceles. Suppose $\sin(\alpha/2)$ is irrational. Let N be arbitrary. Then there is no N -tiling of triangle ABC by a tile with angles α and β .*

Proof. Suppose, for proof by contradiction, that there is a triangle ABC and an N -tiling as in the lemma. Then the equations of the previous lemma hold. From (27) we have

$$\begin{aligned} N(m+r) &\leq -\Delta - Nh\frac{p}{g+h} \\ N(m+r) + Nh\frac{p}{g+h} &\leq -\Delta \\ N\left(m+r + \frac{hp}{g+h}\right) &\leq -\Delta \\ &\leq dgr + pfl + hme - pmr - dfh - egl \\ &\leq mhe + rdg + p(\ell f - mr) - dfh - egl \end{aligned}$$

From the third row of the \mathbf{d} matrix we have $ha < Z = \lambda c$. Equality cannot hold since $\ell \neq 0$ (and we have $ha + lb < Z$). From the first row we have $ec \leq \lambda at$. Multiplying these two inequalities (which is legal since all these quantities are positive) and dividing by ac we have $he \leq \lambda^2 t < N$. Similarly, we have $dg \leq \lambda^2 t = N$. Putting these results into the inequality above, we have

$$N(m+r) + N\frac{hp}{g+h} < N(m+r) + p(\ell f - mr) - dfh - egl$$

²After finding these equations in March 2010, it was still some time before I could show that they had no solutions; in the meantime, I wrote a short C program that checked there are no solutions for $N \leq 400$. That put an end to my attempts to find a tiling of this kind by placing paper triangles on a card table.

Note that the inequality has become strict. Subtracting $N(m+r)$ from both sides we have

$$N \frac{hp}{g+h} < p(\ell f - mr) - dfh - egl$$

We have $p \neq 0$ and $mr - \ell f < 0$, since otherwise the right side would be negative, which would be a contradiction. It follows from (24) that

$$g\ell - hm < 0 \tag{28}$$

From the second row of the \mathbf{d} matrix equation we have $fc \leq \lambda b$. From the third row we have $lb \leq \lambda c$. Hence $\ell f \leq \lambda^2$. Hence

$$N \frac{hp}{g+h} < p\lambda^2 - pmr - dfh - egl$$

Now add $Npg(g+h)$ to both sides of this inequality:

$$\begin{aligned} N \left(\frac{hp}{g+h} + \frac{gp}{g+h} \right) &< p\lambda^2 + N \frac{pg}{g+h} - pmr - dfh - egl \\ Np &< p\lambda^2 + N \frac{pg}{g+h} - pmr - dfh - egl \end{aligned}$$

Subtracting $p\lambda^2$ from both sides we have

$$p(N - \lambda^2) < N \frac{pg}{g+h} - pmr - dfh - egl$$

Dividing by p (which we proved above is not zero), we have

$$N - \lambda^2 < N \frac{g}{g+h} - mr - \frac{dfh}{p} - \frac{egl}{p} \tag{29}$$

The left side is positive, and only the first term on the right can be positive, and it is small when h is large. This observation motivates us to investigate the ratio h/g . From (28) we have $g\ell < mh$. Then $\ell/m < h/g$. From (24) and (28) we see that the numerator of (24) is negative, i.e. $mr - \ell f < 0$. Then $r < (\ell/m)f < (h/g)f$. Hence $r/f < h/g$. Now we have ℓ/m and r/f each less than h/g . From the bottom row of the \mathbf{d} matrix equation we then have

$$\begin{aligned} Z &= ha + lb + rc \\ &= ga \frac{h}{g} + m \frac{\ell}{m} b + f \frac{r}{f} c \\ &< \left(\frac{h}{g} \right) (ga + mb + fc) \\ &= \left(\frac{h}{g} \right) Y \\ \frac{Z}{Y} &< \frac{h}{g} \end{aligned}$$

By the law of sines we have $Z/Y = \sin(\gamma)/\sin(\beta) = c/b$. Hence

$$\frac{c}{b} < \frac{h}{g}$$

Applying (22) to write $c/b = \lambda^2/(N - \lambda^2)$, we have

$$\frac{h}{g} > \frac{\lambda^2}{N - \lambda^2} \tag{30}$$

From (29) we have

$$\begin{aligned} N - \lambda^2 &< N \frac{g}{g+h} \\ &= N \frac{1}{1+h/g} \end{aligned}$$

Now we replace h/g in the denominator on the right by the smaller quantity from (30). Then the right hand side increases.

$$\begin{aligned} N - \lambda^2 &< N \frac{1}{1 + \frac{\lambda^2}{N - \lambda^2}} \\ &= N \frac{N - \lambda^2}{N - \lambda^2 + \lambda^2} \\ &= N \frac{N - \lambda^2}{N} \\ &= N - \lambda^2 \end{aligned}$$

We have derived $N - \lambda^2 < N - \lambda^2$, which is a contradiction. That completes the proof of the lemma.

7.3 The case when ABC has angles 2α , β , and $\beta + \alpha$, and α is a rational multiple of π

Note that our work above has assumed that α is not a rational multiple of π only to limit the possible shapes of ABC . The following lemma shows that there are only two possible rational multiples of π that we might need to consider.

Lemma 20 *Let α be a rational multiple of π , and suppose $3\alpha + 2\beta = \pi$. Let triangle ABC have one angle β , one angle 2α , and one angle $\beta + \alpha$. Suppose that there is an N -tiling of ABC by the tile with angles α and β . Then $\alpha = 2\pi/7$ and $\beta = \pi/14$, or $\alpha = 2\pi/9$ and $\beta = \pi/6$. In either case, $\sin(\alpha/2)$ is irrational.*

Proof. Suppose $\alpha = 2m\pi/n$. Then the degree d of $\mathbb{Q}(e^{i\alpha})$ over \mathbb{Q} is finite and by Lemma 1, we have $d = \varphi(n)$. If both $\sin(\alpha)$ and $\cos(\alpha)$ are rational we have $d = 2$, so $\varphi(n) = 2$, so $n = 3$ or 4 . Remember we are not assuming $\alpha < \beta$; so the smaller one of α and β is less than $\pi/5$, and $\alpha < \pi/3$ and $\beta < \pi/2$. Hence α cannot be $2\pi/3$ or $2\pi/4$. Hence not both $\sin \alpha$ and $\cos \alpha$ are rational.

Assume first that $\sin(\alpha/2)$ is not rational. By (20), the degree of $\mathbb{Q}(s)$ over \mathbb{Q} is at most 3. By Lemma 14, under the assumption that $\sin(\alpha/2)$ is irrational, that degree is at least 3, and hence it is exactly 3. Since $s = 2\sin(\alpha/2)$, $\mathbb{Q}(s)$ is the real subfield of $\mathbb{Q}(e^{i\alpha/2})$, so the degree of that field is 6. Since $\alpha = 2m\pi/n$, we have $\alpha/2 = 2m\pi/(2n)$. Hence by Lemma 1, the degree of $\mathbb{Q}(e^{i\alpha/2})$ over \mathbb{Q} is $\varphi(2n)$. The possibilities are exactly these: $2n = 7, 9, 14$, or 18 . Since n is an integer, we cannot have $2n = 7$ or 9 ; therefore $2n = 14$ or 18 , i.e. $n = 7$ or $n = 9$. Hence $\alpha = 2m\pi/7$ or $2m\pi/9$. Since $\alpha < \pi/3$, we must have $m = 1$ and $\alpha = 2\pi/7$ or $\alpha = 2\pi/9$. In case $\alpha = 2\pi/7$, we have $\beta = (\pi - 3\alpha)/2 = \pi/14$, and in case $\alpha = 2\pi/9$ we have $\beta = \pi/6$. That completes the proof in case $\sin(\alpha/2)$ is irrational.

Suppose then that $\sin(\alpha/2)$ is rational. If $\cos(\alpha/2)$ is also rational, then the degree of $\mathbb{Q}(e^{i\alpha/2})$ over \mathbb{Q} is 2, so by Lemma 1, we have $2 = \varphi(2n)$, so $2n = 3$ or 4 ; so $n = 2$, so $\alpha = 2\pi/2 = \pi$, which is impossible. Hence $\cos(\alpha/2)$ has degree 2 over \mathbb{Q} , so $\mathbb{Q}(e^{i\alpha/2})$ has degree 4 over \mathbb{Q} . Hence we have $4 = \varphi(2n)$, so by Lemma 1, $2n = 5, 8, 10$, or 12 . Hence $n = 4, 5$, or 6 . Hence $\alpha = \pi/2, 2\pi/5$, or $\pi/3$. Since $\alpha < \pi/3$, none of these is possible. That completes the proof of the lemma.

8 The general case when T is not similar to ABC

We have exhibited a few tilings in which the tile T is not similar to ABC . In this section we shall prove that if there are such tilings, then the number of tiles and the shape of the tile and triangle correspond to the exhibited families. In previous sections we have dealt with the cases when T is isosceles (and isosceles includes equilateral), and when T is similar to ABC . Therefore we now assume that T is not isosceles and not similar to ABC , and ABC is N -tiled by T . These assumptions will be in force for this entire section.

Let P , Q , and R be the total number of α , β , and γ angles (respectively) occurring at the vertices of triangle ABC . These numbers control the “vertex splitting”, i.e., the way the vertices of ABC are divided into the angles of T . We have

$$P\alpha + Q\beta + R\gamma = \pi.$$

Together with $\alpha + \beta + \gamma = \pi$, that makes two equations for α , β , and γ . We seek another equation that would enable us to solve for the three angles. At each vertex V of the tiling, which is not a vertex of ABC , there meet several copies of the tile with vertices at V . If V is a non-strict or a boundary vertex, then the angle sum of the vertex angles is π , and if it is a strict interior vertex, the angle sum is 2π . We set $k = 1$ for a non-strict or boundary vertex, and $k = 2$ for a strict interior vertex. Then we have

$$p\alpha + q\beta + r\gamma = k\pi$$

where p , q , and r are the numbers of copies of T with their α , β , or γ angles at V .

The numbers P , Q , and R depend only on ABC and the tiling, but p , q , and r depend on the particular vertex V . Our plan is to analyze the possibilities for P , Q , and R , and given those possibilities, to analyze the possibilities for p , q , and r , in such a way as to eliminate all but a manageable (small finite) number of special cases for the tile T .

Lemma 21 $P + Q + R \geq 5$.

Proof. Suppose $P + Q + R < 5$. Then there must be two vertices of ABC that are not split, since each vertex contributes at least one to the sum $P + Q + R$, and split vertices contribute at least 2. Each vertex that does not split is equal to one of the angles of T . Since T is not isosceles, that means that T has two angles equal to angles of ABC . Hence T is similar to ABC , contrary to hypothesis. That completes the proof of the lemma.

Lemma 22 $R < 2$, i.e. $R = 0$ or $R = 1$.

Proof. If $\gamma = \pi/3$, then T is equilateral, contrary to hypothesis. Since $\gamma \geq \pi/3$ for any triangle, we have $\gamma > \pi/3$. Hence $R \leq 3$. But if $R = 3$, then each angle of ABC is at least γ . Since the sum of the angles of ABC is π , this is possible only if $\gamma = \pi/3$, which we have shown is false. Hence $R < 3$; but R is an integer, so $R \leq 2$. Hence, it suffices to show $R \neq 2$. Assume, for proof by contradiction, that $R = 2$. Then $P\alpha + Q\beta + 2\gamma = \pi$. Subtracting $\alpha + \beta + \gamma = \pi$, we find $(P-1)\alpha + (Q-1)\beta + \gamma = 0$. We must have $Q = 0$ or $P = 0$, since if not the left side is positive. If $P = 0$ then $\alpha = (Q-1)\beta + \gamma \geq \beta$, contradiction. Hence $Q = 0$. Then our equation simplifies to $\beta = (P-1)\alpha + \gamma \geq \gamma$, contradicting $\beta < \gamma$. That completes the proof of the lemma.

Lemma 23 If $R = 1$ then $Q = 0$ and $\beta = (P-1)\alpha$.

Proof. Assume $R = 1$. Then $P\alpha + Q\beta + \gamma = \pi$. Subtracting $\alpha + \beta + \gamma = \pi$, we find $(P-1)\alpha + (Q-1)\beta = 0$. Since there is vertex splitting and T is not similar to ABC , we have $P+Q \geq 4$ by Lemma 21. Hence we must have $Q = 0$ or $P = 0$. If $P = 0$ then $\alpha = (Q-1)\beta \geq \beta$, contradiction. Hence $Q = 0$ and $P \geq 4$, and $\beta = (P-1)\alpha$. That completes the proof of the lemma.

Theorem 1 Suppose that triangle ABC is N -tiled by triangle T , and T is not similar to ABC . Suppose also that if $\gamma = 2\pi/3$ then T is isosceles. Then one of the following holds:

(i) N has the form $3m^2$ or $6m^2$ for some integer m and ABC is equilateral; and the tile is (in case $N = 3m^2$) the tile used in the equilateral 3-tiling, that is, $\gamma = 2\pi/3$, $\alpha = \beta = \pi/6$, or (in case $N = 6m^2$) it is half of that tile.

(ii) N is twice a square, or six times a square, or twice a sum of two squares; $\gamma = \pi/2$; and ABC is isosceles, having angles α , α , and 2β , or angles β , β , and 2α , so that T is a right triangle similar to half of ABC .

(iii) $3\alpha + 2\beta = \pi$, triangle ABC has angles 2α , β , and $\beta + \alpha$, $s = \sin(\alpha/2)$ is rational, and $N \geq 28$. Here we do not assume $\alpha < \beta$.

Remark. Case (iii) is taken up in [2], where a new family of tilings is presented, and a necessary and sufficient condition on N is given for such tilings to exist.

Corollary 2 If N is not divisible by 2 or by 3 (or even if it is, and the quotient is not a square), there are no N -tilings in which T is not similar to ABC .

Proof. Consider the case $\gamma = \pi/2$ and $\alpha = \pi/4$, i.e. T is an isosceles right triangle. If two or more angles α occur at a vertex of ABC , then that vertex angle is at least $\pi/2$. Hence at most one vertex angle of ABC can be split. Hence two angles of ABC are equal to $\pi/2$ or $\pi/4$. We cannot have two angles equal to $\pi/2$; if two angles are equal to $\pi/4$ then ABC is similar to T , contradiction. Hence one angle of ABC is $\pi/2$ and one is $\pi/4$; but then triangle ABC is similar to T , contradiction. Hence the case $\gamma = \pi/2$ and $\alpha = \pi/4$ is ruled out.

Now consider the case $\gamma = \pi/2$ and $\alpha < \beta$; that is, the tile is a non-isosceles right triangle. By Theorem 5 of [1], triangle ABC must be isosceles and T is similar to half of ABC . By Theorem 5 of [1], N is twice a square, six times a square, or twice a sum of two squares, so conclusion (ii) of the theorem holds.

Therefore we may assume, for the rest of the proof, that $\gamma \neq \pi/2$. In Lemma 4 we have ruled out the case $\gamma = 2\pi/5$. Hence we can assume that γ is not equal to $2\pi/3$, $\pi/2$, or $2\pi/5$. Moreover, γ is not equal to $\pi/3$, since in that case T is equilateral, and in that case no vertex splitting can occur, so T must be similar to ABC .

We now continue with the proof of the theorem. The inequality $R < P + Q$ means that, at the vertices of ABC (taken together), there are more α and β angles than γ angles. Then there exists a vertex V at which there are more γ angles than α and β angles. Let n , m , and ℓ be the number of α angles, β angles, and γ angles at V , respectively. Then $\ell > n + m$. Let the angle sum at V be $k\pi$ (so $k = 1$ or $k = 2$). It is not the case that both n and m are zero, since then $k\pi = \ell\gamma$; since $\gamma \geq \pi/3$ the only possibilities are $\gamma = 2\pi/3$, $\gamma = \pi/2$, $\gamma = 2\pi/5$, and $\gamma = \pi/3$, all of which have been ruled out already.

If $\ell = 1$ then (since $\ell > n + m$) we have $n = m = 0$, contradiction. If $\ell = 2$ then $m + n = 0$ or $m + n = 1$; if $m + n = 0$ then $m = n = 0$, which we have ruled out above. If $\ell = 2$ and $m + n = 1$ then $2\gamma + \alpha = k\pi$ or $2\gamma + \beta = k\pi$. If $k = 1$ we can subtract $\alpha + \beta + \gamma = \pi$ to obtain $\gamma = \beta$ or $\gamma = \alpha$; but since ABC is not isosceles this is a contradiction. Hence $k = 2$ and we have $2\gamma + \alpha = 2\pi$ or $2\gamma + \beta = 2\pi$. Writing $2\pi = 2\alpha + 2\beta + 2\gamma$ and subtracting the left-hand side we have $0 = \alpha + 2\beta$ or $0 = \beta + 2\alpha$, so $\alpha = \beta = 0$, which is a contradiction. That rules out the case $\ell = 2$.

Since $\gamma > \pi/3$, we have $\ell \leq 5$. That leaves the possibilities 3, 4, and 5 for ℓ . We will rule out each of these in turn; each one requires a detailed argument.

We first assume $\ell = 3$. Since $\gamma > \pi/3$, we have $k = 2$, i.e. the angle sum at V is 2π ; and we cannot have $n = m = 0$, since then γ would be $\pi/3$. Since $m + n < \ell$, we have $m + n \leq 2$. If $n = 1$ and $m = 1$ then

$$2\pi = 3\gamma + \alpha + \beta$$

$$\begin{aligned}
&= 2\gamma + (\gamma + \alpha + \beta) \\
&= 2\gamma + \pi
\end{aligned}$$

Hence $\gamma = \pi/2$, contradiction. Hence $n = 1$ and $m = 1$ is impossible.

If $n = 2$ and $m = 0$, then

$$\begin{aligned}
2\pi &= 3\gamma + 2\alpha \\
&= \gamma + 2\gamma + 2\alpha \\
2(\gamma + \alpha + \beta) &= \gamma + 2\gamma + 2\alpha \\
2\beta &= \gamma
\end{aligned}$$

Assume $R = 1$. Then by Lemma 23, $\beta = (P - 1)\alpha$. Then since $\gamma = 2\beta$, we have

$$\begin{aligned}
\pi &= \alpha + \beta + \gamma \\
&= \alpha + 3\beta \\
&= \alpha + 3(P - 1)\alpha \\
&= \alpha(3P - 2)
\end{aligned}$$

We can then solve for all three angles:

$$\begin{aligned}
\alpha &= \frac{\pi}{3P - 2} \\
\beta &= \frac{P - 1}{3P - 2}\pi \\
\gamma &= \frac{2(P - 1)}{3P - 2}\pi
\end{aligned}$$

In that case the inequality $\gamma \leq \pi/2$ is equivalent to $2(P - 1)/(3P - 2) \leq 1/2$, which is equivalent to $P \leq 2$. Since $P \geq 4$ we have $\gamma > \pi/2$ in this case.

If $R = 0$ then we have

$$\begin{aligned}
\pi &= P\alpha + Q\beta \\
2\pi &= 3\gamma + 2\alpha \quad \text{since } \ell = 3, n = 2, \text{ and } m = 0 \\
\gamma &= 2\beta \quad \text{by subtracting } 2\gamma + 2\alpha \\
\pi &= \alpha + 3\beta \quad \text{since } 3\gamma + 2\alpha = 2\pi \text{ and } \gamma = 2\beta
\end{aligned}$$

Since $P + Q + R \geq 5$, and $R = 0$, we do not have $P = 1$ and $Q = 3$. If $3P - Q = 0$ and $(P, Q) \neq (1, 3)$, then the equations are contradictory. Hence $3P - Q \neq 0$, and the equations are uniquely solvable:

$$\begin{aligned}
\alpha &= \frac{3 - Q}{3P - Q}\pi \\
\beta &= \frac{P - 1}{3P - Q}\pi \\
\gamma &= \frac{2P - 2}{3P - Q}\pi
\end{aligned}$$

Since $\alpha < \beta$, if $3P - Q < 0$ we have $Q - 3 < 1 - P$; hence $P + Q < 4$, contradiction. Hence $3P - Q > 0$. We have $P - 1 > 0$, hence $3P > Q$; hence $3 - Q > 0$, hence $Q < 3$.

If $n = 1$ and $m = 0$ then

$$\begin{aligned}
2\pi &= \alpha + 3\gamma && \text{since } \ell = 3, n = 1, \text{ and } m = 0 \\
2(\alpha + \beta + \gamma) &= \alpha + 3\gamma \\
\gamma &= \alpha + 2\beta \\
\alpha + \beta + (\alpha + 2\beta) &= \pi && \text{since } \alpha + \beta + \gamma = \pi \\
2\alpha + 3\beta &= \pi
\end{aligned}$$

If $R = 1$, then by Lemma 23, $\beta = (P - 1)\alpha$, so $(2 + 3(P - 1))\alpha = \pi$, and we have

$$\begin{aligned}
\alpha &= \frac{\pi}{3P - 1} \\
\beta &= \frac{P - 1}{3P - 1}\pi \\
\gamma &= \frac{2P - 1}{3P - 1}\pi
\end{aligned}$$

If $R = 0$ then we have

$$\begin{aligned}
2\alpha + 3\beta &= \pi && \text{as shown above} \\
P\alpha + Q\beta &= \pi && \text{since } R = 0
\end{aligned} \tag{31}$$

If $P = 2$ and $Q = 3$ then case (iii) of the theorem holds. Therefore we may assume $(P, Q) \neq (2, 3)$. It follows that $2Q - 3P \neq 0$, since if $2Q - 3P = 0$ and $(P, Q) \neq (2, 3)$, subtracting the two equations (31) yields a contradiction. Hence the determinant of the system (31) is nonzero, and we have

$$\begin{aligned}
\alpha &= \frac{3 - Q}{3P - 2Q}\pi \\
\beta &= \frac{P - 2}{3P - 2Q}\pi \\
\gamma &= \frac{2P - Q - 1}{3P - 2Q}\pi
\end{aligned}$$

Since $\alpha < \beta$, if $3P - 2Q < 0$ then we have $Q - 3 < 2 - P$, or $P + Q < 5$, contradicting $P + Q + R \geq 5$. Hence $3P - 2Q > 0$. Then since $\alpha > 0$ we have $Q \leq 2$, and since $\beta > 0$ we have $P \geq 3$, and since $\alpha < \beta$ we have $3 - Q < P - 2$, or $P + Q > 5$.

If $n = 0$ and $m = 1$ then

$$\begin{aligned}
2\pi &= \beta + 3\gamma \\
2(\alpha + \beta + \gamma) &= \beta + 3\gamma \\
\gamma &= 2\alpha + \beta \\
\alpha + \beta + (2\alpha + \beta) &= \pi && \text{since } \alpha + \beta + \gamma = \pi \\
3\alpha + 2\beta &= \pi
\end{aligned}$$

If $R = 1$, then by Lemma 23, $\beta = (P - 1)\alpha$, so $(3 + 2(P - 1))\alpha = \pi$, and we have

$$\begin{aligned}
\alpha &= \frac{\pi}{2P + 1} \\
\beta &= \frac{P - 1}{2P + 1}\pi \\
\gamma &= \frac{P + 1}{2P + 1}\pi
\end{aligned}$$

Since $\alpha < \beta$ we have $2P + 1 < P - 1$, which is impossible. Hence this case does not occur. If $R = 0$ then we have

$$\begin{aligned} 3\alpha + 2\beta &= \pi \\ P\alpha + Q\beta &= \pi \end{aligned}$$

If $P = 3$ and $Q = 2$ then case (iii) of the theorem holds. Therefore we may assume $(P, Q) \neq (3, 2)$. Therefore $3Q - 2P \neq 0$ and

$$\begin{aligned} \alpha &= \frac{2 - Q}{2P - 3Q}\pi \\ \beta &= \frac{P - 3}{2P - 3Q}\pi \\ \gamma &= \frac{P - 2Q + 1}{2P - 3Q}\pi \end{aligned}$$

Since $\alpha < \beta$, if $2P - 3Q < 0$, we have $Q - 2 < 3 - P$, or $P + Q < 5$, contradiction. Hence $2P - 3Q \geq 0$. Then since $\alpha > 0$ we have $Q = 0$ or $Q = 1$.

If $m = 2$ and $n = 0$ we have

$$\begin{aligned} 2\pi &= 3\gamma + 2\beta \\ &= 3(\pi - \alpha - \beta) + 2\beta \\ 3\alpha + \beta &= \pi \end{aligned}$$

If $R = 1$ then $\beta = (P - 1)\alpha$ so $3\alpha + (P - 1)\alpha = \pi$, and we have

$$\begin{aligned} \alpha &= \frac{\pi}{P + 2} \\ \beta &= \frac{P - 1}{P + 2}\pi \\ \gamma &= \frac{2}{P + 2}\pi \end{aligned}$$

Since $\beta < \gamma$ we have $P - 1 < 2$, or $P < 3$; since $Q = 0$ this makes $P + Q + R < 4$, contradiction; so this case cannot occur.

If $R = 0$ then we have

$$\begin{aligned} 3\alpha + \beta &= \pi \\ P\alpha + Q\beta &= \pi \end{aligned}$$

We do not have $P = 3$ and $Q = 1$ since $P + Q \geq 5$. Hence

$$\begin{aligned} \alpha &= \frac{1 - Q}{P - 3Q}\pi \\ \beta &= \frac{P - 3}{P - 3Q}\pi \end{aligned}$$

Since $\alpha < \beta$, if $P - 3Q < 0$ we have $Q - 1 < 3 - P$, or $P + Q < 4$, contradiction. Hence $P - 3Q > 0$. Since $\alpha > 0$ we have $Q = 0$. Hence

$$\begin{aligned} \alpha &= \frac{\pi}{P} \\ \beta &= \frac{P - 3}{P}\pi \\ \gamma &= \frac{2}{P}\pi \end{aligned}$$

Since $\alpha < \beta$, we have $P > 4$. Since $\beta < \pi/2$, we have $P < 6$. Hence $P = 5$. That makes $\alpha = \pi/5$ and $\beta = \gamma = 2\pi/5$, a case which has already been ruled out. The following table summarizes the results obtained above for $\ell = 3$:

n	m	R	P	Q	α/π	β/π	γ/π	info
2	0	1		0	$\frac{1}{3P-2}$	$\frac{P-1}{3P-2}$	$\frac{2P-2}{3P-2}$	$\gamma = 2\beta$
2	0	0		$Q \leq 2$	$\frac{3-Q}{3P-Q}$	$\frac{P-1}{3P-Q}$	$\frac{2P-2}{3P-Q}$	$\gamma = 2\beta$
1	0	1		0	$\frac{1}{3P-1}$	$\frac{P-1}{3P-1}$	$\frac{2P-1}{3P-1}$	$\gamma = 2\beta + \alpha$
1	0	0	$P \geq 3$	$Q \leq 2$	$\frac{3-Q}{3P-2Q}$	$\frac{P-2}{3P-2Q}$	$\frac{2P-Q-1}{3P-2Q}$	$\gamma = 2\beta + \alpha$
0	1	0	$P \geq 6 - Q$	0 or 1	$\frac{2-Q}{2P-3Q}$	$\frac{P-3}{2P-3Q}$	$\frac{P-2Q+1}{2P-3Q}$	$\gamma = 2\alpha + \beta$

We have no further need of vertex V and the associated numbers n and m , so we reprint the table without the first two columns. The point is, that the vertex-splitting numbers P , Q , and R determine α , β , and γ uniquely.

R	P	Q	α/π	β/π	γ/π	info
1		0	$\frac{1}{3P-2}$	$\frac{P-1}{3P-2}$	$\frac{2P-2}{3P-2}$	$\gamma = 2\beta$
0		$Q \leq 2$	$\frac{3-Q}{3P-Q}$	$\frac{P-1}{3P-Q}$	$\frac{2P-2}{3P-Q}$	$\gamma = 2\beta$
1		0	$\frac{1}{3P-1}$	$\frac{P-1}{3P-1}$	$\frac{2P-1}{3P-1}$	$\gamma = 2\beta + \alpha$
0	$P \geq 3$	$Q \leq 2$	$\frac{3-Q}{3P-2Q}$	$\frac{P-2}{3P-2Q}$	$\frac{2P-Q-1}{3P-2Q}$	$\gamma = 2\beta + \alpha$
0	$P \geq 6 - Q$	0 or 1	$\frac{2-Q}{2P-3Q}$	$\frac{P-3}{2P-3Q}$	$\frac{P-2Q+1}{2P-3Q}$	$\gamma = 2\alpha + \beta$

Now that we see how P , Q , and R determine the angles, we will show that each of these determinations leads to a contradiction. Let the vertices of the tiling, other than A , B , and C , be V_1, V_2, \dots ; let the angle sum at V_i be $k_i\pi$, so that $k_i = 1$ for a non-strict or boundary vertex, and 2 for a strict interior vertex. At each vertex V_i , let n_i , m_i , and ℓ_i be the number of α , β , and γ angles at that vertex. Since there are N tiles altogether, each having one α , one β , and one γ angle, we have $P + \sum n_i = Q + \sum m_i = R + \sum \ell_i = N\pi$. Consider the quantity $q_i = 2\ell_i + m_i$. We will show below that, for the first three rows of the table, at each vertex V_i we have $3n_i \geq q_i$. Once that is proved, we finish the proof as follows: Adding over all vertices, we have

$$\sum 3n_i \geq \sum (2\ell_i + m_i)$$

The sum on the left is three times the total number of α angles at the vertices V_i . This must be equal to $3(N - P)$. The sum on the right is the total number of β angles plus twice the number of γ angles. This is equal to $N - Q + 2(N - R) = 3N - Q - 2R$. Hence

$$\begin{aligned} 3(N - P) &\geq 3N - Q - 2R \\ 3N - Q - 2R &\leq 3(N - P) \\ -Q - 2R &\leq -3P \\ 3P &\leq Q + 2R \end{aligned}$$

But the table above shows that in every line of the table, we have $Q + 2R \leq 2$. Hence $3P \leq 2$. Since P is a positive integer, this is a contradiction.

It remains to supply the proof that $3n_i \geq q_i$, for the cases given in the first three rows of the table.

We first claim that in case $k_i = 1$ (i.e. the angle sum at vertex V_i is π), we can assume $\ell_i \leq 1$, i.e. if $\ell_i > 1$, the conclusion $3n_i \geq q_i$ holds. We certainly have $\ell_i \leq 2$, since $3\gamma > \pi$. Now assume that $\ell_i = 2$. Then $m_i = 0$, since $2\gamma + \beta > \pi$. Hence $q_i = 4$. Then $3n_i \geq q_i$ unless $n_i = 0$ or $n_i = 1$. If $n_i = 0$, then the equation $n_i\alpha + m_i\beta + \ell_i\gamma = \pi$ becomes $2\gamma = \pi$, so $\gamma = \pi/2$; but we have proved that (we can assume that) γ is not a right angle, so $n_i \neq 0$. Hence we may

assume $n_i = 1$. Then, we have $2\gamma + \alpha = \pi$. But this, together with $\alpha + \beta + \gamma = \pi$, implies $\beta = \gamma$, which contradicts the assumption that T is not isosceles. This contradiction shows that $n_i \neq 1$, and hence disposes of the case $k_i = 1$ and $\ell_i = 2$. That completes that proof of the claim that we can assume $\ell_i \leq 1$ when $k_i = 1$.

For the first two rows of the table, we have $\gamma = 2\beta$. In that case $\pi = \alpha + \beta + \gamma = \alpha + 3\beta$, so $4\beta > \pi$. Then $2\gamma + \beta = 5\beta > \pi$ as well.

If $k_i = 1$ and $\ell_i = 0$ then $q_i = m_i \leq 3$, since $4\beta > \pi$. If $k_i = 1$ and $\ell_i = 1$ then $q_i = 2 + m_i$, and since $\gamma = 2\beta$ and $4\beta > \pi$, we have $\gamma + 2\beta > \pi$, so $m_i \leq 1$. Hence $q_i \leq 3$. Since we showed above that we can assume $\ell_i \leq 1$ when $k_i = 1$, we have proved that (for the first two rows of the table) $q_i \leq 3$ when $k_i = 1$ (or else the conclusion of the lemma holds).

Now we assume, for proof by contradiction, that $k_i = 2$ and the first row of the table applies. Then

$$\begin{aligned} q_i\beta + n_i\alpha &= (2\ell_i + m_i)\beta + n_i\alpha \\ &= \ell_i\gamma + m_i\beta + n_i\alpha \quad \text{since } 2\beta = \gamma \\ q_i\beta + n_i\alpha &= 2\pi \end{aligned} \tag{32}$$

We wish to bound q_i . First we take up the first case (first row of the table). Then in addition to $\gamma = 2\beta$ we have $\beta = (P - 1)\alpha$. We have

$$\begin{aligned} 6\beta + 2\alpha &= 2\pi \quad \text{since } 6\beta = 2\beta + 2\gamma \\ q_i\beta + n_i\alpha &= 2\pi \quad \text{by (32)} \end{aligned}$$

Subtracting these two equations we have $(6 - q_i)\beta = (n_i - 2)\alpha$. If $q_i > 6$ then $n_i < 2$. Then $n_i = 0$ or $n_i = 1$, and $\beta = ((2 - n_i)/(q_i - 6))\alpha$. But $\beta = (P - 1)\alpha$, so $(2 - n_i)/(q_i - 6) = P - 1$, an integer at least 4. This is impossible since the numerator $2 - n_i$ is at most 2. This shows that in the first row of the table, we cannot have $k_i = 2$.

Now we give a bound for q_i in case $k_i = 2$ and the second row of the table applies, so $\gamma = 2\beta$ and $\beta = ((P - 1)/(3 - Q))\alpha$. Assume $q_i > 6$. Then as in the previous paragraph, we have $n_i = 0$ or $n_i = 1$ and $\beta = ((2 - n_i)/(q_i - 6))\alpha$. But now also $\beta = ((P - 1)/(3 - Q))\alpha$. Hence

$$\begin{aligned} \frac{2 - n_i}{q_i - 6} &= \frac{P - 1}{3 - Q} \\ \frac{(2 - n_i)(3 - Q)}{q_i - 6} &= P - 1 \end{aligned}$$

On the left side, the factor $2 - n_i$ is either 1 or 2, and the factor $3 - Q$ is either 1, 2, or 3, since $Q \leq 2$. Since $R = 0$ in the second row of the table, we have $P + Q \geq 5$. If $Q = 2$ then we have $(2 - n_i)/(q_i - 6) = P - 1$ and $P - 1 \geq 2$; but then $(2 - n_i)/(q_i - 6)$ is an integer at least 2, with numerator at most 2. This is possible only if $q_i = 7$ and $n_i = 0$ and $(P, Q) = (3, 2)$. But then case (iii) of the theorem holds. Therefore the case $q_i > 6$ has led to a contradiction or to conclusion (iii) of the theorem. Therefore we may assume $q_i \leq 6$ in case $k_i = 2$ and the second row of the table applies.

Now we will prove $q_i \leq 3n_i$ for the first two rows of the table.

Case (i) $k_i = 1$. We have

$$q_i = 2\ell_i + m_i \quad \text{definition of } q_i$$

Multiplying by β and adding $n_i\alpha$ we have

$$\begin{aligned} q_i\beta + n_i\alpha &= 2\ell_i\beta + m_i\beta + n_i\alpha \\ &= \ell_i\gamma + m_i\beta + n_i\alpha \quad \text{since } \gamma = 2\beta \\ &= \pi \end{aligned}$$

Now if $n_i = 0$ then $q_i\beta = \pi$. As shown above, we can assume $q_i \leq 3$. Then, since $q_i\beta = \pi$, we have $\beta \geq \pi/3$. Hence $\gamma = 2\beta \geq 2\pi/3$ and $\beta + \gamma \geq \pi$, contradiction. Hence $n_i \neq 0$. Therefore $n_i \geq 1$. Since $q_i \leq 3$, we have $q_i \leq 3n_i$ as desired.

Case (ii) $k_i = 2$. We showed above that this is impossible for the first row of the table, so the second row applies. We have $3\beta + \alpha = \pi$ and $q_i\beta + n_i\alpha = 2\beta + n_i\alpha = \pi$. Therefore

$$2\beta + n_i\alpha = 3\beta + \alpha$$

From the second row of the table, we have $\beta = ((P-1)/(3-Q))\alpha$. Hence we obtain

$$\begin{aligned} 2\beta + n_i\alpha &= 2\beta + \left(1 + \frac{P-1}{3-Q}\right)\alpha \\ n_i &= \left(1 + \frac{P-1}{3-Q}\right) \end{aligned}$$

Since $Q \leq 2$ the denominator is positive and at most 2; since $P+Q \geq 5$, we have $P \geq 3$ so the numerator is at least 2. It follows that $n_i \geq 2$. Since we proved above that $q_i \leq 6$ in case $k_i = 2$ and the second row of the table applies, we have $3n_i \geq q_i$ as desired. That completes the proof for the first two rows of the table.

Next we take up the third row of the table. In this case we have $\gamma = 2\beta + \alpha$ instead of $\gamma = 2\beta$. Again we consider $q_i = 2\ell_i + m_i$ and try to prove $3n \geq q$. We start by observing that $\pi = \alpha + \beta + \gamma = 2\alpha + 3\beta$. Hence $2\gamma = 4\beta + 2\alpha > \pi$, so $\gamma > \pi/2$. Therefore, if $k_i = 1$, we have $\ell_i \leq 1$, and if $k_i = 2$, we have $\ell_i \leq 3$.

We argue by (more) cases.

Case (iii). $k_i = 1$ and $\ell_i = 1$. Then we have $m_i \leq 1$, since $\gamma + 2\beta = 2\beta + \alpha + 2\beta = 4\beta + \alpha > 3\beta + 2\alpha = \pi$. Then

$$\begin{aligned} \pi &= \gamma + m_i\beta + n_i\alpha \\ 3\beta + 2\alpha &= \gamma + m_i\beta + n_i\alpha \\ 3\beta + 2\alpha &= (2 + m_i)\beta + (n_i + 1)\alpha \quad \text{since } \gamma = 2\beta + \alpha \\ (1 - m_i)\beta &= (n_i - 1)\alpha \\ (1 - m_i)(P-1)\alpha &= (n_i - 1)\alpha \\ n_i - 1 &= (1 - m_i)(P-1) \end{aligned}$$

If $m_i = 1$, so the right side is zero, then $n_i = 1$ also, and we have $n_i = m_i = \ell_i = 1$. In that case $3n_i \geq q_i$ and we are finished. Hence (since $m_i \leq 1$) we may assume $m_i = 0$. Then we have $n_i = P$. Then $q = 2\ell + m_i = 2$, and $3n_i = 3P > q_i$ since $P \geq 2$, since $Q \leq 2$ and $R \leq 1$ and $P+Q+R \geq 5$. That disposes of Case (iii).

Case (iv) $k_i = 1$ and $\ell_i = 0$ and $m_i > 4$. This case can be ruled out, because $5\beta > 3\beta + 2\alpha = \pi$.

Case (v) $k_i = 1$ and $\ell_i = 0$ and $m_i = 4$. Then $\pi = 4\beta + n_i\alpha > 3\beta + (n_i + 1)\alpha$. Since $3\beta + 2\alpha = \pi$, this implies $n_i = 0$ and $\beta = \pi/4$. Then $\gamma = 2\beta + \alpha = \pi/2 + \alpha$. Hence

$$\begin{aligned} \pi &= \alpha + \beta + \gamma \\ &= \alpha + \pi/4 + \pi/2 + \alpha \\ \alpha &= \pi/8 \\ \gamma &= 5\pi/8 \end{aligned}$$

For this special case of T , we cannot establish the desired bound. But we can rule out this case by considering the possibilities for P and Q , as follows. We have assumed that row 3 of the table applies. Then $R = 1$, so $P+Q \geq 4$, and $\alpha = \pi/(3P-1)$, and since $\alpha = \pi/8$, we have $P = 3$,

and since $P + Q \geq 4$, we have $Q \geq 1$. But in row 3 of the table, we have $Q = 0$, contradiction. That disposes of this case.

Case (vi) $k_i = 1$ and $\ell_i = 0$ and $m_i \leq 3$. Then $\pi = m_i\beta + n_i\alpha$; and since $\pi = 3\beta + 2\alpha$, we have $n_i \geq 2$, and so $3n_i \geq 6$, and $q_i = 2\ell_i + m_i = m_i \leq 3$, so $3n_i > q_i$ as required.

This disposes of all cases where $k_i = 1$ and the third row of the table applies. Now suppose $k_i = 2$ and the third row of the table applies, so we have $\gamma = 2\beta + \alpha$ and $\pi = 2\alpha + 3\beta$. Then we have

$$\begin{aligned} 2\pi &= \ell_i\gamma + m_i\beta + n_i\alpha \\ &= \ell_i(2\beta + \alpha) + m_i\beta + n_i\alpha \\ &= q_i\beta + (\ell_i + n_i)\alpha \end{aligned}$$

Subtracting from this $2\pi = 4\alpha + 6\beta$ (which is twice $\pi = 2\alpha + 3\beta$), we obtain

$$0 = (q_i - 6)\beta + (\ell_i + n_i - 4)\alpha \quad (33)$$

Case (vii) The third row of the table applies, and $k_i = 2$, and $q_i > 6$. Then we have $\beta = (P - 1)\alpha$, so we find

$$\begin{aligned} 0 &= (q_i - 6)(P - 1)\alpha + (\ell_i + n_i - 4)\alpha && \text{by (33)} \\ 4 - \ell_i - n_i &= (P - 1)(q_i - 6) \end{aligned}$$

In the third row of the table, we have $Q = 0$, and since $P + Q + R \geq 5$ and $R = 1$, we have $P \geq 4$, so $P - 1 \geq 3$.

Since $q_i > 6$, the factor $q_i - 6$ is positive. The left side, on the other hand, is an integer at most 4. Hence, $q_i - 6$ must be 1, so $q_i = 7$, and both sides are equal either to 3 or to 4. Hence, looking at the left side $4 - \ell_i - n_i$ is 3 or 4, so $\ell_i + n_i$ is 0 or 1. In case it is zero, we have $\ell_i = n_i = 0$ and $P = 5$. Since $q_i = 2\ell_i + m_i = 7$, we have $m_i = 7$, so

$$\begin{aligned} \beta &= 2\pi/7 \\ \alpha &= \frac{\beta}{P-1} \\ &= \beta/4 \\ &= \pi/14 \\ \gamma &= 2\beta + \alpha \\ &= 9\pi/14 \end{aligned}$$

This case is impossible, by Lemma 6.

Now suppose $\ell_i = 1$ and $n_i = 0$. Then $q_i = 7$ implies $m_i = 1$, so the equation $\ell_i\gamma + m_i\beta + n_i\alpha = 2\pi$ becomes $\gamma + \beta = 2\pi$; but since $\gamma + \beta < \pi$, that is a contradiction. Suppose $\ell_i = 0$ and $n_i = 1$. Then $q_i = 7$ implies $m_i = 7$, so the equation $\ell_i\gamma + m_i\beta + n_i\alpha = 2\pi$ becomes $7\beta + \alpha = 2\pi$. In this case we have $P = 4$, so $\beta = (P - 1)\alpha = 3\alpha$. Hence

$$\begin{aligned} 2\pi &= 7\beta + \alpha \\ &= 21\alpha + \alpha \\ &= 22\alpha \\ \alpha &= \frac{\pi}{11} \\ \beta &= \frac{3\pi}{11} \\ \gamma &= \frac{7\pi}{11} \end{aligned}$$

Since $R = 1$, the largest angle of ABC must be at least γ , but according to Lemma 5, that is not possible.

Case (viii) The third row of the table applies, and $k_i = 2$, and $q_i = 6$. Then by (33) we have

$$\begin{aligned} 0 &= (\ell_i + n_i - 4)\alpha \\ \ell_i + n_i &= 4 \end{aligned}$$

Since $q_i = 2\ell_i + m_i$, we have $\ell_i = (q_i - m_i)/2$. Hence

$$\begin{aligned} q_i - m_i + 2n_i &= 8 \\ q_i &\leq 2n_i + m_i \\ &\leq 3n_i \end{aligned}$$

which disposes of this case.

Case (ix) The third row of the table applies, and $k_i = 2$, and $q_i < 6$. By (33) we have

$$4 - \ell_i - n_i = (P - 1)(q_i - 6)$$

The right side is negative, since $P \geq 5$ and $q_i < 6$. Hence the left side $4 - \ell_i - n_i$ must also be negative. Since $\ell_i \leq 3$, we must have $n_i \geq 2$. But then $3n_i \geq 6 > q_i$, and we have disposed of this case.

That completes all cases involving the third row of the table.

Now we take up the fourth row of the table. Here we also have $\gamma = 2\beta + \alpha$, but the formulas for β and α in terms of P and Q are different, and we have $R = 0$. It is simplest just to enumerate the possibilities for ℓ_i , m_i , and n_i . We have $3\gamma + \alpha = 2\pi$ (since $\ell = 3$, $m = 0$, and $n = 1$, from the first table); that gives the first row in the following table. The others are obtained by using the equation $\gamma = 2\beta + \alpha$ and the equation $\beta = (P - 2)/(Q - 3)\alpha$ to “trade in” some larger angles for smaller ones. The entries containing fractions apply only if the fraction shown is an integer. This table assumes $k_i = 2$, so the angle sum is 2π .

ℓ_i	m_i	n_i	$2\ell_i + m_i$	$2m_i + \ell_i$
3	0	1	6	3
2	2	2	6	6
2	0	$1 + \frac{2(P-2)}{3-Q}$	4	2
2	1	$1 + \frac{P-2}{3-Q}$	1	4
1	4	3	4	9
1	4-r	$3 + r\frac{P-2}{3-Q}$	$6 - r$	$9 - 2r$
0	6	4	6	12
0	6-r	$3 + r\frac{P-2}{3-Q}$	$6 - r$	$12 - 2r$

We note that the estimate $q_i = 2\ell_i + m_i \leq 3n_i$, which we used for the first three rows of the table, fails here, e.g. in the first row. But we can replace it by $r_i = 2m_i + \ell_i \leq 3n_i$, shown in the last column of the table.

Remembering that the rows with fractions occur only if the fraction is an integer, and that $P - 2$ and $Q - 3$ are positive integers since $Q \leq 2$ and $P + Q \geq 5$, inspection of the table shows that $r_i \leq 3n_i$ in each row. This table applies only when $k_i = 2$, so we also need to consider the case $k_i = 1$. Since $3\gamma + \alpha = 2\pi$, we have $\gamma > \pi/2$, so when $k_i = 1$, we have $\ell_i = 0$ or 1. If $\ell_i = 1$, then since $\alpha \neq \beta$, we have $m_i = n_i = 1$ (in which case $r_i = 3 \leq 3n_i$), or $m_i = 0$ and β is an integer multiple of α . In that case, n_i is at least 3, since $\beta > \alpha$, and $r_i = 1$, so $r_i \leq 3n_i$. Now suppose $k_i = 1$ and $\ell_i = 0$. Since $\pi = (2\beta + \alpha) + \beta + \alpha = 3\beta + 2\alpha$, we have $m_i = 3$ and $n_i = 2$ as one possibility, in which case $r_i = 6 \leq 3n_i$; and other possibilities may arise if 3β is a multiple of α , but if that happens, then $r_i < 6$ and $n_i > 2$, so we still have $r_i \leq 3n_i$.

Hence, for the fourth row of Table 2, we always have $r_i \leq 3n_i$. Now we sum over all vertices of the tiling that are not vertices of ABC . We have

$$\begin{aligned} \sum r_i &= \sum (2m_i + \ell_i) \\ &= 2 \sum m_i + \sum \ell_i \\ &= 2(N - Q) + N \quad \text{since } R = 0 \\ &= 3N - 2Q \end{aligned}$$

On the other hand we have

$$3 \sum n_i = 3(N - P)$$

Since $\sum r_i \leq 3 \sum n_i$, we have $3N - 2Q \leq 3N - 3P$. Hence $3P \leq 2Q$. But $Q \leq 2$ and $P \geq 3$, so $3P \geq 9 > 4 \geq 2Q$, contradiction. That completes the proof that the fourth row of Table 2 is impossible.

Now we turn to the fifth (and last) row of Table 2. Here we have $\gamma = 2\alpha + \beta$. The following table shows the possible vertices. The rows containing fractions represent possible vertices only in case the fraction shown is actually an integer. Since $Q \leq 1$ in the fifth row of table 2, the denominator of the fractions shown is either 1 or 2.

k_i	ℓ_i	m_i	n_i
2	3	1	0
2	3	0	$\frac{P-3}{2-Q}$
2	2	2	2
2	2	1	$2 + \frac{P-3}{2-Q}$
2	2	0	$2 + 2\frac{P-3}{2-Q}$
2	1	3	6
2	1	3-r	$6 + r\frac{P-3}{2-Q}$
2	0	4	8
2	0	4-r	$8 + r\frac{P-3}{2-Q}$
1	1	1	1
1	0	0	$\frac{P-3}{2-Q}$
1	0	1	2
1	0	1	$\frac{P-3}{2-Q}$
1	0	0	$2\frac{P-3}{2-Q}$

For this row of Table 2, the estimates used in the previous rows do not work, because of the possibility shown in the first row of this table, namely $\ell_i = 3$ while $n_i = 0$. Therefore we give a different argument. Note that the first four rows of the table correspond to vertices with at least two γ angles. Since there are altogether N γ angles, there are at most $N/2$ such vertices. Hence (and since $R = 0$) there are at least $N/2$ vertices other than those of ABC , each contributing at most one γ angle. Now consider the difference $m_i - n_i$ at the two kinds of vertex. At the vertices with two or three γ angles, $m_i - n_i \leq 1$; indeed only the first row has $m_i - n_i > 0$, and in that row it is 1. In all the other rows of the table, we have $n_i - m_i \geq 1$. Hence, adding $n_i - m_i$ over all the vertices, we obtain at least one from each of at least $N/2$ vertices, and at least -1 from the remaining (smaller number of) vertices, so the sum of $n_i - m_i$ over all the vertices (not counting the vertices of ABC) is non-negative. But when we add in the contributions of the vertices of ABC , we add $P - Q$, which is a positive number (since $Q \leq 1$ and $P + Q \geq 5$). That is a contradiction, since altogether there are N each of α angles and β angles, so the sum over

all vertices, including those of ABC , of $n_i - m_i$ must be zero. That completes the proof that the fifth row of Table 2 is impossible, and it also completes the proof that $\ell = 3$ is impossible.

We still have the cases $\ell = 4$ and $\ell = 5$ to deal with. We now assume $\ell = 4$. That is, there is at least one vertex V where four γ angles meet, and there are at most 3 other angles, i.e. $n + m \leq 3$. Since $\gamma > \pi/3$, we have $k = 2$, i.e. the angle sum at V is 2π , not π . If $n + m = 0$, then $\gamma = \pi/2$, contradiction. If $n = 0$ then we have

$$\begin{aligned} 2\pi &= 4\gamma + m\beta \\ 2(\alpha + \beta + \gamma) &= 4\gamma + m\beta \\ 2\alpha + (2 - m)\beta &= 2\gamma \end{aligned}$$

Since $\alpha < \gamma$, we must have $2 - m > 0$, or $m < 2$. Since $n = m = 0$ is impossible, that leaves $m = 1$ as the only possibility when $n = 0$, and in that case $2\gamma = 2\alpha + \beta$.

We will prove that in this case ($\ell = 4$ and $n = 0$) we have $R = 0$. By Lemma 22, we have $R \leq 1$. Assume, for proof by contradiction, that $R = 1$. Then by Lemma 23 we have $Q = 0$ and $\beta = (P - 1)\alpha$, so $2\gamma = 2\alpha + \beta = (P + 1)\alpha$. We then have

$$\begin{aligned} \beta &< \gamma \\ (P - 1)\alpha &< \frac{P + 1}{2}\alpha \\ P - 1 &< \frac{P + 1}{2} \\ P &< 3 \end{aligned}$$

contradicting $P + Q \geq 5$, since $Q = 0$. That completes the proof that $R = 0$.

Having proved that $R = 0$, we then have

$$\begin{aligned} \pi &= \alpha + \beta + \gamma \\ &= \alpha + \beta + \frac{1}{2}(2\alpha + \beta) \quad \text{since } 2\gamma = 2\alpha + \beta \\ &= 2\alpha + \frac{3}{2}\beta \\ &= P\alpha + Q\beta \end{aligned}$$

This gives us two equations in α and β . The determinant of the system is (a multiple of) $4Q - 3P$. Assume, for proof by contradiction, that this determinant is zero. Then

$$\begin{aligned} \pi &= P\alpha + Q\beta \\ 4\pi &= 4P\alpha + 4Q\beta \\ &= 4P\alpha + 3P\beta \quad \text{since } 4Q = 3P \\ &= (4\alpha + 3\beta)P \\ &= 2\pi P \quad \text{since } 4\alpha + 3\beta = 2\pi \end{aligned}$$

Thus $4\pi = 2\pi P$. Hence $P = 2$. Now we have

$$\begin{aligned} 2\pi &= 4\alpha + 3\beta \\ \pi &= P\alpha + Q\beta \\ &= 2\alpha + Q\beta \quad \text{since } P = 2 \end{aligned}$$

Multiplying the last equation by 2 and subtracting the first equation, we find $2Q = 3$, which is impossible since Q is an integer. This contradiction shows that the determinant is not zero.

Hence the equations are solvable:

$$\begin{aligned}\alpha &= \frac{2Q-3}{4Q-3P}\pi \\ \beta &= \frac{4-2P}{4Q-3P}\pi \\ \gamma &= \alpha + \frac{1}{2}\beta \\ &= \frac{2Q-P-1}{4Q-3P}\pi\end{aligned}$$

If $4Q > 3P$ then, since $\alpha < \beta$, we have

$$\begin{aligned}2Q-3 &< 4-2P \\ 2(P+Q) &< 7\end{aligned}$$

contradicting $P+Q \geq 5$. Hence $3P < 4Q$. Writing the equations for α , β , and γ with positive denominator, we have

$$\begin{aligned}\alpha &= \frac{3-2Q}{3P-4Q}\pi \\ \beta &= \frac{2P-4}{3P-4Q}\pi \\ \gamma &= \frac{P+1-2Q}{3P-4Q}\pi\end{aligned}$$

Then the numerators must also be positive, so $2Q < 3$, which implies $Q = 0$ or $Q = 1$. Assume, for proof by contradiction, that $Q = 0$. Then $\alpha = \pi/P$ and $\beta = 2\pi/3 - 4\pi/(3P)$. Since $\beta < \gamma$ we have

$$\begin{aligned}\beta &< \gamma \\ \frac{2}{3}\pi - \frac{4}{3P}\pi &< \frac{P+1}{3P}\pi \\ \frac{2}{3} &< \frac{P+5}{3P} \\ 6P &< 3(P+5) \\ 3P &< 15 \\ P &< 5\end{aligned}$$

contradicting $P+Q \geq 5$. Hence $Q \neq 0$. Hence $Q = 1$. Then

$$\begin{aligned}\alpha &= \frac{1}{3P-4}\pi \\ \beta &= \frac{2P-4}{3P-4}\pi \\ \gamma &= \frac{P-1}{3P-4}\pi\end{aligned}$$

Since $\beta < \gamma$, we have $2P-4 < P-1$, which implies $P < 5$, contradicting $P+Q \geq 5$. Hence every subcase of $\ell = 4$, $n = 0$ has led to a contradiction. Hence $\ell = 4$ and $n = 0$ cannot occur.

Next we take up the case $\ell = 4$ and $n = 1$. Then we have

$$\begin{aligned}2\pi &= 4\gamma + m\beta + \alpha \\ 2(\alpha + \beta + \gamma) &= 4\gamma + m\beta + \alpha \\ \alpha + (2-m)\beta &= 2\gamma\end{aligned}$$

and since $\alpha < 2\gamma$, we must have $2 - m > 0$, which implies $m < 2$. The case $m = 1$ can also be ruled out, for then we would have $\alpha + \beta = 2\gamma$, and adding γ to both sides, we would have $\pi = 3\gamma$, contradicting $\gamma > \pi/3$. Hence $m = 0$ when $n = 1$, and we have $2\gamma = \alpha + 2\beta$. Adding γ to both sides we have

$$\begin{aligned} 3\gamma &= \alpha + 2\beta + \gamma \\ &= \pi + \beta \\ \gamma &= \frac{\pi}{3} + \frac{\beta}{3} \\ \gamma &< \frac{\pi}{2} \quad \text{since } 4\gamma + \alpha = 2\pi \end{aligned}$$

Therefore

$$\beta + \frac{\pi}{3} < \frac{\pi}{2}$$

Subtracting $\pi/3$ from both sides we have

$$\begin{aligned} \beta &< \frac{\pi}{6} \\ \alpha &= \pi - \gamma - \beta \\ &> \pi - \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3} \quad \text{since } \gamma < \pi/2 \text{ and } \beta < \pi/6 \end{aligned}$$

Therefore $\alpha > \pi/3$, contradiction. Hence the case $\ell = 4$ and $n = 1$ is not possible.

If $\ell = 4$ and $n = 2$ then since $n + m \leq 3$, we have $m = 0$ or $m = 1$ and

$$\begin{aligned} 2\pi &= 4\gamma + m\beta + 2\alpha \\ 2(\alpha + \beta + \gamma) &= 4\gamma + m\beta + 2\alpha \\ (2 - m)\beta &= 2\gamma \\ \beta &= \frac{2}{2 - m}\gamma \end{aligned}$$

Since m is 0 or 1, the fraction on the right is either 2 or 1, contradicting $\beta < \gamma$. Hence the case $n = 2$ cannot occur.

If $\ell = 4$ and $n = 3$ then since $n + m \leq 3$ we have $m = 0$, and

$$\begin{aligned} 2\pi &= 4\gamma + 3\alpha \\ 2(\alpha + \beta + \gamma) &= 4\gamma + 3\alpha \\ 2\beta &= 2\gamma + 3\alpha \end{aligned}$$

contradicting $\beta < \gamma$. So the case $n = 3$ cannot occur either. Since $m + n < \ell$, this is the last possibility with $\ell = 4$; hence $\ell = 4$ has been ruled out.

The last possibility for ℓ is $\ell = 5$. Assume $\ell = 5$. That is, there is at least one vertex V where five γ angles meet, and there are at most 4 other angles, i.e. $n + m \leq 4$. Since $\gamma > \pi/3$, we have $k = 2$, i.e. the angle sum at V is 2π , not π . If $n + m = 0$, then $\gamma = 2\pi/5$, which has already been ruled out by Lemma 4. Assume, for proof by contradiction, that $n = 0$. Then we have

$$\begin{aligned} 2\pi &= 5\gamma + m\beta \\ 2(\alpha + \beta + \gamma) &= 5\gamma + m\beta \\ 2\alpha + (2 - m)\beta &= 3\gamma \\ (2 - m)\beta &= 3\gamma - 2\alpha \\ &= 2(\gamma - \alpha) + \gamma \\ &> \gamma \quad \text{since } \gamma > \alpha \end{aligned}$$

Since $\beta < \gamma$, this implies $2 - m > 1$, which implies $m = 0$. Since we have assumed $n = 0$, and we have proved $n + m \neq 0$, that is a contradiction. This contradiction completes the proof that $n \neq 0$.

Now assume $\ell = 5$ and $n = 1$. Then we have

$$\begin{aligned} 2\pi &= 5\gamma + m\beta + \alpha \\ 2(\alpha + \beta + \gamma) &= 5\gamma + m\beta + \alpha \\ \alpha + (2 - m)\beta &= 3\gamma \\ (2 - m)\beta &= 3\gamma - \alpha \\ (2 - m)\beta &> 2\gamma \quad \text{since } \alpha < \gamma \end{aligned}$$

Since $\beta < \gamma$, this implies $2 - m > 2$, which is not possible, since $m \geq 0$. Hence the case $\ell = 5$ and $n = 1$ is impossible.

Now assume $\ell = 5$ and $n \geq 2$. Then we have

$$\begin{aligned} 2\pi &= 5\gamma + m\beta + n\alpha \\ 2(\alpha + \beta + \gamma) &= 5\gamma + m\beta + n\alpha \\ (2 - m)\beta &= 3\gamma + (n - 2)\alpha \\ (2 - m)\beta &\geq 3\gamma \quad \text{since } n \geq 2 \end{aligned}$$

Since $\beta < \gamma$, this implies $2 - m > 3$, which is not possible, since $m \geq 0$. Hence the case $\ell = 5$ is impossible. That was the last possibility for ℓ , so the proof of the lemma is complete.

9 Conclusions

The aim of this series of papers is to classify the triples (ABC, T, N) such that triangle ABC can be N -tilled by T . We have completed this classification, except for the case when the tile T has a 120° angle and is isosceles. For example, aside from that shape of tile, given an integer N , if there is any N -tiling, then N is either a square, or a sum of two squares, or is 2,3, or 6 times a square, or twice a sum of two squares. For example, there are no N -tilings for $N = 7, 11, \text{ or } 19$. The following theorem gives more information about the possibilities for the shapes of the tile and the tiled triangle.

Theorem 2 (Main Theorem) *Suppose triangle ABC is tiled by triangle T . Suppose ABC is not similar to T and T is not a right triangle. Then one of the following conclusions holds:*

- (i) ABC is equilateral, T is isosceles with base angles $\pi/6$, and N is three times a square, or
- (ii) $3\alpha + 2\beta = \pi$, where α and β are the two smallest angles of the tile, in either order, and α is not a rational multiple of π , and $\sin(\alpha/2)$ is rational (which implies that the sides of the tile have rational ratios), and two of the angles of ABC are 2α and β .
- (iii) $\gamma = 2\pi/3$ (120 degrees) and $\alpha \neq \beta$ (the tile is not isosceles).

Theorems covering the case when ABC is similar to T and the case when T is a right triangle are in [1]. Theorems covering the case in conclusion (ii) are in [2], where a necessary and sufficient condition on N is given for an N -tiling to exist in that case. We conjecture there are no tilings falling under case (iii); some partial results in that direction are in [3].

References

- [1] Michael Beeson. Triangle tiling I: the tile is similar to ABC or has a right angle.
To appear, available on the author's website.
- [2] Michael Beeson. Triangle tiling III: the triquadratic tilings.
To appear, available on the author's website.
- [3] Michael Beeson. Triangle tiling IV: A non-isosceles tile with a 120 degree angle.
To appear, available on the author's website.