Triangle Tiling I: the tile is similar to ABC or has a right angle

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Abstract

An N-tiling of triangle ABC by triangle T is a way of writing ABC as a union of N triangles congruent to T, overlapping only at their boundaries. The triangle T is the "tile". The tile may or may not be similar to ABC. This paper is the first of four papers, which together seek a complete characterization of the triples (ABC, N, T) such that ABC can be N-tiled by T. In this paper, we consider the case in which the tile is similar to ABC, the case in which the tile is a right triangle, and the case when ABC is equilateral. We use (only) techniques from linear algebra and elementary field theory, as well as elementary geometry and trigonometry.

Our results (in this paper) are as follows: When the tile is similar to ABC, we always have "quadratic tilings" when N is a square. If the tile is similar to ABC and is not a right triangle, then N is a square. If N is a sum of two squares, $N = e^2 + f^2$, then a right triangle with legs e and f can be N-tiled by a tile similar to ABC; these tilings are called "biquadratic". If the tile and ABC are 30-60-90 triangles, then N can also be three times a square. If T is similar to ABC, these are all the possible triples (ABC, T, N).

If the tile is a right triangle, of course it can tile a certain isosceles triangle when N is twice a square, and in some cases when N is six times a square. Equilateral triangles can be 3-tiled and 6-tiled and hence they can also be $3n^2$ and $6n^2$ tiled for any n. We also discovered a family of tilings when N is 3 times a square, which we call the "hexagonal tilings." These tilings exhaust all the possible triples (ABC, T, N) in case T is a right triangle or is similar to ABC. Other cases are treated in subsequent papers.

1   Examples of Tilings

We consider the problem of cutting a triangle into N congruent triangles. Figures 1 through 4 show that, at least for certain triangles, this can be done with $N = 3$, 4, 5, 6, 9, and 16. Such a configuration is called an N-tiling.
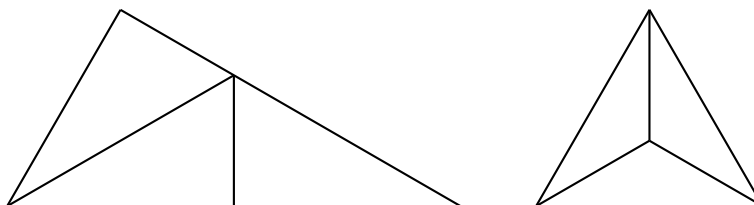
Figure 1: Two 3-tilings
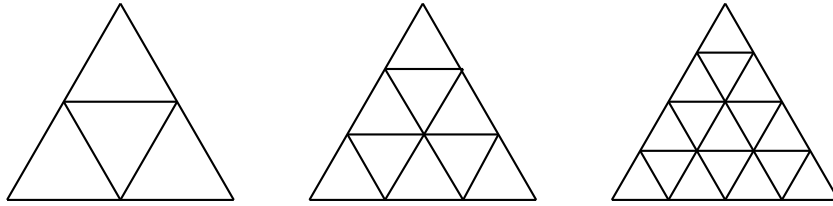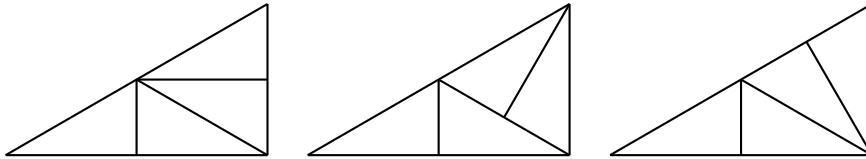
Figure 2: A 4-tiling, a 9-tiling, and a 16-tiling

Figure 3: Three 4-tilings

The method illustrated for $N = 4$,9, and 16 clearly generalizes to any perfect square N. While the exhibited 3-tiling, 6-tiling, and 5-tiling clearly depend on the exactly angles of the triangle, *any* triangle can be decomposed into n^2 congruent triangles by drawing $n - 1$ lines, parallel to each edge and dividing the other two edges into n equal parts. Moreover, the large (tiled) triangle is similar to the small triangle (the "tile"). We call such a tiling a *quadratic tiling.* Fig. 4 illustrates a quadratic tiling of an arbitrary triangle.

Figure 4: A quadratic tiling of an arbitrary triangle
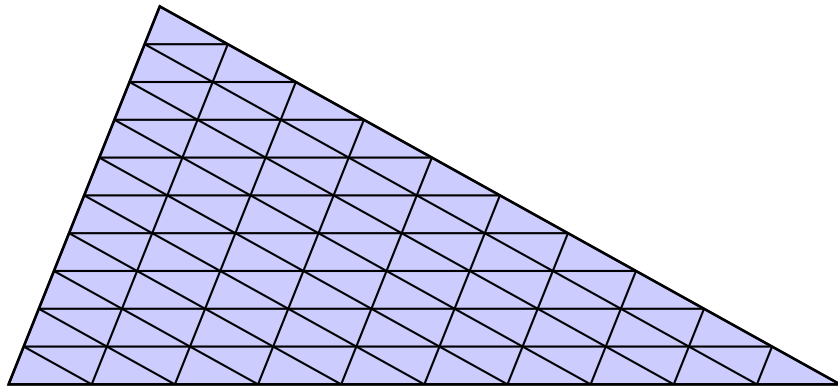
It follows that if we have a tiling of a triangle ABC into N congruent triangles, and m is any integer, we can tile ABC into Nm^2 triangles by subdividing the first tiling, replacing each of the N triangles by m^2 smaller ones. Hence the set of N for which an N-tiling of some triangle exists is closed under multiplication by squares.

Let N be of the form $n^2 + m^2$. Let triangle T be a right triangle with perpendicular sides n and m, say with $n \geq m$. Let ABD be a right triangle with base AD of length m^2, the right angle at D and altitude mn, so side BD has length mn. Then ABD can be decomposed into m triangles congruent to T, arranged with their short sides (of length m) parallel to the base AD.

Now, extend AD to point C, located n^2 past D. Triangle ADC can be tiled with n^2 copies of T, arranged with their long sides parallel to the base. The result is a tiling of triangle ABC by $n^2 + m^2$ copies of T. The first 5-tiling exhibited in Fig. 5 is the simplest example, where $n = 2$ and $m = 1$. The case $N = 13 = 3^2 + 2^2$ is illustrated in Fig. 6. We call these tilings "biquadratic." More generally, a *biquadratic tiling* of triangle ABC is one in which ABC has a right angle at C, and can be divided by an altitude from C to AB into two triangles, each similar to ABC, which can be tiled respectively by n^2 and m^2 copies of a triangle similar to ABC. The second 5-tiling in Fig. 5 shows that this can be sometimes be done more generally than by combining two quadratic tilings.
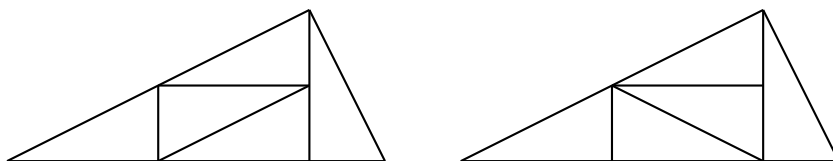
Figure 5: Two 5-tilings

Figure 6: A 13-tiling
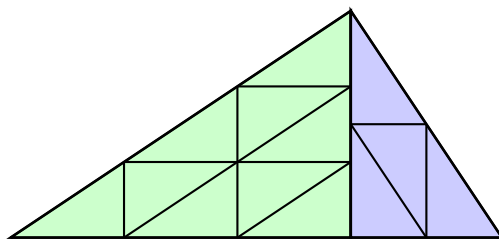
A larger biquadratic tiling, with $n = 5$ and $m = 7$ and hence $N = 74$, is shown in Fig. 7.

Figure 7: Biquadratic tiling with $N = 74 = 5^2 + 7^2$
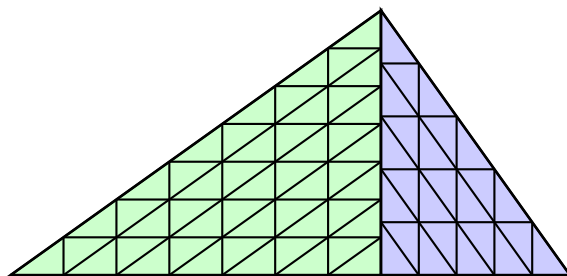
If the original triangle ABC is chosen to be isosceles, then each of the n^2 triangles can be divided in half by an altitude; hence any isosceles triangle can be decomposed into $2n^2$ congruent triangles. If the original triangle is equilateral, then it can be first decomposed into n^2 equilateral triangles, and then these triangles can be decomposed into 3 or 6 triangles each, showing that any equilateral triangle can be decomposed into $3n^2$ or $6n^2$ congruent triangles.

These tilings are neither quadratic nor biquadratic. For example we can 12-tile an equilateral triangle in two different ways, starting with a 3-tiling and then subdividing each triangle into 4 triangles ("subdividing by 4"), or starting with a 4-tiling and then subdividing by 3.

Figure 8: A 6-tiling, an 8-tiling, and a 12-tiling
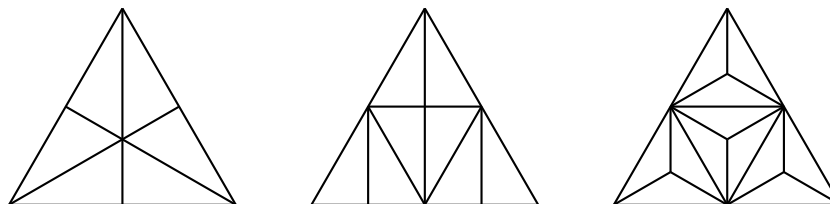
Examples like these led us to the following definitions:

A tiling E of triangle ABC (with tile T_2) is a *subtiling* of another tiling F of ABC (with tile T), if T can be tiled by the tile T_2 and the tiling E is obtained by tiling each copy of T in F with triangle T_2. It is not required that the *same* tiling be used for each copy of T. For example, we could take F to be one of the two five-tilings, and then tile each of the tiles in that tiling by one of its two five-tilings. In this way we can obtain 64 different 25-tilings, none of them quadratic.

A tiling of ABC is called *composite* if it is a subtiling of some tiling. It is called *prime* if it is not composite. Note that a quadratic N^2-tiling is prime if and only if N is a prime number.

The examples above do not exhaust all possible tilings, even when N is a square. For example, Fig. 9 shows a 9-tiling that is not produced by those methods:

Figure 9: Another 9-tiling
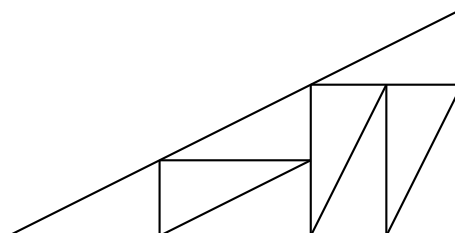
There is another family of N-tilings, in which N is of the form $3m^2$, and both the tile and the tiled triangle are 30-60-90 triangles. We call these the "triple-square" tilings. The case $m = 1$ is given in Fig. 1; the case $m = 2$ makes $N = 12$. There are two ways to 12-tile a 30-60-90 triangle with 30-60-90 triangle. One is to first quadratically 4-tile it, and then subtile the four triangles with the 3-tiling of Figure 1. This produces the first 12-tiling in Fig. 10. Somewhat surprisingly, there is another way to tile the same triangle with the same 12 tiles, also shown in Fig. 10; the second tiling is prime. The next member of this family is $m = 3$, which makes $N = 27$. Two 27-tilings are shown in Fig. 11; the first obtained by subtiling a quadratic tiling, and the second one prime. Similarly, there are two 48-tilings (not shown).

Until October 12, 2008, we did not know any more complicated tilings than those illustrated above (and there also none in [10]). Then we found the beautiful 27-tiling shown in Fig. 12. This tiling is one of a family of $3k^2$ tilings (the case $k = 3$). The next case is a 48-tiling, made from six hexagons (each containing 6 tiles) bordered by 4 tiles on each of 3 sides. In general one can arrange $1 + 2 + \ldots + k$ hexagons in bowling-pin fashion, and add $k + 1$ tiles on each of three

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Figure 10: Two 12-tilings
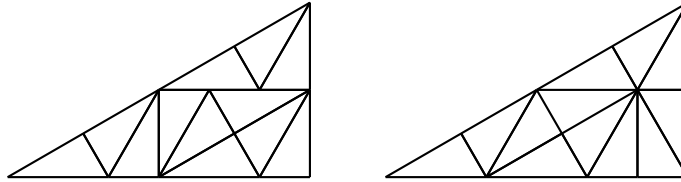
Figure 11: Two 27-tilings
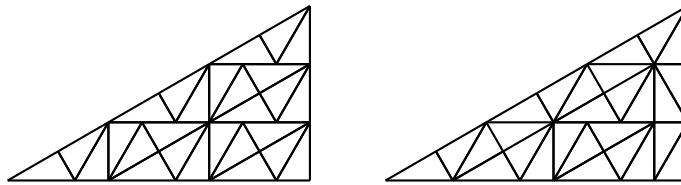
sides, for a total number of tiles of $6(1+2+\ldots+k)+3(k+1) = 3k(k+1)+3(k+1) = 3(k+1)^2$. Fig. 13 shows more members of this family, which we call the "hexagonal tilings."[1]

Figure 12: A prime 27-tiling
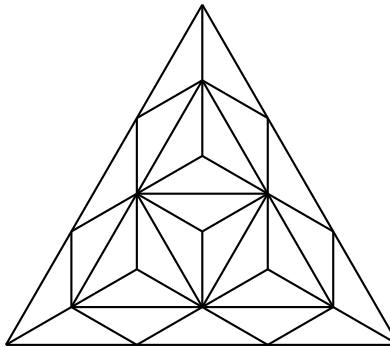
In the third paper in this series, [2], we exhibit a new family of tilings called the "triquadratic tilings." The tiles and triangles ABC involved do not fall into the cases considered in this paper. The main result of this paper is that, when the tile is similar to ABC, or the tile is a right triangle, or ABC is equilateral, there are no more triples (ABC, N, T) than we have mentioned.

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[1]In January, 2012, I bought a puzzle at the exhibition at the AMS meeting, which contained the tiling in Fig. 12 as part of a tiling of a larger hexagon. The tiling is attributed to Major Percy Alexander MacMahon (1854-1929) [9].

Figure 13: $3m^2$ (hexagonal) tilings for $m = 4$ and $m = 5$
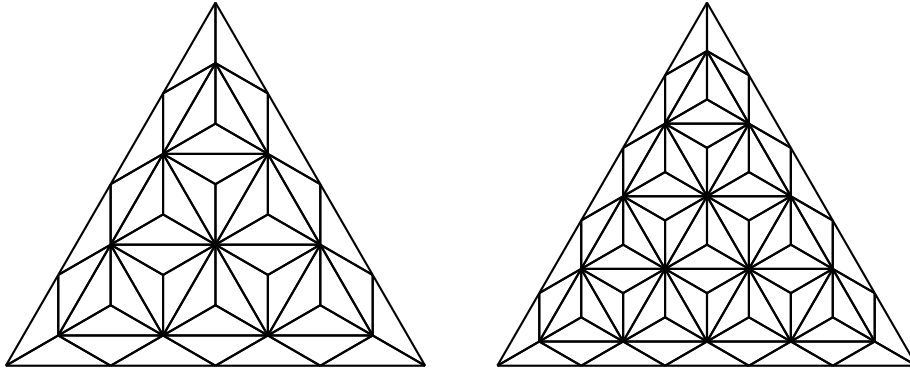
2   Previous work

The examples given in Figures 1 through 6 are well-known. They have been discussed, in particular, in connection with "rep-tiles" [8]. A "rep-tile" is a set of points X in the plane (not necessarily just a triangle) that can be dissected into N congruent sets, each of which is similar to S. An N-tiling in which the tiled triangle ABC is similar to the triangle T used as the tile is a special case of this situation. That is the case, for example, for the n^2 family and the $n^2 + m^2$ family, but not for the 3-tiling, 6-tiling, or the 12-tiling exhibited above. Thus the concepts of an N-tiling and rep-tiles overlap, but neither subsumes the other. The paper [7] also contains a diagram showing the n^2 family of tilings, but the problem considered there is different: one is allowed to cut N copies of the tile first, before assembling the pieces into a large figure, but the large figure must be similar to the original tile. The two books [5] and [4] have tantalizing titles, but deal with other problems.

Only after completing the work in this paper did I encounter Soifer's book [10], when the second edition came out, although the first edition had been out for 19 years. The book contains the observation that if the tile T is similar to the tiled triangle then $\sqrt{N}$ is an eigenvalue of a certain matrix, so that observation is, as it turns out, not new. The book, however, does not contain any examples of tilings beyond the quadratic tilings, though it gives an indication that at least the biquadratic tilings were known, since it says that the 1989 Russian Mathematical Olympiad contained the problem to show that if N is a sum of two squares then there is a triangle that can be N-tiled. Soifer states (p. 48) the open problem solved in this series of three papers, and says that Paul Erdős offered a $25 prize for the first solution. He does not state where or when Erdős mentioned these problems. The problem statement is: Find all positive integers N such that at least one triangle can be cut into N triangles congruent to each other. This is Soifer's "Problem 6.7."

Soifer also states some related problems. His "Problem 6.5" is: For each triangle ABC, find all positive integers N such that T can be cut into N triangles congruent to each other, and the number of distinct partitions of T into N congruent triangles. Actually this is two problems, and we have made serious progress on them both in these papers. Given a triangle ABC, our results succinctly describe the pairs (N, T) such that ABC can be N-tiled by triangle T, with the caveat that when the tile T has a $120°$ angle and is not isosceles, we have not yet (as of April, 2012) proved that there are no N-tilings of ABC by T (except, of course, if ABC is similar to T).

Describing the pairs (N, T) such that T can N-tile ABC is better than just finding the possible N, but not as good as completely classifying and counting the tilings. Soifer says that his Problem 6.5 is "open and very difficult."

Soifer's "Problem 6.6" is also a \$25 Erdős problem: Find (and classify) all triangles that can only be cut into n^2 congruent triangles for any integer n. We have solved this problem, except for the unsolved case when has a $120°$ angle and is not isosceles. The solution is presented in [2]. In fact, our main theorem, plus the conjectured absence of tilings by non-isosceles tiles with a $120°$ angle, would yield the stronger statement: Given triangle ABC, a necessary and sufficient condition for ABC to have N-tilings only when N is a square is that ABC be not isosceles, and not a right triangle whose angles have rational tangents, and not of the form $N = 2K^2 - M^2$ with K dividing M^2. We note that it is not the case that for triangles ABC meeting these conditions, any tiling must be the quadratic tiling; there can still be integral relations between the sides, such as $3a = 2b$, while the angles have irrational tangents. See Lemma 8 below, and the accompanying Fig. 14, for an example.

Soifer claims, without publishing a proof, that if the sides and angles of ABC are integrally independent, then ABC admits only quadratic tilings. It follows (as he points out) that the perfect squares are exactly the N for which *every* triangle ABC can be N-tiled by some triangle.

Dima Fan-Der-Flaas informed me that the problem of finding an N-tiling of some triangle when $N = 1989$ was posed on the Russian Mathematical Olympiad in 1989; it was solved by a few students, who had to discover what we call here the "biquadratic tilings", and realize that 1989 is a sum of two squares and the relevance of that fact. I would like to thank Dima for his careful reading of parts of an early draft of this paper and [1].

3   Definitions, notation, and some simple lemmas

We give a mathematically precise definition of "tiling" and fix some terminology and notation. Given a triangle T and a larger triangle ABC, a "tiling" of triangle ABC by triangle T is a set of triangles $T_1, \ldots, T_n$ congruent to T, whose interiors are disjoint, and the closure of whose union is triangle ABC.

Let a, b, and c be the sides of the tile T, and angles α, β, and γ be the angles opposite sides a, b, and c. The letter "N" will always be used for the number of triangles used in the tiling. An N-tiling of ABC is a tiling that uses N copies of some triangle T. The meanings of N, α, β, γ, a, b, c, A, B, and C will be fixed throughout these three papers. In this paper and the next, we assume $\alpha \le \beta \le \gamma$; in the third paper we only assume $\alpha \le \gamma$ and $\beta \le \gamma$, but not $\alpha \le \beta$.

An *interior vertex* in a tiling of ABC is a vertex of one of T_i that does not lie on the boundary of ABC. A *boundary vertex* is a vertex of one of the T_i that lies on the boundary of ABC. A "strict vertex" of the tiling is a vertex of one of the T_i that does not lie on the interior of an edge of another T_j. A "strict tiling" is one in which no T_i has a vertex lying on the interior of an edge of another T_j, i.e. every vertex is strict. For example, the biquadratic tilings (illustrated above for $N = 5$, 13, and 74) are not strict, but all the other tilings shown above are strict. A "non-strict vertex" V is one that lies on an edge of T_j, with T_j on one side of the edge and (more than one) T_i having vertex V on the other side.

Another way of describing a vertex is to say it is of "type π" or "type 2π", or "has angle sum π" or "has angle sum 2π". Strict interior vertices have angle sum 2π, or are "of type 2π"; boundary vertices and non-strict interior vertices have angle sum π, or are "of type π."

By the law of sines we have

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$$

Up to similarity then we may assume

$$a = \sin \alpha$$
$$b = \sin \beta$$

$$c \quad = \quad \sin\gamma$$

Definition 1 *A **maximal segment** in a tiling is a line segment S contained in the union of the edges of the tiling, that cannot be extended to a longer line segment so contained.*

Let S be a maximal segment. Since there are triangles on each side of S, there are triangles on each side of S at every point of S (since S cannot extend beyond the boundary of ABC). Hence the length of S is a sum of lengths of sides of triangles T_i in two different ways (though the summands may possibly be the same numbers in a different order). Let us assume for the moment that the summands are not the same numbers. Then it follows that some linear relation of the form

$$pa + qb + rc = 0$$

holds, with p, q, and r integers not all zero (one of which must of course be negative), and the sum of the absolute values of p, q, and r is less than or equal to N, since there are no more than N triangles. This is called an "edge relation." Of course since $a = \sin\alpha$, etc., we can express an edge relation in terms of the sines of the angles. We can do that without mentioning γ: Since $\gamma = \pi - (\alpha + \beta)$ we have $\sin(\gamma) = \sin(\alpha + \beta)$, so

$$p\sin\alpha + q\sin\beta + r\sin(\alpha + \beta) = 0.$$

A *quadratic tiling* is one in which N is a perfect square, say $N = m^2$, and the tiling is produced by drawing $m - 1$ equally spaced lines parallel to each side, dividing each edge into m equal segments. In such a tiling, the tile T is similar to the large triangle ABC. An *angle relation* is an equation

$$p\alpha + q\beta + r\gamma = 2\pi$$

where p, q, and r are non-negative integers, not all equal. (Since we always have $\alpha + \beta + \gamma = \pi$, we do not count that equation or its multiples as an angle relation.)

A *split vertex* occurs when two copies of the tile in a triangle share one of the vertices of the large triangle. Split vertices give rise to "angle relations", by which we mean an equation $p\alpha + q\beta + r\gamma = n\pi$, with integers p, q, and r. We always have the angle relation $\alpha + \beta + \gamma = \pi$, but the question of whether a tiling has or might have additional angle relations will be important in our work.

The following lemma is simple and fundamental:

Lemma 1 *If, in a tiling, P is a boundary vertex (or a non-strict interior vertex) and only one interior edge emanates from P, then both angles at P are right angles and $\gamma = \pi/2$.*

Proof. If the two angles at P are different, then their sum is less than π, since the sum of all three angles is π. Therefore the two angles are the same. But $2\alpha \le \alpha + \beta < \pi$ and $2\beta \le \beta + \gamma < \pi$. Therefore both angles are γ. But then $2\gamma = \pi$, so $\gamma = \pi/2$.

The following result we call "Euler's equation", because it is related to Euler's famous formula for the numbers of vertices, edges, and faces of a polygon, although we have not derived it that way here.

Lemma 2 *Let a tiling contain N_b boundary vertices, N_s strict interior vertices, and N_n non-strict interior vertices. We then have*

$$N - 1 \quad = \quad N_b + N_n + 2N_s \tag{1}$$

Proof. We count vertices. At each strict interior vertex, the sum of the angles of the tiles sharing that vertex is 2π. At each non-strict interior vertex and at each boundary vertex (that is, vertex lying on the boundary of triangle ABC but not equal to A, B, or C), the angle sum is π. The total angle sum of all N copies of the tile is $N\pi$, which must be accounted for by the π at the vertices of ABC, plus the contributions at the other vertices.

The following lemma identifies those relatively few rational multiples of π that have rational tangents or whose sine and cosine satisfy a polynomial of low degree over $\mathbb{Q}$.

Lemma 3 *Let $\zeta = e^{i\theta}$ be algebraic of degree d over $\mathbb{Q}$, where θ is a rational multiple of π, say $\theta = 2m\pi/n$, where m and n have no common factor.*

Then $d = \varphi(n)$, where φ is the Euler totient function. In particular if $d = 4$, which is the case when $\tan\theta$ is rational and $\sin\theta$ is not, then n is 5, 8, 10, or 12; and if $d = 8$ then n is 15, 16, 20, 24, or 30.

Remark. For example, if $\theta = \pi/6$, we have $\sin\theta = 1/2$, which is of degree 1 over $\mathbb{Q}$. Since $\cos\theta = \sqrt{3}/2$, the number $\zeta = e^{i\theta}$ is in $\mathbb{Q}(i, \sqrt{3})$, which is of degree 4 over $\mathbb{Q}$. The number ζ is a 12-th root of unity, i.e. n in the theorem is 12 in this case; so the minimal polynomial of ζ is of degree $\varphi(12) = 4$. This example shows that the theorem is best possible.

Remark. The hypothesis that θ is a rational multiple of π cannot be dropped. For example, $x^4 - 2x^3 + x^2 - 2x + 1$ has two roots on the unit circle and two off the unit circle.

Proof. Let f be a polynomial with rational coefficients of degree d satisfied by ζ. Since $\zeta = e^{i2m\pi/n}$, ζ is an n-th root of unity, so its minimal polynomial has degree $d = \varphi(n)$, where φ is the Euler totient function. Therefore $\varphi(n) \leq d$. If $\tan\theta$ is rational and $\sin\theta$ is not, then $\sin\theta$ has degree 2 over $\mathbb{Q}$, so ζ has degree 2 over $Q(i)$, so ζ has degree 4 over $\mathbb{Q}$. The stated values of n for the cases $d = 4$ and $d = 8$ follow from the well-known formula for $\varphi(n)$. That completes the proof of (ii) assuming (i).

Corollary 1 *If $\sin\theta$ or $\cos\theta$ is rational, and $\theta < \pi$ is a rational multiple of π, then θ is a multiple of $2\pi/n$ where n is 5, 4, 8, 10, or 12.*

Proof. Let $\zeta = \cos\theta + i\sin\theta = e^{i\theta}$. Under the stated hypotheses, the degree of $\mathbb{Q}(\zeta)$ over $\mathbb{Q}$ is 2 or 4. Hence, by the lemma, θ is a multiple of $2\pi/n$, where $n = 5$, 8, 10, or 12 (if the degree is 4) or $n = 3$ or 6 (if the degree is 3). But the cases 3 and 6 are superfluous, since then θ is already a multiple of $2\pi/12$.

Lemma 4 *If N is a sum of two squares of rational numbers, then N is the sum of two squares of integers.*

Proof. See for example, Proposition 5.4.9, p. 314 of [6], where the theorem is attributed to Fermat. The proof given there is a one-paragraph appeal to the Hasse-Minkowski theorem, which was not available to Fermat. A simpler proof was pointed out to me by Will Sawin (on MathOverflow), which only uses the theorem that N is a sum of two integer squares if and only no prime p congruent to 3 mod 4 divides N to (exactly) an odd power. (Fermat knew that theorem.) Here is the proof: Suppose $N = p^2 + q^2$. Then, clearing denominators, $w^2 N = u^2 + v^2$ for some integers w, u, and v. Then any prime p congruent to 3 mod 4 divides the right hand side to an even power; and hence p also divides N to an even power. It follows that N is a sum of two integer squares. That completes the proof of the lemma.

Lemma 5 *Let $\mathcal{S}$ be the set of sums of two rational squares. Then $\mathcal{S}$ is closed under multiplication and division.*

Proof. The members of $\mathcal{S}$ are just the norms $u^2 + v^2$ of complex numbers $u + iv$ with rational real and imaginary parts. Thus

$$\begin{aligned}
(u^2 + v^2)(p^2 + q^2) &= |u + iv|^2 |p + iq|^2 \\
&= |(u + iv)(p + iq)|^2 \\
&= (up - vq)^2 + (vp + uq)^2
\end{aligned}$$

and

$$\frac{u^2 + v^2}{p^2 + q^2} = \frac{|u + iv|^2}{|p + iq|^2}$$

$$= \left| \frac{u + iv}{p + iq} \right|^2$$

$$= \left(\frac{up + vq}{p^2 + q^2} \right)^2 + \left(\frac{vp - uq}{p^2 + q^2} \right)^2$$

That completes the proof of the lemma.

A product of two sums of integer squares is a sum of two integer squares, as is shown directly by the proof of the previous lemma, or follows from the previous two lemmas together.

Lemma 6 *Let N be an integer. If $N/2$ is a sum of two rational squares, then $N/2$ is an integer, i.e. N is even.*

Proof. Suppose $N/2 = p^2 + q^2$. Then $2N = (2p)^2 + (2q)^2$ is also a sum of two rational squares. Then by Lemma 4, there are integers u and v such that $2N = u^2 + v^2$. If u and v are both even, then the right hand side is divisible by 4, so N is even. If u is even, then $v^2 = 2N - u^2$ is even, so v is even too; so N is even in that case. Similarly if v is even. Finally, if u and v are both odd, then $p = u + v$ and $q = u - v$ are even integers, and we have $p^2 + q^2 = 2(u^2 + v^2) = 4N$. But each of p^2 and q^2 is divisible by 4, so $p^2 + q^2$ is divisible by 8. Hence $4N$ is divisible by 8, which means N is even. That completes the proof of the lemma.

4   Quadratic and non-quadratic tilings

In this section we give a simple sufficient condition for a tiling to be quadratic.

Lemma 7 *Suppose tile T strictly tiles triangle ABC. If the tile T is similar to the triangle ABC, and there are no angle relations, then the tiling is a quadratic tiling.*

Proof. Note that since there are no angle relations, the three angles α, β, and γ are pairwise unequal: for example, if $\alpha = \beta$, then the relation $\alpha + \beta + \gamma = \pi$ implies $2\alpha + \gamma = \pi$, which is an angle relation.

Since T is similar to triangle ABC, and angle A is the smallest angle of ABC, angle $A = \alpha$. Then consider the copy T_1 of the tile that shares vertex A. Its two sides lie on the sides of triangle ABC. We can relabel the vertices B and C if necessary so that the angle of T_1 at its vertex P_1 on side AB is β, and its angle at its vertex Q_1 on side AC is γ.

There must be exactly three copies of the tile meeting at P_1, and the three angles at P_1 are (in some order) α, β, and γ, because any other vertex behavior gives rise to an angle relation. Let the tiles meeting at P_1 be T_1, T_2, and T_3, numbered so that T_2 and T_1 share a side. That shared side is a, since it is opposite angle A in T_1. Then T_2 does not have angle α at P_1, since the α vertex of T_2 has to be opposite side P_1Q_1. T_2 does not have angle β at P_1, since T_1 has angle β there, and only one β can occur at P_1. Therefore T_2 has angle γ at P_1. Therefore T_3 has angle α at P_1. Since the tiling is strict, the angle of T_3 at its second vertex P_2 on side AB must be β; otherwise the shared sides of T_2 and T_3 will have different length, since the length of that side of T_2 is b. But now, we are in the same situation with T_2 as we originally were with T_1: the two angles along side AB are α and β (in that order). We can argue as before that the three triangles T_3, T_4, and T_5 meeting at P_2 have angles β, γ, and α at P_2, in that order. Continuing down side AB in this fashion, we eventually reach a tile T_{2m-1} that has B for a vertex. Tile T_{2m-1} has its β angle at B; then the angle of ABC at B must be β (not γ, which is *a priori* possible since we may have relabeled B and C), since if the angle at B is γ, it splits, and γ is a sum of β plus some other angles, giving an angle relation, contrary to the hypothesis. There will be m copies of the tile sharing a side with AB; there will be $m - 1$ vertices $P, \ldots, P_{m-1}$ along AB, each shared by three triangles; the number of tiles used is $2m - 1$. The third vertices of these triangles are points $Q_1, \ldots, Q_{m-1}$, lying on a line parallel to AB, and the last point Q_{m-1} lies on BC, since the angle at B is β. The triangle $Q_1 C Q_{m-1}$ is thus tiled by the restriction of

the original tiling to that triangle. This restricted tiling is still strict and has no angle relations. By induction, we can assume that this restricted tiling is quadratic. Since it has $m - 1$ tiles along side $Q_1 Q_{m-1}$, we have $(m-1)^2 = N - (2m-1)$. Then $N = (m-1)^2 + 2m - 1 = m^2$. That completes the proof.

Remark. The 5-tiling in Figure 5 has T similar to ABC, but it has an angle relation $2\alpha + 2\beta = \pi$, and it also has a non-strict vertex. It is natural to ask if the hypotheses of the lemma can be weakened by dropping one or the other of the hypotheses. Does there exist a strict non-quadratic tiling in which T is similar to ABC? (Angle relations are OK.) The following lemma shows that the hypothesis that there are no edge relations cannot be dropped.

Lemma 8 *There exists a triangle ABC and integer $N = m^2$ such that ABC can be N-tiled by a tile similar to ABC, and there are no angle relations, and the sines, cosines, and tangents of α and β are irrational, but the tiling is not quadratic. (There is, however, an edge relation.)*

Proof. Take the tile to have $a = 2$ and $b = 3$. Then $3a = 2b$, which as illustrated in Fig. 14 allows us to construct a nonquadratic tiling. We are still free to choose the angle γ, or equivalently the side c (as long as they are not too large). Then α, β, and c depend continuously and monotonically on γ, so there are only countably many values of γ that satisfy an angle relation $p\alpha + q\beta + r\gamma = n\pi$, and only countably many values of γ that make any of the trig functions of α, β, or γ rational. Pick a value of γ that is not one of these countably many values. That completes the proof.

Whether or not γ avoids the countable set where there are angle relations, we can still draw a non-quadratic tiling in which all vertices are standard. Fig. 14 illustrates such a tiling for $\gamma = 60°$. (Probably there are no angle relations in that case, but we could not prove it. If there are, then there are other values of γ arbitrarily close to $60°$ with no angle relations.)

Figure 14: Here there is an edge relation $3a = 2b$, but no angle relations.
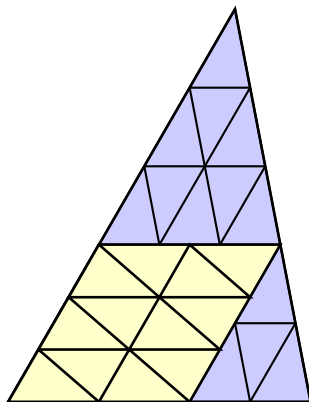
The tiling in Fig. 14 has another interesting feature: it has "one degree of freedom". That is, with its base AC fixed, point B can be varied, but not arbitrarily (as with a quadratic tiling); the requirement that $3a = 2b$ imposes a restriction on the possible variation of B. The other tilings we have seen (besides the quadratic tilings) have zero degrees of freedom.

5   The d-matrix, and a related eigenvalue problem

Let triangle ABC be tiled by the tile T, whose sides are a, b, and c. Let the sides of ABC be X, Y, and Z. We assume the triangle is labeled so that angles A, B, and C are listed in

non-decreasing order; hence also $X \leq Y \leq Z$. In case triangle ABC is similar to the tile, this implies that angle $A = \alpha$, angle $B = \beta$, and angle $C = \gamma$.

Each side X, Y, and Z is a linear combination of a, b, and c, the coefficients specifying how many tiles share sides of length a, b, and c with X, Y, or Z. These nine numbers are the entries of the matrix $\mathbf{d}$, such that

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \mathbf{d} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

If the triangle ABC is similar to the tile, then we have

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \sqrt{N} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

because each side of ABC must be $\sqrt{N}$ times the corresponding side of the tile T, in order that the area of ABC can be N times the area of T. Therefore

$$\mathbf{d} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \sqrt{N} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

That is, $\sqrt{N}$ is an eigenvalue of $\mathbf{d}$, and (a, b, c) is an eigenvector for that eigenvalue. If triangle T is isosceles, then $\mathbf{d}$ is not (yet) uniquely defined. In that case we have either $a = b$ or $b = c$; our convention is to ignore b, so that when T is isosceles, the middle column of the $\mathbf{d}$ matrix is zero. We will not make use of the $\mathbf{d}$ matrix when T is equilateral, but for completeness, we define the $\mathbf{d}$ matrix in that case to have non-zero entries only in the first column. If T is not isosceles, then the coefficients in the $\mathbf{d}$ matrix are integers between 0 and $N - 1$, inclusive, assuming $N > 2$: Not all N triangles can share a side of triangle ABC, since if $N > 2$, there would be two adjacent vertices along that side at which only two triangles meet; but then by Lemma 1, the copy of the tile between those vertices would have two right angles.

For example, consider the 5-tiling shown in Figure 5. Here the shortest side of the large triangle consists of one c, so the top row of the $\mathbf{d}$ matrix is 0 0 1. The middle side of the large triangle consists of two c's, so the middle row of the $\mathbf{d}$ matrix is 0 0 2. The longest side of the large triangles consists of one a and two b's, so the bottom row is 1 2 0. Thus the $\mathbf{d}$ matrix for this example is

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$$

and the eigenvalue equation is

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \sqrt{5} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

In this example we have $\alpha = \pi/6$, $\beta = \pi/3$, and $\gamma = \pi/2$, so $a = \sin \pi/6 = 1/2$, $b = \sin \pi/3 = \sqrt{3}/2$, and $c = \sin \pi/2 = 1$. One can check the eigenvalue equation numerically with these values.

Note that the $\mathbf{d}$ matrix for a quadratic tiling is $\sqrt{N}$ times the identity. At an early stage in this research, we conjectured that if N is a perfect square, say m^2, and $\mathbf{d}$ is m times the identity, then the tiling is quadratic. But now we know that this conjecture is not correct. Observe the tiling in Fig. 14, which was constructed starting with a quadratic tiling that, due to the edge relation $3a = 2b$, contains a rhombus. That rhombus can be tiled differently by turning the tiles of the quadratic tiling "the other way." Had we started with a larger quadratic tiling, such a rhombus could be found entirely in the interior of the tiling, leaving the $\mathbf{d}$ matrix the same as for a quadratic tiling. The moral is that the $\mathbf{d}$ matrix only tells us about the "boundary tiling", not about the interior of the tiling; and the larger N is, the more interior there is.

6 Tilings with T similar to ABC

In this section, we assume triangle ABC is N-tiled by triangle T similar to ABC. In case N is a square, we have the quadratic tiling of ABC; in this section we assume N is not a square. Let the sides of T be a, b, and c, in non-decreasing order; these are opposite the angles α, β, and γ of T. We start by disposing of a special case.

Lemma 9 *Suppose T and ABC are both equilateral, and there is an N-tiling of ABC by T. Then N is a square and the tiling is a quadratic tiling.*

Proof. Since all the angles of T and ABC are equal, and all the sides of T are equal, there is only one way to place tile T_1 at vertex B. Along side BC there must be a certain number m of copies of T; hence the side of ABC is mc, where c is the side of X. We prove by induction on m that such a tiling is a quadratic tiling using m^2 triangles. There are m tiles that share sides with BC. Call them $T_1, \ldots, T_m$. This sawtooth-like configuration requires the placement of $m - 1$ copies of T, one between each adjacent pair of triangles $T_1, \ldots, T_m$. Now we have identified a total of $2m - 1$ triangles that participate in the original tiling, and the remaining triangles tile the smaller equilateral triangle formed by deleting the tiles identified so far from ABC. The base of this triangle is smaller than the original base BC by c, the side of T. By the induction hypothesis, the tiling of this triangle is quadratic, using $(m - 1)^2$ tiles. Combining this with the row of $2m - 1$ triangles along BC, we have a quadratic tiling with a total of $(m - 1)^2 + 2m - 1 = m^2$ tiles, completing the inductive proof.

Next we review the computation of eigenvectors by cofactors. To find an eigenvector of the $\mathbf{d}$ matrix with eigenvalue $\sqrt{N}$, consider the matrix $X := \mathbf{d} - \sqrt{N}I$. An eigenvector can be found by picking any row, and then arranging the cofactors of the elements of that row as a (column) vector. If these cofactors do not all vanish, then the result is an eigenvector. (The reader may either verify this or just check directly that the particular eigenvalues produced this way below are indeed eigenvectors.)

Now we take up the general case of a tiling T with ABC similar to T, when N is not a square.

Lemma 10 *Let triangle ABC be N-tiled by tile T similar to ABC, and suppose N is not a square. Then the diagonal entries of the $\mathbf{d}$ matrix are zero.*

Proof. Since the area of ABC is N times the area of T, and T is similar to ABC, the sides of ABC are $\sqrt{N}$ times a, b, and c. Then (as discussed in a previous section) we have the eigenvalue equation

$$\mathbf{d} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \sqrt{N} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

The characteristic polynomial $f(x)$ of the $\mathbf{d}$ matrix, the determinant of $\mathbf{d} - xI$, is a cubic polynomial with integer coefficients, yet has for a zero the number $\sqrt{N}$. This is only possible if it factors into a quadratic factor and a linear factor. Since N is not a square, the quadratic factor must be a multiple of $\lambda^2 - N$. The coefficient of x^3 is -1, and so for some q we have

$$f(x) = (x^2 - N)(q - x)$$

In general the coefficient of x^2 in the characteristic polynomial of any 3 by 3 matrix $\mathbf{d}$ is the trace of $\mathbf{d}$, and the constant term is the determinant of $\mathbf{d}$. Hence q is the trace of $\mathbf{d}$ and $-Nq$ is the determinant of $\mathbf{d}$. Since the entries of $\mathbf{d}$ are non-negative integers, the trace is non-negative, so $q \geq 0$.

To avoid so many subscripts, we use separate letters for the entries in the $\mathbf{d}$-matrix, writing it as

$$\mathbf{d} = \begin{pmatrix} p & d & e \\ g & m & f \\ h & \ell & r \end{pmatrix}$$

Since the similarity factor between ABC and T is $\sqrt{N}$, there cannot be more than $\sqrt{N}$ tiles with a sides along X, the short side of ABC. That is, $p \leq \sqrt{N}$. More formally, $a\sqrt{N} = X = pa + db + ec \geq pa$, so $p \leq \sqrt{N}$. Similarly $m \leq \sqrt{N}$ and $r \leq \sqrt{N}$. Since N is not rational, we have strict inequalities: $p < \sqrt{N}$, $r < \sqrt{N}$, and $r < \sqrt{N}$. It follows that $pm < N$, etc.

We also note that there is just one tile sharing vertex A, where ABC has its α angle. That tile must have its b and c sides along AB and AC, or along AC and AB, we don't know which. Thus either f and ℓ are nonzero, or m and r are nonzero.

Suppose, for proof by contradiction, that q, the trace of $\mathbf{d}$, is not zero. Then $q = p+m+r > 0$. Since the three eigenvalues are distinct (because q is rational and $\sqrt{N}$ is not), the eigenspace corresponding to $\sqrt{N}$ is one-dimensional. The eigenvalue equation is

$$(\mathbf{d} - \lambda I)\begin{pmatrix} u \\ v \\ w \end{pmatrix} = 0$$

or showing the coefficients

$$\begin{pmatrix} p - \sqrt{N} & d & e \\ g & m - \sqrt{N} & f \\ h & \ell & r - \sqrt{N} \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = 0$$

We claim that there exists an eigenvector (u, v, w) whose components lie in $\mathbb{Q}(\sqrt{N})$. To prove this we will use the cofactor method described above.

The resulting eigenvector is (u, v, w), provided all three components are nonzero, where

$$u = \begin{vmatrix} d & e \\ m - \sqrt{N} & f \end{vmatrix}$$

$$v = -\begin{vmatrix} p - \sqrt{N} & e \\ g & f \end{vmatrix}$$

$$w = \begin{vmatrix} p - \sqrt{N} & d \\ g & m - \sqrt{N} \end{vmatrix}$$

Although we have not given a proof of the cofactor method's correctness, one can easily verify directly that the exhibited vector is indeed an eigenvector for $\sqrt{N}$; this also provides a check that no algebraic mistake has been made. The fact that all three cofactors are nonzero is really only needed to conclude directly that the eigenspace of (u, v, w) is one-dimensional; but we know that directly in our case since the eigenvalues $\sqrt{N}$, $-\sqrt{N}$, and q are distinct. It therefore suffices to check that one of the cofactors u, v, w is nonzero; then the others must automatically be nonzero because (u, v, w) is a nonzero multiple of (a, b, c). But we give direct proofs that all three cofactors are nonzero anyway, as it takes only one more paragraph.

We have $u = df - em + e\sqrt{N}$. If $u = 0$ then $e = 0$ and hence $df = 0$. If $v = 0$ then similarly $f = 0$ and $eg = 0$. Finally if $w = 0$ then $p + m = 0$ and hence $p = m = 0$, so $N = dg$.

Assume, for proof by contradiction, that $w = 0$. Then

$$\begin{aligned} (m - \sqrt{N})(p - \sqrt{N}) &= dg \\ mp + N - \sqrt{N}(p + m) &= dg \end{aligned}$$

Since $\sqrt{N}$ is irrational this means $p + m = 0$, and since p and m are nonnegative, that implies $p = 0$ and $m = 0$. Hence $dg = N$. But $d \leq (a/b)\sqrt{N}$, with equality implying that $p = d = 0$, and $g \leq (b/a)\sqrt{N}$ with equality implying $m = f = 0$. Since $dg = N$, equality must hold in both inequalities. Hence $d = (a/b)\sqrt{N}$ and $g = (b/a)\sqrt{N}$ and $p = e = m = f = 0$. But we showed above that either m and r are both nonzero, or f and ℓ are both nonzero. That is now contradicted by $m = f = 0$. This contradiction shows that $w \neq 0$.

Next we give the proof that $u \neq 0$; as remarked above, this is technically superfluous, but still it is interesting because the proof we give is not simply an abstract argument about projecting onto the one-dimensional eigenspace. Assume, for proof by contradiction, that $u = 0$. Then $df - em + e\sqrt{N} = 0$. Since $\sqrt{N}$ is not rational, $e = 0$ and $df - em = 0$. Then $df = 0$, so $d = 0$ or $f = 0$. If $d = 0$, then since both d and e are zero, side X of triangle ABC is composed of all a sides and $X = (p - \sqrt{N})a$. But since $\sqrt{N}$ is the similarity factor between T and ABC, we have $X = \sqrt{N}a$. Hence $p - \sqrt{N} = \sqrt{N}$. Hence $\sqrt{N} = p/2$, so $N^2 = p^2/4$. Hence $4N^2 = p^2$ and p is even, so N is a square, contradiction. This contradiction proves $d \neq 0$. Since $d = 0$ or $f = 0$, we have $f = 0$. Now assume, for proof by contradiction, that $g = 0$. Then since $f = 0$, side Y is composed entirely of b sides of tiles, so $Y = mb$. But $Y = b\sqrt{N}$ since $\sqrt{N}$ is the similarity factor between T and ABC. Hence $m = \sqrt{N}$, contradiction. That proves $g \neq 0$. As shown above, either f and ℓ are both nonzero or m and r are both nonzero. But $f = 0$. Hence both m and r are nonzero. Now

$$
\begin{aligned}
X &= pa + db \qquad \text{since } e = 0 \\
X &= \sqrt{N}a \\
(\sqrt{N} - p)a &= db \\
\frac{b}{a} &= \frac{\sqrt{N} - p}{d} \\
Y &= \sqrt{N}b \\
Y &= ga + mb \qquad \text{since } f = 0 \\
\frac{b}{a} &= \frac{g}{\sqrt{N} - m} \\
\frac{b}{a} &= \frac{\sqrt{N} - p}{d} = \frac{g}{\sqrt{N} - m}
\end{aligned}
$$

Cross-multiplying we have

$$
\begin{aligned}
dg &= N - (m + p)\sqrt{N} + mp \\
m + p &= 0 \qquad \text{as the coefficient of } \sqrt{N} \text{ must be zero} \\
m &= p = 0 \qquad \text{as } m \text{ and } p \text{ are nonnegative}
\end{aligned}
$$

But m was proved above to be nonzero. This contradiction completes the proof that $u \neq 0$.

Now assume $v = 0$. Then

$$
pf - eg + f\sqrt{N} = 0
$$

Since N is irrational we have $f = 0$ and $pf = eg$, but since $f = 0$ we have $eg = 0$. Hence either $e = 0$ or $g = 0$. Assume, for proof by contradiction, that $g = 0$. Then the middle side of ABC (corresponding to the middle row) is equal to mb but also to $b\sqrt{N}$, so $m = \sqrt{N}$, contradiction. This contradiction proves $g \neq 0$. Hence $e = 0$. Since either f and ℓ are both nonzero or m and r are both nonzero, and we have proved $f = 0$, then m and r are both nonzero. Now that we have $e = 0 = f$, and $m \neq 0$, we reach a contradiction by the same computation as in the case $u = 0$, shown in the series of displayed equations above. Hence $v \neq 0$.

Thus none of the three cofactors is zero. That completes the proof that there is an eigenvector (u, v, w) for the eigenvalue $\sqrt{N}$ with components in $\mathbb{Q}(\sqrt{N})$. Since the eigenspace is one-dimensional, this eigenvector is a (not necessarily rational) multiple of (a, b, c).

Recall that the third eigenvalue of the $\mathbf{d}$ matrix is the trace $q = p + m + r$. We can use the cofactor method to find an eigenvector for this eigenvalue as well, namely

$$
V = \left(\begin{vmatrix} d & e \\ m - q & f \end{vmatrix}, -\begin{vmatrix} p - q & e \\ g & f \end{vmatrix}, \begin{vmatrix} p - q & d \\ g & m - q \end{vmatrix} \right)
$$

$$= \begin{pmatrix} df - em + e(p+m+r) \\ -pf + eg + f(p+m+r) \\ pm - dg - (m+p)(m+p+r) + (m+p+r)^2 \end{pmatrix}$$

$$= \begin{pmatrix} df + e(p+r) \\ eg + f(m+r) \\ pm - dg + r(m+p+r) \end{pmatrix}$$

Technically, it is not an eigenvector until we prove that the components are not zero, but we do not need that right now; it suffices that it satisfy the eigenvalue equation. The eigenvalue equation $\mathbf{d}V = (p+m+r)V$ is

$$\begin{pmatrix} p & d & e \\ g & m & f \\ h & \ell & r \end{pmatrix} \begin{pmatrix} df - e(p+r) \\ eg + f(m+r) \\ pm - dg + r(m+p+r) \end{pmatrix} = (p+m+r) \begin{pmatrix} df - e(p+r) \\ eg + f(m+r) \\ pm - dg + r(m+p+r) \end{pmatrix} \quad (2)$$

The first component of this vector equation is

$$p(df - e(p+r)) + d(eg + f(m+r)) + e(pm - dg + r(m+p+r)) \quad = \quad (p+m+r)(df - e(p+r))$$

Multiplying out and cancelling like terms, and dividing by 2, we find

$$epm + er(m+p+r) = 0.$$

We argue by cases, according to whether $e = 0$ or not. We first take up the case that $e \neq 0$. Then $pm + r(m+p+r) = 0$. Since these terms are nonnegative, they are both zero. Hence $pm = 0$ and $r(m+p+r) = 0$. Hence $r = 0$ or $m+p+r = 0$. In either case $r = 0$. Writing out the third component of the eigenvalue equation, and setting $r = 0$, we have

$$\begin{aligned} h(df - ep) + \ell(eg + fm) &= (p+m)(pm - dg) \\ h(df - ep) + \ell(eg + fm) &= -(p+m)dg \qquad \text{since } pm = 0 \\ hdf - hep + \ell eg + \ell fm &= -pdg - mdg \end{aligned}$$

Now we write out the second component of the eigenvalue equation (2), setting $r = 0$:

$$\begin{aligned} g(df - ep) + m(eg + fm) + f(pm - dg) &= (p+m)(eg + fm) \\ gdf - gep + m(eg + fm) + fpm - fdg &= peg + pfm + m(eg + fm) \\ peg &= 0 \\ pg &= 0 \qquad \text{since } e \neq 0 \end{aligned}$$

Assume, for proof by contradiction, that $m \neq 0$. Then since $mp = 0$ we have $p = 0$. The equation $N(p+m) = hdg + \ell pf + \ell eg$ becomes

$$Nm = hdg + \ell eg.$$

The third component of the eigenvalue equation becomes, with $p = 0$,

$$hdf + \ell eg + \ell fm \quad = \quad -mdg$$

The left side is ≥ 0 and the right side is ≤ 0. Hence both sides are equal to zero. Since $m \neq 0$ and $e \neq 0$, we have $dg = 0$ and $hdf = 0$ and $\ell g = 0$ and $\ell f = 0$. We derived above (by observing that the b and c sides of the tile at vertex A lie on the two adjacent sides of ABC) that either f and ℓ are both nonzero or m and r are both nonzero. Since $r = 0$ we must have f and ℓ both nonzero. Hence $\ell f = 0$ is a contradiction. That contradiction completes the proof that $m = 0$.

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Now assume, for proof by contradiction, that $p \neq 0$. Then since $pg = 0$ we have $g = 0$. Then the equation $N(p + m) = hdg + \ell pf + \ell eg$ becomes $Np = \ell pf$. Canceling p we have $N = \ell f$. But as proved above, $\ell f \leq N$, and equality holds if and only if AC is composed only of c sides of tiles and AB is composed only of b sides. Therefore we have $h = 0$ as well as $g = m = r = 0$. Then since $h = 0$ and $r = 0$, the long side AB of ABC is composed entirely of b sides of tiles.

If T is isosceles, then by convention the middle column of the **d**-matrix is zero. Since $\ell f = N$, we now have $\ell \neq 0$, so the middle column is not zero, and T is not isoceles.

Since side AB is composed entirely of b sides of tiles, there are equally spaced vertices $V_0 = A, V_1, \ldots, V_\ell$, spaced b apart, each one of which is one side of a tile T_i. Tile T_1, which has vertices at A and V_1, has its α angle at V_0. All these tiles have their β angles in the interior of ABC, and their α and γ angles at the V_i. If $\gamma > \pi/2$ then there is only one possible orientation for these tiles, as two γ angles will not fit at any V_i. In that case the angle of the last tile at vertex B must be γ, contradiction, since the angle there cannot exceed β, and $\beta \neq \gamma$ since then T would be isosceles. Hence $\gamma \leq \pi/2$.

In particular, the tile that shares vertex B has its b side along AB. Therefore the tile sharing vertex B and part of side AB has its α angle at B, and the angle β at vertex B splits into some number of α angles, so for some number J, we have $\beta = J\alpha$. Somewhere along AB there must occur a vertex V_k at which both the tile T_k and the tile T_{k+1} have angle γ. There is not room at V_k for a third tile, since $2\gamma + \alpha > \alpha + \beta + \gamma = \pi$. Hence there are exactly those two tiles at V_k, and we have $\gamma = \pi/2$.

Since γ is a right angle, we must have $a^2 + b^2 = c^2$. Since (u, v, w) is a multiple of (a, b, c) we also have $u^2 + v^2 = w^2$. We now compute these expressions from the formulas for (u, v, w). In view of $m = g = 0$ we have

$$
\begin{aligned}
u &= ef - e\sqrt{N} \\
v &= f\sqrt{N} - fp \\
w &= N - p\sqrt{N}
\end{aligned}
$$

Squaring these equations we have

$$
\begin{aligned}
u^2 &= e^2(f^2 + N - 2f\sqrt{N}) \\
v^2 &= f^2(N + p^2 - 2p\sqrt{N}) \\
w^2 &= N^2 + p^2 N - 2pN\sqrt{N}
\end{aligned}
$$

Setting $u^2 + v^2 = w^2$ we find

$$
e^2 f^2 + e^2 N + f^2 N + f^2 p^2 - 2(e^2 f - f^2 p)\sqrt{N} = N^2 + p^2 N - 2pN\sqrt{N}
$$

Equating the coefficients of $\sqrt{N}$ and equating the rational parts, we have

$$
\begin{aligned}
e^2 f - f^2 p &= pN \\
e^2 f^2 + e^2 N + f^2 N + f^2 p^2 &= N^2 + p^2 N
\end{aligned}
$$

Since γ is a right angle, $\alpha + \beta = \pi/2$. Since $\beta = J\alpha$, we have $\alpha = \pi/(2(J+1))$, so α is a rational multiple of 2. We have $\tan \alpha = b/a = v/u$, which belongs to $\mathbb{Q}(\sqrt{N})$. We have

$$
\cos \alpha = \frac{u}{u^2 + v^2}
$$

which is also in $\mathbb{Q}(\sqrt{N})$. Similarly $\sin \alpha$ belongs to $\mathbb{Q}(\sqrt{N})$. Then $\zeta = d^{i\alpha}$ is of degree 4 over $\mathbb{Q}$, since $\mathbb{Q}(\zeta) = \mathbb{Q}(i, \sqrt{N})$. By Lemma 3, $4(J+1)$ is 5, 8, 10, or 12. Since 5 and 10 are not divisible by 4, we have $4(J + 1) = 8$ or 10. But if $4(J + 1) = 8$ then $J = 1$, while we have $J \geq 2$ since $\beta = J\alpha$. The only remaining possibility is $4(J + 1) = 12$, which makes $J = 2$. Then $\alpha = \pi/6$

and $2\beta = \alpha$, so $\beta = \pi/3$. Then $a = \sin\alpha = 1/2$, $b = \sqrt{3}/2$ and $c = 1$. But now $\overline{AC} = fc = f$, and

$$
\begin{aligned}
\ell b &= \overline{AB} \\
&= \frac{2}{\sqrt{3}}\overline{AC} \\
&= \frac{2}{\sqrt{3}}fc
\end{aligned}
$$

Now we put in $c = 1$ and $b = \sqrt{3}/2$:

$$
\begin{aligned}
\ell\frac{\sqrt{3}}{2} &= \frac{2}{\sqrt{3}}f \\
3\ell &= 4f
\end{aligned}
$$

We have $N = \ell f = (4/3)f^2$, so $3N = 4f^2$, so f is divisible by 3, say $f = 3k$; then $N = 3(2k)^2$ is three times a square. It remains to show that $p = 0$; in fact we claim $p = 0$ and $e = 0$, so side AC is also composed entirely of b sides of triangles. We have

$$
\begin{aligned}
\overline{BC} &= \frac{1}{2}\overline{AB} \\
&= \frac{1}{2}\ell b \\
&= \frac{\ell\sqrt{3}}{4} \\
&= pa + db + ec
\end{aligned}
$$

Now we put in the values $a = 1/2$, $b = \sqrt{3}/2$, and $c = 1$.

$$
\frac{\ell\sqrt{3}}{4} = \frac{p}{2} + d\frac{\sqrt{3}}{2} + e
$$

This is an equation in $\mathbb{Q}(\sqrt{3})$. Equating the rational parts we have $0 = p/2 + e$. Since both $p/2$ and e are nonnegative, we have $p = 0$ and $e = 0$, as claimed. In particular $p = 0$ so the diagonal elements are nonzero, which is the conclusion of the theorem; or we could say, in particular $e = 0$, contradicting the assumption $e \neq 0$ and completing the analysis of that case.

Therefore we may now assume $e = 0$. Remember that $r = 0$ was derived only under the assumption $e \neq 0$, so the equation $r = 0$ is no longer in force. The third component of the eigenvalue equation (2) is (substituting $e = 0$)

$$
hdf + \ell f(m + r) + r(pm - dg + r(m + p + r)) = (p + m + r)(pm + r(m + p + r))
$$

Subtracting $r^2(m + p + r)$ from both sides we have

$$
\begin{aligned}
hdf + \ell fm + \ell fr + rpm - rdg &= (p + m + r)pm + (p + m)r(m + p + r) \\
hdf + \ell fm + \ell fr + rpm - rdg &= (p + m + r)(pm + pr + mr) \qquad (3)
\end{aligned}
$$

To get rid of h and ℓ, we expand the determinant of the $\mathbf{d}$ matrix by cofactors on the bottom row. That determinant is $-Nq = -N(p + m + r)$, so we have (remembering $e = 0$)

$$
\begin{aligned}
-N(p + m + r) &= h\begin{vmatrix} p & d \\ g & m \end{vmatrix} - \ell\begin{vmatrix} p & e \\ g & f \end{vmatrix} + r\begin{vmatrix} p & d \\ g & m \end{vmatrix} \\
&= hdf + \ell pf + rpm - rdg
\end{aligned}
$$

Adding and subtracting ℓpf to the left side of (3) the expression for the determinant appears, and we have

$$
\begin{aligned}
hdf + \ell pf + rpm - rdg + \ell fm + \ell fr - \ell pf &= (p+m+r)(pm+pr+mr) \\
-N(p+m+r) + \ell fm + \ell fr - \ell pf &= (p+m+r)(pm+pr+mr)
\end{aligned}
$$

Moving everything to the right side we have

$$
0 = (p+m+r)(pm+pr+mr) + (N-\ell f)(m+r) + (N+\ell f)p \tag{4}
$$

Since ℓ is the number of b sides of tiles on the long side $\sqrt{N}c$ of ABC, we have $\ell b \le \sqrt{N}c$, or $\ell \le (c/b)\sqrt{N}$. Since f is the number of c sides of tiles on AC, whose length is $\sqrt{N}b$, we have $fc \le \sqrt{N}b$, or $f \le (b/c)\sqrt{N}$. Hence

$$
\begin{aligned}
\ell f &\le \left(\frac{c}{b}\sqrt{N}\right)\left(\frac{b}{c}\sqrt{N}\right) \\
\ell f &\le N
\end{aligned}
$$

Hence all the terms on the right of (4) are nonnegative. Hence each of them is zero. In particular $(N+\ell f)p = 0$; but $N + \ell f > 0$, so $p = 0$. Then then equation becomes

$$
(m+r)mr + (N - \ell f)(m+r) = 0
$$

If $m + r = 0$ then $m = 0 = r$ and the lemma is proved. Hence we may assume $mr = 0$ and $N = \ell f$. But if $N = \ell f$ then we must have equality in the two inequalities $\ell \le (c/b)\sqrt{N}$ and $f \le (b/c)\sqrt{N}$. This implies that side AC is composed only of c sides of tiles and side AB is composed only of b sides of tiles, so $g = m = h = r = 0$. In particular $m = r = 0$. That completes the proof of the lemma.

We pause to observe that the $\mathbf{d}$ matrix for a biquadratic tiling, in case $N = m^2 + n^2$, has the form

$$
\mathbf{d} = \left(\begin{array}{ccc} 0 & 0 & n \\ 0 & 0 & m \\ n & m & 0 \end{array}\right)
$$

which does satisfy the conditions above (as it must). The hypothesis that N is not a square is necessary, as shown by the 9-tiling in Fig. 9. Its $\mathbf{d}$ matrix is

$$
\left(\begin{array}{ccc} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 3 \end{array}\right)
$$

and as predicted, the determinant is zero, but the trace is not zero, and the characteristic polynomial is $-x(x-3)^2$.

Continuing with the general case of N not a square, some further conclusions can be drawn about the $\mathbf{d}$ matrix. We have shown that $p = m = r = 0$. The determinant is then given by

$$
\det \mathbf{d} = dfh + eg\ell
$$

Since the matrix entries are nonnegative, that means that each of these two terms must contain a zero factor. In particular, at most four entries in the $\mathbf{d}$ matrix are nonzero.

The negated coefficient of λ in the characteristic equation is (since the diagonal elements are zero) the sum of paired products of off-diagonal elements:

$$
N = dg + eh + fl \tag{5}
$$

But at least one of these three terms will be zero, as shown above.

In view of Lemma 10, the $\mathbf{d}$ matrix becomes

$$\mathbf{d} = \begin{pmatrix} 0 & d & e \\ g & 0 & f \\ h & \ell & 0 \end{pmatrix} \tag{6}$$

and the matrix equation

$$\mathbf{d} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \sqrt{N} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

becomes the three equations

$$\begin{aligned} db + ec &= \sqrt{N}a \\ ga + fc &= \sqrt{N}b \\ ha + \ell b &= \sqrt{N}c \end{aligned}$$

Lemma 11 *Suppose ABC is N-tiled by tile T similar to ABC, and N is not a square. Then γ is a right angle.*

Proof. First we note that T and ABC are not equilateral, by Lemma 9. Next we will prove that T and ABC are not isosceles with $\beta = \gamma$. Assume, for proof by contradiction, that $\beta = \gamma$. Then, by our definition of the $\mathbf{d}$ matrix, the middle column of the $\mathbf{d}$ matrix is zero, i.e. b is counted as c. Then we have $d = \ell = 0$ and

$$\mathbf{d} = \begin{pmatrix} 0 & 0 & e \\ g & 0 & f \\ h & 0 & 0 \end{pmatrix} \tag{7}$$

That implies that the short side BC of triangle ABC has only c sides of tiles on it, and the long side AB has only a sides of tiles on it. At the vertex A, there can only be one tile, since the angle at A is the smallest angle α so there can be no vertex splitting. This tile has one side of length a opposite angle A and another along side AB. Hence $a = b$. Since T is not equilateral, we must have $b < c$ and $\beta < \gamma$. This contradicts the assumption $\beta = \gamma$, and thus completes the proof by contradiction that $\beta < \gamma$.

Since the $\mathbf{d}$ matrix has zeroes on the diagonal, no c sides of tiles occur along the longest side AB of triangle ABC; only a and b sides occur there. Since there are no c edges of tile on AB, every tile with an edge on AB has a γ angle on AB. No γ angle occurs at the endpoints A and B, since the angles there are α and β respectively, and both are less than γ. By the pigeon-hole principle, there is a vertex V on AB such that two tiles, say T_1 and T_2, have their γ angles at V. At that point we know $\gamma \leq \pi/2$. Assume, for proof by contradiction, that $\gamma \neq \pi/2$. Then there is at least one additional copy T' of the tile between T_1 and T_2, sharing vertex V. If T' has its γ angle at V then there are exactly those three tiles meeting at V (else $\gamma < \pi/3$) and we have $\gamma = \pi/3$, and hence T is equilateral, which as noted above is impossible. Hence none of the additional tiles T' meeting at V have a γ angle at V. None of the tiles T' can contribute a β angle at V either, since $2\gamma + \beta > \pi$. Hence there is an angle relation $2\gamma + p\alpha = \pi$, where p additional tiles contribute α each to the angle sum at V, and $p > 0$. But $2\gamma + p\alpha > \gamma + \beta + \alpha = \pi$, since $\gamma > \beta$. This contradiction completes the proof of the lemma.

Lemma 12 *Suppose ABC is N-tiled by tile T similar to ABC, and N is not a square. Then $f = \mathbf{d}_{21}$ is not zero.*

Proof. Suppose, for proof by contradiction, that $f = 0$. Then the middle row of the $\mathbf{d}$ matrix is $(g, 0, 0)$, which means that all the tiles along side AC of triangle ABC share their a sides with AC. At vertex A, where ABC has its smallest angle α, there is exactly one tile T_1, with angle

α at a. Hence both the side of T_1 opposite that angle, and the side shared with AC, are equal to a. Thus T is isosceles. In that case, by convention we have agreed to write the $\mathbf{d}$ matrix with zeroes in the second column, so the $\mathbf{d}$ matrix has the form

$$\mathbf{d} = \begin{pmatrix} 0 & 0 & e \\ g & 0 & 0 \\ h & 0 & 0 \end{pmatrix} \tag{8}$$

Now the bottom row is $(h, 0, 0)$, which means that all the tiles along side AB share their a sides with AB. In particular the tile at vertex A has an a side along AB. But we have already seen that its other two sides are a. Hence the tile is equilateral, contradicting Lemma 9, since N is not a square. That completes the proof.

There are six letters for coefficients in the $\mathbf{d}$ matrix (since the three diagonal elements are zero), but for any specific tiling, at most four of those coefficients are nonzero. We will analyze some special cases. The case corresponding to the biquadratic tilings is $d = 0$ and $g = 0$. We call that the "biquadratic case". In the biquadratic case the $\mathbf{d}$ matrix has the form

$$\mathbf{d} = \begin{pmatrix} 0 & 0 & e \\ 0 & 0 & f \\ h & \ell & 0 \end{pmatrix} \tag{9}$$

Equation (5) now becomes

$$eh + \ell f \quad = \quad N \tag{10}$$

We compute the eigenvector in the biquadratic case, using the cofactor method described above. Let

$$X = \begin{pmatrix} -\sqrt{N} & 0 & e \\ 0 & -\sqrt{N} & f \\ h & \ell & -\sqrt{N} \end{pmatrix}$$

Taking the cofactors of the bottom row (notice the minus sign in the second component, which comes from the definition of "cofactor") we find the eigenvector

$$\left(\begin{vmatrix} 0 & e \\ -\sqrt{N} & f \end{vmatrix}, - \begin{vmatrix} -\sqrt{N} & e \\ 0 & f \end{vmatrix}, \begin{vmatrix} -\sqrt{N} & 0 \\ 0 & -\sqrt{N} \end{vmatrix} \right) \quad = \quad \begin{pmatrix} e\sqrt{N} \\ f\sqrt{N} \\ N \end{pmatrix}$$

Note that $e \neq 0$ and $f \neq 0$, since the first two sides of ABC are given by ec and fc. Hence the cofactors do not vanish.

We claim that the bottom two rows of $\mathbf{d} - \sqrt{N}I$, namely $(0, -\sqrt{N}, f)$ and $(h, \ell, -\sqrt{N})$, are linearly independent. Indeed, suppose that for some constants p and q we have $p(0, -\sqrt{N}, f) + q(h, \ell, -\sqrt{N}) = 0$. From the first component we see that $qh = 0$. From the third component we see that $pf = q\sqrt{N}$. If q is not zero, then $\sqrt{N} = pf/q$, contradicting the irrationality of $\sqrt{N}$. Hence $q = 0$. Hence from the second component, $p\sqrt{N} = 0$. Hence $p = 0$. This proves that the bottom two rows of $\mathbf{d} - \sqrt{N}I$ are linearly independent. Hence $\mathbf{d} - \sqrt{N}I$ has rank 2; hence the eigenspace associated with the eigenvalue $\sqrt{N}$ is one-dimensional. It follows that the eigenvector computed above is a multiple of (a, b, c). That is, for some constant μ we have

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad = \quad \mu \begin{pmatrix} e\sqrt{N} \\ f\sqrt{N} \\ N \end{pmatrix} \tag{11}$$

The constant μ is an arbitrary scale factor; changing μ just changes the size of the tile T and the triangle ABC by the same factor. We are therefore free to choose μ to suit our convenience.

We choose to take $\mu = \sqrt{N}$; then we have

$$
\begin{pmatrix} a \\ b \\ c \end{pmatrix} \;=\; \begin{pmatrix} e \\ f \\ \sqrt{N} \end{pmatrix}
\tag{12}
$$

Lemma 13 *Let triangle ABC be N-tiled by T, and suppose N is not a square and T is similar to ABC, and $d = g = 0$ (the biquadratic case). Then N is a sum of squares, specifically $N = e^2 + f^2$ where e and f are as above, and $\tan \alpha = e/f$. In particular $\tan \alpha$ is rational.*

Proof. By Lemma 11, we have $\gamma = \pi/2$. By the Pythagorean theorem and (12) we see that $\gamma = \pi/2$ if and only if $e^2 + f^2 = N$. Since $\gamma = \pi/2$, we have $\tan \alpha = a/b$, and by (11), $\tan \alpha = e/f$. That completes the proof of the lemma.

Lemma 14 *Suppose ABC is N-tiled by tile T similar to ABC, and N is not a square, and $d = g = 0$ (the biquadratic case). Then the right angle of ABC is split by the tiling, and the tangents of the other angles of ABC are rational.*

Proof. We suppose, as always, that the γ angle of ABC is at C, the β angle at B, and the α angle at A. Since the $\mathbf{d}$ matrix has the form given in (9), all the tiles along side BC share their c sides with BC (there are e of them) and all the tiles along side AC share their c sides with AC (there are f of them). Suppose, for proof by contradiction, that the vertex at C is not split. Then a single tile shares vertex C, so the tile has two c sides, and hence is isosceles with $b = c$. But by (12), we have $c/b = \sqrt{N}/e$. Hence if $b = c$ we have $N = e^2$, contradicting the hypothesis that N is not a square. Hence the vertex C is split as claimed. The tangents of the other two angles are e/f and f/e, which are rational. This completes the proof of the lemma.

We now turn to another important case, when $e = 0$. We call this the "triple-square case", because it will turn out that in this case N must be three times a square. The following lemma and its proof give a complete analysis of this case.

Lemma 15 *Suppose ABC is not equilateral and is N-tiled by tile T similar to ABC, and N is not a square, and $e = 0$ (the triple-square case). Then $\alpha = \pi/6$, $\beta = \pi/3$, and $N = 3d^2$ is three times a square.*

Remark. There do exist tilings for each N of the form $3d^2$ that fall under the triple square case, as we showed in Fig. 11 and Fig. 12.

Proof. Under the hypotheses of the lemma we have

$$
\mathbf{d} = \begin{pmatrix} 0 & d & 0 \\ g & 0 & f \\ h & \ell & 0 \end{pmatrix}.
$$

In this matrix, d and $h + \ell$ are not zero, since they represent the number of tiles along BC and AB, respectively. By Lemma 12 we have $f \neq 0$. We have

$$
X = \mathbf{d} - \sqrt{N}I = \begin{pmatrix} -\sqrt{N} & d & 0 \\ g & -\sqrt{N} & f \\ h & \ell & -\sqrt{N} \end{pmatrix}
$$

We will prove that the bottom two rows of X are linearly independent. If they are linearly dependent, then for some p and q, we have

$$
\begin{aligned}
0 &= p(g, -\sqrt{N}, f) + q(h, \ell, \sqrt{N}) \\
&= pg + pf + q\ell + qh + \sqrt{N}(q - p)
\end{aligned}
$$

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Since N is not a square, the coefficient of $\sqrt{N}$ is zero, so $q = p$, and

$$
\begin{aligned}
0 &= pg + pf + q\ell + qh \\
&= pg + pf + p\ell + ph \\
&= p(g + f + \ell + h)
\end{aligned}
$$

Since the entries of the $\mathbf{d}$ matrix are non-negative, and $h + \ell$ is strictly positive, we conclude $p = q = 0$. That proves that the bottom two rows of X are linearly independent, so X has rank 2 and the eigenspace of $\sqrt{N}$ is one-dimensional. We then compute the eigenvector by the cofactor method. Taking the cofactors of the bottom row, we find the eigenvector

$$
\left(\begin{vmatrix} 0 & e \\ -\sqrt{N} & f \end{vmatrix}, - \begin{vmatrix} -\sqrt{N} & 0 \\ g & f \end{vmatrix}, \begin{vmatrix} -\sqrt{N} & d \\ g & -\sqrt{N} \end{vmatrix} \right) = \left(\begin{array}{c} df \\ f\sqrt{N} \\ N - dg \end{array} \right)
$$

Since $d \neq 0$ and $f \neq 0$, the first two components are not zero. From the first row of the $\mathbf{d}$ matrix we have $db \leq \sqrt{N}a$, and from the second row, we have $ga < \sqrt{N}b$, with strict inequality because $f \neq 0$. Hence $dg < \sqrt{N}(a/b)\sqrt{N}(b/a) = N$. Hence the third component is nonzero.

Since (a, b, c) is an eigenvector and the eigenspace is one dimensional, (a, b, c) is a multiple of this computed eigenvector. By scaling the triangle appropriately we can assume (a, b, c) is actually equal to the computed eigenvector:

$$
\left(\begin{array}{c} a \\ b \\ c \end{array} \right) = \left(\begin{array}{c} df \\ f\sqrt{N} \\ N - dg \end{array} \right)
$$

It follows that $\sin \alpha = a/c = fd/(N - dg)$ is rational and $\tan \alpha = a/b = d/\sqrt{N}$ is of degree 2 over $\mathbb{Q}$.

According to the first row of the $\mathbf{d}$ matrix, the tiles along BC have only b sides on BC. Assume, for proof by contradiction, that vertex B does not split. Then there is a single tile T_1 at vertex B, which therefore shares one side with AB and one side with BC. Triangle T is not isosceles, since then by definition the $\mathbf{d}$ matrix would have zeroes in the middle column. Hence the unique b side of T_1 must be opposite angle B; but it must also lie on BC, which is a contradiction. Hence vertex B does split. Therefore for some integer P we have $\beta = P\alpha$. Since by Lemma 11, $\gamma = \pi/2$, we have

$$
\begin{aligned}
\frac{\pi}{2} &= \alpha + \beta \\
&= \alpha + P\alpha \\
&= (P + 1)\alpha
\end{aligned}
$$

Therefore

$$
\alpha = \frac{\pi}{2(P + 1)} = \frac{2\pi}{4(P + 1)}.
$$

By Lemma 3 we conclude that $4(P + 1)$ is one of the numbers $n = 3, 4, 5, 8, 10,$ or 12 for which $\phi(n) = 4$. Of these numbers, only $4, 1,$ and 12 are divisible by 4, which implies $P = 2$, since the values $P = 0$ and $P = 1$ do not correspond to vertex splitting. Hence $P = 2$ and we have $\beta = 2\alpha$, so $\alpha + \beta = \pi/2 = 3\alpha$, and $\alpha = \pi/6$. Hence $\tan \alpha = df/(f\sqrt{N}) = d/\sqrt{N} = 1\sqrt{3}$. Hence $N = 3d^2$. That completes the proof of the lemma.

Now we have dealt with the biquadratic case (when $d = g = 0$) and the triple-square case (when $e = 0$). It remains to show that these are the only two possible cases, when N is not a square and T is similar to ABC. Recall that $dfh = 0$ and $eg\ell = 0$ since $\det \mathbf{d} = 0$; that leaves only a few possibilities to consider. We begin by showing that if $d = 0$ then we are already in the biquadratic case.

Lemma 16 *Assume triangle ABC is N-tiled by T, that N is not a square, that T is similar to ABC, and that d and g are two entries in the $\mathbf{d}$ matrix of the tiling, in the notation used above (the ones that are zero in the biquadratic case). Then $d = 0$ implies $g = 0$, i.e. we are in the biquadratic case as soon as $d = 0$.*

Proof. We have

$$X = \mathbf{d} - \sqrt{N}I = \begin{pmatrix} -\sqrt{N} & d & e \\ g & -\sqrt{N} & f \\ h & l & -\sqrt{N} \end{pmatrix}$$

By the cofactor method described above we compute the eigenvector

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} df + e\sqrt{N} \\ f\sqrt{N} + eg \\ N - dg \end{pmatrix}$$

Suppose, for proof by contradiction, that $d = 0$ and $g \neq 0$. Then

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} e\sqrt{N} \\ f\sqrt{N} + eg \\ N \end{pmatrix}$$

By Lemma 11, γ is a right angle, so by the Pythagorean theorem, we have $a^2 + b^2 = c^2$. That is,

$$e^2 N + (f\sqrt{N} + eg)^2 = N^2$$
$$e^2 N + f^2 N + 2egf\sqrt{N} + e^2 g^2 = N^2$$

Since N is not a square, the coefficient of $\sqrt{N}$ is zero; that is, $egf = 0$. By hypothesis, $g \neq 0$, so $ef = 0$. The first row of the $\mathbf{d}$ matrix is $(0, d, e) = (0, 0, e)$, so $e \neq 0$ because there must be some triangles on the first side of ABC. Therefore $f = 0$. Then the $\mathbf{d}$ matrix is

$$\begin{pmatrix} 0 & 0 & e \\ g & 0 & 0 \\ h & \ell & 0 \end{pmatrix}$$

Hence all the tiles on the middle side AC of ABC have their a side on AC, and all the tiles on the hypotenuse AB do not have their c side on AB. Consider the tile T_1 sharing vertex A (there is only one, since ABC has angle α there). It has its a side on AC and does not have its c side on AB. Hence its b side is on AB and its c side opposite angle A, which is α. Hence $a = c$ and triangles T and ABC are equilateral, which is a contradiction since $\gamma = \pi/2$. This contradiction shows that the assumption $d = 0$ and $g \neq 0$ is untenable, which completes the proof of the lemma.

Lemma 17 *Assume triangle ABC is N-tiled by T, that N is not a square, that T is similar to ABC, and that $d \neq 0$. Then $e = 0$, i.e. we are in the triple-square case as soon as $d \neq 0$.*

Proof. We have as in the proof of the previous lemma

$$\mathbf{d} = \begin{pmatrix} 0 & d & e \\ g & 0 & f \\ h & \ell & 0 \end{pmatrix}$$

$$X = \mathbf{d} - \sqrt{N}I = \begin{pmatrix} -\sqrt{N} & d & e \\ g & -\sqrt{N} & f \\ h & \ell & -\sqrt{N} \end{pmatrix}$$

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$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} df + e\sqrt{N} \\ eg + f\sqrt{N} \\ N - dg \end{pmatrix}$$

By Lemma 11 and the Pythagorean theorem we have

$$\begin{aligned} 0 &= c^2 - a^2 - b^2 \\ &= (N - dg)^2 - (df + e\sqrt{N})^2 - (eg + f\sqrt{N})^2 \\ &= -2(def + egf)\sqrt{N} + \text{ rational} \end{aligned}$$

Since N is not a square and the entries of the $\mathbf{d}$ matrix are nonnegative integers, we have $def = 0$ and $egf = 0$. Since $f \neq 0$ by Lemma 12, and $d \neq 0$ by hypothesis, we have $e = 0$. That completes the proof of the lemma.

The following theorem completely answers the question, "for which N does there exist an N-tiling in which the tile is similar to the tiled triangle?"

Theorem 1 *Suppose ABC is N-tiled by tile T similar to ABC. Then either N is a square, or a sum of two squares, or three times a square.*

Proof. Suppose N is not a square. Then by Lemma 11, γ is a right angle. Now consider the $\mathbf{d}$ matrix. By Lemma 10 the diagonal entries are zero, so as stated in (6) the $\mathbf{d}$ matrix has the form

$$\mathbf{d} = \begin{pmatrix} 0 & d & e \\ g & 0 & f \\ h & \ell & 0 \end{pmatrix}$$

By Lemma 16, if $d = 0$ then also $g = 0$, i.e. we are in the "biquadratic case". Then by Lemma 13, N is a sum of squares. If $e = 0$ then by Lemma 15, N is three times a square and T is a 30-60-90 triangle. Finally, Lemma 17 shows that the cases $d = 0$ and $e = 0$ are exhaustive. That completes the proof of the theorem.

As for characterizing the triples (ABC, N, T), we have the following results:

Theorem 2 *Suppose ABC is N-tiled by tile T similar to ABC. If N is not a square, then T and ABC are right triangles. Then either*

(i) N is three times a square and T is a 30-60-90 triangle, or

(ii) N is a sum of squares $e^2 + f^2$, the right angle of ABC is split by the tiling, and the acute angles of ABC have rational tangents e/f and f/e,

*and these two alternatives are mutually exclusive.*

Proof. First we prove that N cannot be both a sum of squares and three times a square, since the equation $x^2 + y^2 = 3z^2$ has no integer solutions. To see that, we can assume without loss of generality that x, y, and z are not all even. Note that squares are always congruent to 0 or 1 mod 4, so the left side is 0, 1, or 2 mod 4. Then z^2 must be congruent to 0 mod 4, since if not, the right side is congruent to 3 mod 4. Hence z is even. But x and y must also be even to make the left side congruent to 0 mod 4, contradiction. Hence the equation has no solutions. Thus the alternatives in the theorem are mutually exclusive, as claimed. The rest of the theorem follows from lemma:gammaright-sumofsquares, Lemma 11 and Lemma 15.

Note that the 9-tiling in Fig. 9 shows that not every m^2-tiling is a quadratic tiling, so we have not even classified all the m^2-tilings. Briefly we conjectured that a tiling in which the $\mathbf{d}$-matrix is m times the identity should be a quadratic tiling, but that is not true. One can extend the 9-tiling in Fig. 9 by adding more triangles to the right and below, producing a 25-tiling in which the $\mathbf{d}$-matrix is 5 times the identity.

7 Tilings of an isoceles triangle by a right triangle

Let us first review the known examples of tilings of an isosceles triangle ABC by a right triangle T. There is always the "double quadratic" tiling, in which one divides ABC into two halves by the altitude and then quadratically tiles each half. One might conjecture that *any* tiling contains the altitude, i.e. is a subtiling of the 2-tiling defined by the altitude. But this is easily refuted: consider the second tiling in Fig. 6. In that case ABC is equilateral; take the base BC as one of the slanting sides in Fig. 6. This is not a very satisfying counterexample since the triangle can be rotated so that, with a different vertex and base, the tiling does contain the altitude. Fig. 15 gives a counterexample in which ABC is not equilateral.

Figure 15: A tiling that does not include the altitude of ABC
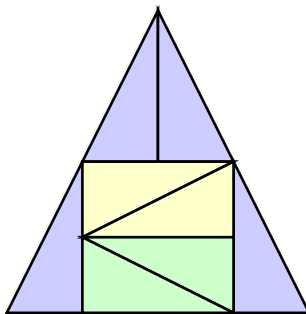
Whenever there is an N-tiling of the right triangle ABM, there is a $2N$-tiling of the isosoceles triangle ABC. Using the biquadratic tilings (see Fig. 5 and Fig. 6 and triple-square tilings (see Fig 10 and Fig. 11), we can produce $2N$-tilings when N is a sum of squares or three times a sum of squares. We call these tilings "double biquadratic" and "hexquadratic". For example, one has two 10-tilings and two 26-tilings, obtained by reflecting Figs. 4 and 5 about either of the sides of the triangles shown in those figures; and one has 24-tilings and 54-tilings obtained from Figs. 8 and 9. Note that in the latter two cases, ABC is equilateral.

In the case when the sides of the tile T form a Pythogorean triple $n^2 + m^2 + k^2 = N/2$, then we can tile one half of ABC with a quadratic tiling and the other half with a biquadratic tiling. The smallest example is when the tile has sides 3, 4, and 5, and $N = 50$. See Fig. 16. One half is 25-tiled quadratically, and the other half is divided into two smaller right triangles which are 9-tiled and 16-tiled quadratically. This shows that the tiling of ABC does not have to be symmetric about the altitude.

One looks for other ways to tile the two halves of ABC differently. We cannot use two different biquadratic tilings, even if $N/2$ can be written as a sum of squares in two different ways, because the tiles would have to be different shapes. We cannot use a biquadratic tiling and a triple-square tiling, since $m^2 + n^2 = 3k^2$ has no solutions. In addition to the quadratic-biquadratic tilings corresponding to Pythagorean triples, we could, for certain isosceles ABC, tile the two halves with tilings differing by having the tiles in a certain internal square "turned the other way." For example, one half could be 9-tiled by a quadratic tiling, and the other half 9-tiled as in Fig. 9.

In this section, our normal convention that A is the smallest angle of ABC is temporarily suspended. Instead, A is the vertex of the isosceles triangle and BC is the base. Similarly, we also suspend the convention that $\alpha < \beta$. The following lemma shows that the base angles of ABC are either α or β; we will make the convention for this section that β is the base angle.

The area of triangle ABC must be N times the area of the tile T. By hypothesis, the triangle T is similar to half the triangle ABC; the similarity factor is the square root of the ratio of the

Figure 16: A tiling related to a Pythagorean triple $a^2 + b^2 = c^2$.
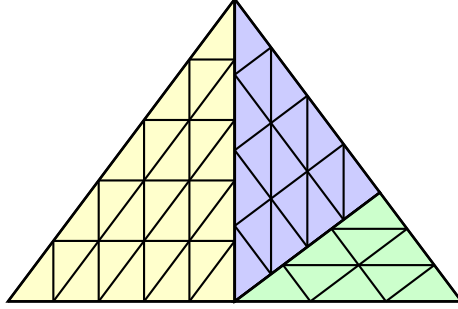
half the area of ABC to the area of T, namely $\sqrt{\frac{N}{2}}$. The sides of T are $a = \sin\alpha$, $b = \cos\alpha$, and $c = 1$. Let $\overline{AB}$ be the length of AB. Since AB is opposite the right angle of one of the halves of ABC that is similar to T, we have

$$\overline{AB} = \sqrt{\frac{N}{2}}\sin\gamma = \sqrt{\frac{N}{2}} \qquad \text{since } \gamma = \pi/2$$
$$\frac{\overline{BC}}{2} = \sqrt{\frac{N}{2}}\sin\alpha$$

On the other hand, $\overline{AB}$ and $\overline{BC}$ must be integer linear combinations of a, b, and c. So we have, for some non-negative integers p, d, and e, that $\overline{AB} = pa + db + ec$. Since $c = 1$ and $\overline{AB} = \sqrt{\frac{N}{2}}$, we have

$$p\sin\alpha + d\cos\alpha + e = pa + db + e = \sqrt{\frac{N}{2}} \tag{13}$$

The notation in this equation will be used throughout this section. In this section we also assume that $X = \overline{AB} = Y = \overline{AC}$ are the equal sides, and $Z = \overline{BC}$ is the base of ABC, and the $\mathbf{d}$ matrix satisfies

$$\mathbf{d}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

This differs from the convention in other sections since we do not know the relative sizes of the angles at the vertices A and B. With this convention, (p, d, e) is the first row of the $\mathbf{d}$ matrix.

Lemma 18 *Let ABC be an isosceles triangle with base BC, N-tiled by a right triangle T similar to half of ABC. Suppose $N/2$ is not a rational square. Then a belongs to $\mathbb{Q}(b)$, and b belongs to $\mathbb{Q}(a)$, and both a and b belong to $\mathbb{Q}\left(\sqrt{\frac{N}{2}}\right)$.*

Proof. Squaring both sides of (13) and simplifying as a polynomial in a we have

$$(pa + db + e)^2 = \frac{N}{2}$$
$$p^2a^2 + 2pa(db + e) + (db + e)^2 - \frac{N}{2} = 0$$

Since $a = \sin\alpha$ and $b = \cos\alpha$ we have $a^2 + b^2 - 1 = 0$. Replacing a^2 by $1 - b^2$ we find

$$2ap(db + e) + (db + e)^2 + p^2(1 - b^2) - \frac{N}{2} = 0$$

which shows that a belongs to $\mathbb{Q}(b)$. Similarly we find that b belongs to $\mathbb{Q}(a)$, so $\mathbb{Q}(a) = \mathbb{Q}(b) = \mathbb{Q}(a, b)$. Starting again from (13) we have

$$db \quad = \quad \sqrt{\frac{N}{2}} - pa - e$$

For typographical simplicity we write

$$\lambda := \sqrt{\frac{N}{2}}.$$

This equation shows that if $d = 0$, a belongs to $\mathbb{Q}(\lambda))$, and since $\mathbb{Q}(b) = \mathbb{Q}(a)$, the conclusion of the theorem holds. Therefore we may assume without loss of generality that $d \neq 0$. Continuing, we have

$$\begin{aligned} d^2 b^2 &= (\lambda - pa - e)^2 \\ d^2(1 - a^2) &= (\lambda - pa - e)^2 \\ &= p^2 a^2 - 2ap(\lambda - e) + (\lambda - e)^2 \end{aligned}$$

Writing it as a polynomial in a we have

$$0 = a^2(p^2 + d^2) - 2ap(\lambda - e) + (\lambda - e)^2 - d^2, \tag{14}$$

which shows that a has degree 2 over $\mathbb{Q}(\lambda)$ or is in $\mathbb{Q}(\lambda)$. Since $d \neq 0$, the quadratic term in (14) does not vanish.

Solving (14) by the quadratic formula we have

$$\begin{aligned} a &= \quad \frac{p(\lambda - e)}{p^2 + d^2} \pm \frac{\sqrt{(\lambda - e)^2 p^2 + (p^2 + d^2)(d^2 - (\lambda - e)^2)}}{p^2 + d^2} \\ &\quad \frac{p(\lambda - e)}{p^2 + d^2} \pm \frac{\sqrt{p^2 d^2 + d^4 - d^2(\lambda - e)^2)}}{p^2 + d^2} \\ &= \quad \frac{p(\lambda - e)}{p^2 + d^2} \pm \frac{d\sqrt{p^2 + d^2 - (\lambda - e)^2)}}{p^2 + d^2} \end{aligned} \tag{15}$$

Define

$$\mu := \sqrt{p^2 + d^2 - (\lambda - e)^2)}$$

For proof by contradiction, assume that a (and μ) do not belong to $\mathbb{Q}(\lambda)$. Then 1, λ, μ, and $\lambda\mu$ constitute a basis for $\mathbb{Q}(\lambda)$ over $\mathbb{Q}$, as shown by the equation for a above. Let σ be the automorphism of $\mathbb{Q}(\sqrt{\frac{N}{2}})$ that takes λ to $-\lambda$. We extend σ to be defined on $\mathbb{Q}\left(a, \sqrt{\frac{N}{2}}\right)$ and fix μ. Therefore $\mu = \mu\sigma$. We have

$$\mu\sigma = \sqrt{p^2 + d^2 - (-\lambda - e)^2)} \tag{16}$$

Then

$$\begin{aligned} \mu &= \quad \mu\sigma \\ \mu^2 &= \quad (\mu\sigma)^2 \\ p^2 + d^2 - (\lambda - e)^2) &= \quad p^2 + d^2 - (-\lambda - e)^2) \end{aligned}$$

Subtracting the right hand side from both sides we have

$$\begin{aligned} 0 &= \quad p^2 + d^2 - (\lambda - e)^2) - (p^2 + d^2 - (-\lambda - e)^2)) \\ 0 &= \quad -(\lambda - e)^2) + (-\lambda - e)^2)) \\ 0 &= \quad 4e\sqrt{\frac{N}{2}} \\ e &= \quad 0 \end{aligned} \tag{17}$$

But with $e = 0$ the expressions for μ and $\mu\sigma$ simplifies considerably:

$$\begin{aligned}
\mu &= d\sqrt{p^2 + d^2 - \lambda^2} \\
&= d\sqrt{p^2 + d^2 - \frac{N}{2}} \qquad \text{since } \lambda^2 = N/2 \\
\mu\sigma = &= \sqrt{p^2 + d^2 + \frac{N}{2}} \qquad \text{by (16)}
\end{aligned}$$

Since $d \neq 0$ we have $\mu\sigma > \mu$. But this contradicts $\mu = \mu\sigma$. This contradiction completes the proof by contradiction that a belongs to $\mathbb{Q}(\lambda)$, and hence the proof of the lemma.

Lemma 19 *Let ABC be an isosceles triangle with base BC, tiled by triangle T similar to half of ABC, in which angle α is not a rational multiple of π. Then the base angles do not split, and the vertex angle A is shared by exactly two tiles with the same angle at A, i.e. the vertex angle at A splits into exactly two equal angles.*

Proof. We have $\alpha + \beta = \pi/2$. We also have an equation arising from the vertex splitting. Let P, Q, and R be the total number of α, β, and γ angles of tiles at the vertices of ABC. Then because the angles at these vertices must add to π, we have $P\alpha + Q\beta + R\gamma = \pi$, and since $\gamma = \pi/2$ this becomes

$$P\alpha + Q\beta = \pi\left(1 - \frac{R}{2}\right).$$

If $Q \neq P$ then we can put $\beta = \pi/2 - \alpha$ and solve for α:

$$\frac{\alpha}{\pi} = \frac{1}{2}\frac{Q + R - 2}{Q - P}$$

making α a rational multiple of π. But by hypothesis, α is not a rational multiple of π. Therefore we can assume $P = Q$. We claim $P \geq 2$. Certainly P cannot be 1, since if one angle α appears at a vertex of ABC, the rest of the angle cannot be filled without using another α. If $P = 0$ then $Q\beta = \pi(1 - R/2)$, so β is a rational multiple of π, and hence $\alpha = \pi/2 - \beta$ is also a rational multiple of π, contradicting the hypothesis. Hence $P \geq 2$. Therefore $P\alpha + Q\beta \geq 2\alpha + 2\beta = \pi$; but $P\alpha + Q\beta = \pi(1 - \frac{R}{2})$, which is strictly less than π unles $R = 0$. Hence $R = 0$ and $P = Q = 2$.

If neither base angle splits, then the base angles are β (by convention–as mentioned, in this section we do not assume $\alpha \leq \beta$), and the vertex angle splits into two α angles. The only other possibility is that one of the base angles of ABC splits into two α angles. In that case the vertex angle is β and ABC is equilateral. Hence $\alpha = \pi/6$, which is a rational multiple of π, contrary to hypothesis. Therefore it is the vertex angle of ABC that splits. That completes the proof of the lemma.

Fix any vertex V of the tiling, and let n, m, and ℓ count the number of α, β, and γ angles at V, and let $k\pi$ be the angle sum at V, so $k = 1$ at a non-strict vertex or a boundary vertex, and $k = 2$ at a strict interior vertex. Then we have

$$n\alpha + m\beta = \pi\left(k - \frac{\ell}{2}\right)$$

and if $n \neq m$, we can solve this equation together with $\alpha + \beta = \pi/2$, obtaining

$$\frac{\alpha}{\pi} = \frac{1}{2}\frac{m + \ell - 2k}{m - n}$$

Since α/π is not rational, we must have $n = m$ at each vertex. This means that at each boundary or non-strict vertex, there are three possibilities: one each of α, β, and $\gamma = \pi/2$, or two right angles, or two each of α and β.

Lemma 20 *Suppose ABC is isosceles and tiled by triangle T similar to half of ABC, and assume α is a rational multiple of π. Then N is even and either*

(i) N/2 is a square, or

(ii) N/2 is a twice a square (that is, N is a square) and $\alpha = \pi/4$, or

(iii) N/2 is three times a square and $\alpha = \pi/6$.

Proof. Suppose that α is a rational multiple of π. By Lemma 18, $e^{i\alpha}$ has degree 2 or 4 over $\mathbb{Q}$. We can therefore apply Lemma 3 to conclude that $\alpha = 2\pi/n$, where $n = 5, 8, 10$, or 12.

In case $n = 8$ we have $\alpha = \pi/4$; hence the left hand side of (13) belongs to $\mathbb{Q}(\sqrt{2})$; hence $\sqrt{N/2}$ belongs to $\mathbb{Q}(\sqrt{2})$. Then $\sqrt{N/2}$ has the form $u + v\sqrt{2}$ with u and v rational. Squaring both sides we have $N/2 = u^2 + 2v^2 + 2uv\sqrt{2}$. Hence $uv = 0$. In case $v = 0$ then $N/2$ is a square. In case $u = 0$ then $N/2$ is twice a square.

In case $n = 12$, $\alpha = \pi/6$, so $\cos\alpha = \sqrt{3}/2$ and $\sin\alpha = 1/2$; hence the left hand side belongs to $\mathbb{Q}(\sqrt{3})$; hence $\sqrt{N/2}$ belongs to $\mathbb{Q}(\sqrt{3})$. Then $\sqrt{N/2}$ has the form $u + v\sqrt{3}$ with u and v rational. Squaring both sides we have $N/2 = u^2 + 3v^2 + 2uv\sqrt{3}$. Hence $uv = 0$. Hence either $u = 0$ or $v = 0$. In case $u = 0$ then $N/2$ is three times a square (which is possible, for example by bisecting each tile in Fig. 12, producing a 54-tiling); in case $v = 0$ then $N/2$ is a square.

In case $n = 10$ we have $\alpha = \pi/5$. Then $\cos\alpha = (1/4)(1 + \sqrt{5})$, and

$$\sin\alpha = \frac{1}{2}\sqrt{\frac{1}{2}(5 - \sqrt{5})}$$

But by Lemma 18, $\sin\alpha$ must belong to $\mathbb{Q}(\cos\alpha)$; hence $\sqrt{(5 - \sqrt{5})/2}$ belongs to $\mathbb{Q}(\cos\alpha) = \mathbb{Q}(\sqrt{5})$. That is, for some rational numbers u and v, we have $\sqrt{(5 - \sqrt{5})/2} = u + v\sqrt{5}$. A bit of algebra, not reproduced here, shows that this is impossible, so the case $n = 10$ cannot actually arise.

In case $n = 5$ we have $\alpha = 2\pi/5$ and

$$\sin\alpha = \frac{1}{2}\sqrt{\frac{1}{2}(5 + \sqrt{5})}$$
$$\cos\alpha = \frac{1}{4}(-1 + \sqrt{5})$$

and in this case also $\sin\alpha$ does not belong to $\mathbb{Q}(\cos\alpha)$, so by Lemma 18, this case cannot actually arise. That completes the proof of the lemma.

Lemma 21 *Suppose the isosceles triangle ABC is N-tiled by a right triangle similar to half of ABC. Suppose that α is not a rational multiple of π, and N/2 is not an integer square or a sum of two integer squares. Then*

*(i) in the **d** matrix we have $d \neq 0$ and $p \neq 0$ and $e = 0$, i.e. there are no c edges on the two equal sides AB and AC, and there are some a and some b edges there, and*

*(ii) the second row of the **d** matrix is identical to the first, and*

(iii) there are no "edge relations", i.e. no relations $ua + vb + wc = 0$ with rational u, v, and w.

Proof. First, we prove that a half-integer $N/2$ is a square of an integer if and only if it is a rational square. Suppose $N/2 = (P/Q)^2$, with P and Q relatively prime. Then $NQ^2 = 2P^2$, so N is even, since 2 divides the right side to an odd power, and 2 divides Q^2 to an even power. Then $N/2$ is an integer and $(N/2)Q^2 = P^2$. Since P and Q are relatively prime, we must have $Q = 1$ and $N/2 = P^2$.

Next, we note that by Lemma 6 and Lemma 4, a half-integer $N/2$ is a sum of two integer squares if and only if it is a sum of two rational squares. Hence it does not matter whether the hypothesis of the lemma mentions sums of two rational squares, or sums of two integer squares.

For notational simplicity, in this proof we continue to use

$$\lambda := \sqrt{\frac{N}{2}}.$$

Since half of ABC is similar to the tile T, with scale factor λ we have

$$\mathbf{d} \begin{pmatrix} a \\ b \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \\ 2a \end{pmatrix}$$

The first row of this equation says that

$$\lambda = pa + db + 1,$$

from which it follows that if a and b are both rational, $\lambda^2 = N/2$ is a rational square, and hence an integer square. Therefore not both a and b are rational.

Suppose a is rational. Then by Lemma 18, since $N/2$ is not a square, b belongs to $\mathbb{Q}(a)$, so b is also rational, contradiction, since we have shown not both a and b are rational. Therefore a is not rational. Similarly, if b is rational, then a belongs to $\mathbb{Q}(b)$, so a is also rational, contradiction. Hence b is not rational.

Recall that the $\mathbf{d}$ matrix is

$$\mathbf{d} = \begin{pmatrix} p & d & e \\ g & m & f \\ h & \ell & r \end{pmatrix}.$$

We first consider the case $d = 0$. Assume $d = 0$. Since $pa + db + e = \lambda$, with $d = 0$ we have $pa + e = \sqrt{\frac{N}{2}}$. If $p = 0$ then $\lambda = e$, so $N/2$ is a square, and we are finished. Hence we may assume $p \neq 0$. If $e = 0$ then $ap = \lambda$, so $N/2$ is a square, and we are finished.

By Lemma 18, b belongs to $\mathbb{Q}(\lambda)$, so for some integers n and k we have

$$\begin{aligned} b &= \sqrt{1 - a^2} \\ &= n + k\lambda \\ \sqrt{1 - a^2} &= n + k\lambda \end{aligned}$$

Squaring both sides we have

$$1 - a^2 = n^2 + k^2 + 2kn\lambda$$

Since $pa + e = \lambda$, we have

$$a = -\frac{e}{p} + \frac{\lambda}{p}$$

Putting in this expression for a we have

$$1 - \left(-\frac{e}{p} + \frac{\lambda}{p} \right)^2 = n^2 + k^2 + 2kn\lambda$$

$$1 - \frac{e^2}{p^2} - \frac{N}{2p^2} + \frac{2e}{p^2}\lambda = n^2 + k^2 + 2kn\lambda$$

If $N/2$ is a square, we are finished, so we may assume $N/2$ is not a square, and then we can equate the rational parts:

$$1 - \frac{e^2}{p^2} - \frac{N}{2p^2} = n^2 + k^2$$

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Since b is not zero, and $b = n + k\lambda$, we cannot have both $n = 0$ and $k = 0$. Hence the right hand side is a positive integer, so it is at least 1. The left hand side, however, is less than 1, since N is positive (even if $e = 0$). This is a contradiction. We have shown that if $d = 0$, then $N/2$ is either a square or a sum of two squares; but that contradicts our assumptions. Therefore $d \neq 0$.

Our next aim is to prove $e = 0$. Since $d \neq 0$, the expression (15) is valid. That is,

$$a = \frac{p(\lambda - e) \pm d\sqrt{p^2 + d^2 - (\lambda - e)^2}}{p^2 + d^2}.$$

Since a belongs to $\mathbb{Q}(\lambda)$ (by Lemma 18), the expression under the square root is a square in $\mathbb{Q}(\lambda)$. It therefore suffices to prove that if $p^2 + d^2 + (\lambda - e)^2$ is a square in $\mathbb{Q}(\lambda)$, then $e = 0$.

Let σ be the automorphism of $\mathbb{Q}(\lambda)$ determined by $\lambda\sigma = -\lambda$. Define

$$\xi := \sqrt{p^2 + d^2 - (\lambda - e)^2}$$

Then

$$
\begin{aligned}
\xi\sigma &= \pm\sqrt{p^2 + d^2 - (\lambda + e)^2} \\
Norm(\xi) &= \xi(\xi\sigma) \\
\pm(Norm(\xi))^2 &= (p^2 + d^2 - (\lambda - e)^2)(p^2 + d^2 - (\lambda + e)^2) \\
&= (p^2 + d^2)^2 + (\lambda - e)^2(\lambda + e)^2 + (p^2 + d^2)((\lambda + e)^2 - (\lambda - e)^2) \\
&= (p^2 + d^2)^2 + (\lambda^2 - e^2)^2 + 4e(p^2 + d^2)\lambda \\
&= (p^2 + d^2)^2 + \left(\frac{N}{2} - e^2\right)^2 + 4e(p^2 + d^2)\lambda \qquad \text{since } \lambda^2 = N/2
\end{aligned}
$$

The left hand side of this equation is rational. Hence, unless $N/2$ is a square (in which case we are finished), the coefficient of λ on the right is zero. Hence $e(p^2 + d^2) = 0$. But since $d \neq 0$, this implies $e = 0$ as desired.

Next we claim $p \neq 0$ and $r \neq 0$. Consider the tile, say Tile 1, at vertex B. There is only one, since the angle there is β, and if it splits, then β is a multiple of α, and hence α is a rational multiple of π, contrary to hypothesis. Since Tile 1 has its b side opposite its β angle, its a or c side must be on AB. But since $e = 0$, it cannot be the c side. Hence the a edge of Tile 1 is on AB. But since p is the number of a edges on AB, we have $p \neq 0$. Then the c side of Tile 1 is on BC, and since r is the number of c edges on BC, we have $r \neq 0$. This completes the proof of claim (i) of the lemma.

Now that we have proved $e = 0$, (15) simplifies considerably:

$$a = \frac{p\lambda}{p^2 + d^2} \pm \frac{d\sqrt{p^2 + d^2 - N/2}}{p^2 + d^2} \tag{18}$$

We next derive a similar equation for b.

$$
\begin{aligned}
pa + db &= \lambda \qquad \text{(since } e = 0, e \text{ does not appear)} \\
pa &= \lambda - db \\
p^2a^2 &= (\lambda - db)^2 \\
p^2(1 - b^2) &= (\lambda - db)^2 \\
&= d^2b^2 - 2db\lambda + \frac{N}{2} \\
0 &= b^2(p^2 + d^2) - 2bd\lambda + \frac{N}{2} - p^2
\end{aligned}
$$

Solving by the quadratic formula and simplifying, we find the desired formula for b:

$$b = \frac{d\lambda}{p^2 + d^2} \pm \frac{p\sqrt{p^2 + d^2 - N/2}}{p^2 + d^2} \tag{19}$$

We now define
$$\mu := \sqrt{p^2 + d^2 - N/2} \qquad (20)$$
We will show that either $N/2$ is a sum of two squares, or μ is rational. Suppose that μ is not rational. Since a belongs to $\mathbb{Q}(\sqrt{\frac{N}{2}})$, so does μ. But the only rationals that have irrational square roots in $\mathbb{Q}(\sqrt{\frac{N}{2}})$ are rational squares times $\sqrt{\frac{N}{2}}$. Hence for some rational ξ we have $\mu = \xi\sqrt{\frac{N}{2}}$. Then

$$\mu^2 = \xi^2 \frac{N}{2}$$
$$p^2 + d^2 - \frac{N}{2} = \xi^2 \frac{N}{2}$$
$$p^2 + d^2 = (\xi^2 + 1)\frac{N}{2}$$
$$\frac{N}{2} = \frac{p^2 + d^2}{\xi^2 + 1}$$

This is a quotient of sums of two rational squares. By Lemma 5, it is a sum of two rational squares, and since it is a half-integer, it is a sum of two integer squares, by Lemma 6. Hence, as claimed, either $N/2$ is a sum of squares, or μ is rational. But we have earlier assumed that $N/2$ is not a sum of two integer squares; hence μ is rational. We also have $\mu \neq 0$, since if $\mu = 0$ we have $N/2 = p^2 + d^2$.

In terms of λ and μ, our formulas for a and b are

$$a = \frac{p\lambda \pm d\mu}{p^2 + d^2}$$
$$b = \frac{d\lambda \pm p\mu}{p^2 + d^2}$$

We now claim that one must take opposite signs for the $\pm$ in these two formulas. To prove that, we calculate as follows:

$$1 = a^2 + b^2$$
$$= \left(\frac{p\lambda \mp d\mu}{p^2 + d^2}\right)^2 + \left(\frac{d\lambda \pm p\mu}{p^2 + d^2}\right)^2$$
$$= \frac{(p^2 + d^2)(\lambda^2 + \mu^2)}{(p^2 + d^2)^2} + \quad \text{cross terms}$$

Here the "cross terms" are either zero, if a and b have opposite signs on the coefficients of μ, or they are $\pm 2dp\lambda\mu/(p^2 + d^2)$, which is not zero unless $\mu = 0$, since p and d are not zero. Now $\lambda^2 = N/2$ and $\mu^2 = p^2 + d^2 - N/2$, so $\lambda^2 + \mu^2 = p^2 + d^2$ and we get 1 for $a^2 + b^2$ *without* the cross terms. Hence the cross terms *must* cancel out. We will see below that one must take the plus sign for a and the minus sign for b. The fact that the signs must be opposite can also be proved by starting from $pa + db = \lambda$ instead of from $a^2 + b^2 = 1$.

Now we turn to the third row of the **d** matrix, which tells us

$$ha + \ell b + r = 2\lambda a.$$

We will use this to determine the signs in the formulas for a and b, which so far have an ambiguous $\pm$ sign, except that we know the signs must be opposite. Substituting in the formulas for a and b we have

$$2\lambda\left(\frac{p\lambda \pm d\mu}{p^2 + d^2}\right) = h\left(\frac{p\lambda \pm d\mu}{p^2 + d^2}\right) + \ell big(\frac{d\lambda \mp p\mu}{p^2 + d^2}\right) + r$$

Clearing denominators we have

$$2\lambda(p\lambda \pm d\mu) \quad = \quad h(p\lambda \pm d\mu) + \ell(d\lambda \mp p\mu) + r(p^2 + d^2) \tag{21}$$

Equating the coefficients of λ, and noting that μ and λ^2 are real, we have

$$\pm 2d\mu \quad = \quad hp + \ell d$$

Since the right-hand side is nonnegative, and since d and μ are not zero, we must take the positive sign on the left, which means we must take the positive sign in the formula for a, and hence the negative sign in the formula for b. That is,

$$a \quad = \quad \frac{p\lambda + d\mu}{p^2 + d^2} \tag{22}$$

$$b \quad = \quad \frac{d\lambda - p\mu}{p^2 + d^2} \tag{23}$$

We next claim that the second row of the $\mathbf{d}$ matrix is identical to the first, i.e. the numbers of edges of each length are the same on AB as on AC. From the second row of the $\mathbf{d}$ matrix equation we have $ga + mb + fc = \lambda$. This has the same right hand side as the equation $pa + db + ec = \lambda$ from the first row. We can change (p, d, e) to (g, m, f) in all the above calculations, and we find instead of $e = 0$ that $f = 0$, and instead of $d \neq 0$, that $g \neq 0$, and finally we find formulas for a and b:

$$\nu \quad := \quad \sqrt{g^2 + m^2 - N/2}$$

$$a \quad = \quad \frac{g\lambda + m\nu}{g^2 + m^2}$$

$$b \quad = \quad \frac{m\lambda - g\nu}{g^2 + m^2}$$

Equating the coefficients of λ in the two formulas for a we have

$$\frac{d}{d^2 + p^2} = \frac{g}{g^2 + m^2}$$

This implies that the points (p, d) and (g, m) lie on the same line through the origin, and hence for some real t, $m = td$ and $g = tp$. Then let x be the coefficient of λ in a. We have

$$\begin{aligned}
x \quad &= \quad \frac{g}{g^2 + m^2} \\
&= \quad \frac{td}{(td)^2 + (tp)^2} \\
&= \quad \frac{1}{t}\frac{d}{d^2 + p^2} \\
&= \quad \frac{x}{t}
\end{aligned}$$

Thus $x = x/t$, which implies $t = 1$. Hence

$$g \quad = \quad p \tag{24}$$

$$m \quad = \quad d \tag{25}$$

That proves part (ii) of the lemma.

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We now turn our attention to part (iii) of the lemma. Suppose, for proof by contradiction, that $ua + vb + w = 0$ with rational u, v and w. Then

$$
\begin{aligned}
0 &= ua + vb + w \\
&= u\frac{p\lambda + d\mu}{p^2 + d^2} + v\frac{d\lambda - p\mu}{p^2 + d^2} + w \\
0 &= u(p\lambda + d\mu) + v(d\lambda - p\mu) + w(p^2 + d^2) \\
&= \lambda(up + vd) + \mu(ud - vp)
\end{aligned}
$$

Since $p > 0$ and $d > 0$, we must have $u = v = 0$ to make the coefficient of λ zero. But then $0 = ua + vb + w = w$, so w is also zero. We note in passing the consequence that a/b is irrational, since if $a/b = v$ than $a - bv = 0$. That completes the proof of the lemma.

Theorem 3 *Suppose the isosceles triangle ABC is N-tiled by a right triangle similar to half of ABC. Suppose that α is not a rational multiple of π. Then $N/2$ is a square (of an integer) or a sum of two integer squares.*

Proof. Suppose, for proof by contradiction, that there is such a tiling and $N/2$ is neither a square nor a sum of two squares. Then, by Lemma 21, there are no edge relations. That means that each maximal segment in the tiling has equal numbers of a edges on each side, equal numbers of b edges on each side, and equal numbers of c edges on each side, for otherwise an edge relation exists.

Consider two points Q on BA and U on BC such that BQU is similar to the tile, with a right angle at Q and an α angle at U. We say that BQU is "nicely tiled" if the tiling of ABC, restricted to BQU, is obtained from a quadratic tiling of BQU by replacing some pairs of adjacent tiles that form a rectangle by the same rectangle with the other diagonal. Choose segment QU as far to the northeast as possible with BQU nicely tiled. There is some such segment QU because the tile at B has its c side on BC, not on AB, since $e = 0$ by Lemma 21; and since $d \neq 0$ (also by Lemma 21), there are some a edges on AB, so Q lies on the interior of AB. Point U lies on the interior of BC, since the tile at C has its β angle at C, but such a tile will not be part of the lattice tiling of BQU, since those tiles have their α angle to the east. Let P be the next vertex on QA above Q, and let W be the next vertex on BC east of U. Then all the tiles below QU with an edge on QU have their c edge on QU. Since there are no linear integral relations between a, b, and c, all the tiles above QU with an edge on QU also have their c edges on QU. Suppose that PQ has length a.

Let Tile 1 be the tile with a vertex at Q and its southwest edge on QU; let R be its southeast vertex on QU. Then there are two cases: the third vertex of Tile 1 is either P or the point S such that SR is perpendicular to QR and SR has length a. In case the third vertex of Tile 1 is S, then Tile 1 has its α angle at Q, and hence the remaining angle PQR is β, which must be filled by a single tile, Tile 2. Since PQ has length a, Tile 2 must have its c edge along QS and hence the rectangle $PSRQ$ is nicely tiled.

In case the third vertex of Tile 1 is P, then PR has length c, and PR is a maximal segment, so the tile northeast of PR, say Tile 2, shares its c edge with PR. Suppose, for proof by contradiction, that Tile 2 is not PRS. Then it has its β angle at P, and its α angle at R. Tile 1 also has its α angle at R, so the angle between Tile 2 and RU is 2β. That is partly filled by the angle at R of Tile 3, the tile above RU with a b edge on RU. Tile 3 has its b edge on RU, so its angle at R is either α or γ; but since it must fit into 2β, it cannot be γ. It must therefore be α. But that leaves an unfilled angle of $\beta - \alpha$, which cannot be filled by any number of α angles, since α is not a rational multiple of π. That contradiction shows that Tile 2 is in fact PRS. Hence in this case also, rectangle $PSRQ$ is nicely tiled. See Fig. 17.

Now let W be the intersection point of UC and the line containing PS. Let the vertices on QU be $Q, R, R_2, R_3, \ldots$, spaced apart by b, and let the points $P, S_1, S_2, \ldots$ on PW also be spaced apart by b, so $S = S_1$. Let k be the largest integer such that $PS_k R_k Q$ is nicely tiled.

Figure 17: $PQRS$ must be nicely tiled, assuming $\overline{PQ} = a$.
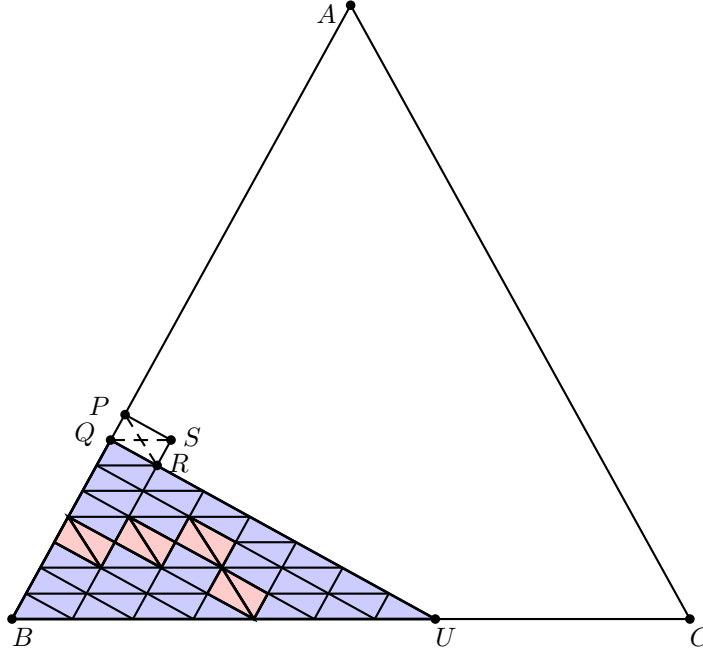
If R_k lies on BC, then (since QW was chosen as far to the northeast as possible), the tile with vertex at R_k and an edge on R_kC has its β angle at R_k, and its c edge against the a edge R_kS_k.

Otherwise point R_{k+1} exists. Let Tile 5 be the tile above QU with its b edge on Q_kU and a vertex at Q_k. The third vertex of Tile 5 is either at S_k or at S_{k+1}. First consider the case when the third vertex of Tile 5 is at S_{k+1}. Then there is exactly one more tile, Tile 6, with a vertex at R_k, and Tile 6 has its β angle at R_k. If Tile 6 has its a side on R_kS_k, then the rectangle $PQR_{k+1}S_{k+1}$ is nicely tiled, contradicting the choice of k. Hence Tile 6 has its c side along R_kS_k. Then there is a maximal segment through R_kS_k with only a edges on the northwest below S_k and at least one c edge on the southeast. See Fig. 18.

Next consider the case when the third vertex of Tile 5 is at S_k. Let Tile 6 be the tile with a vertex at R_{k+1} and sharing an edge with Tile 5. We claim that Tile 6 has its β angle at R_{k+1}. If R_{k+1} lies on BC (so $R_{k+1} = U$), then angle S_kUC is 2β, so it can only be filled by two tiles with their β angles at R_k, one of which is Tile 6. On the other hand, if R_{k+1} does not lie on BC, then R_{k+2} exists, and Tile 7 above $R_{k+1}R_{k+2}$ has its b edge on $R_{k+1}R_{k+2}$. Then if Tile 7 has its right angle at R_{k+1}, that leaves a β angle for Tile 6, and if Tile 7 has its α angle at R_{k+1}, that leaves 2β unfilled at R_{k+1}, which can only be filled by two tiles with their β angles at R_{k+1}, one of which is Tile 6. Hence in any case, Tile 6 has its β angle at R_{k+1}. It does not have its c edge on $R_{k+1}S_k$, as that would make $PQR_{k+1}S_{k+1}$ nicely tiled, contradicting the choice of k. Hence Tile 6 has its a edge on $R_{k+1}S_k$. Then there is a maximal segment through $R_{k+1}S_k$, with only c edges on the southwest side below S_k, and at least one a edge on the northeast side. See Fig. 19.

Thus we are in the same situation, whether R_k lies on BC or not, and regardless of the orientation of Tile 5: to the southeast of a nicely-tiled strip PQS_kR_k there is a line, either containing R_kS_k or $R_{k+1}S_k$, with a c edge on one side and an a edge on the other. Let F be the point R_k or R_{k+1} where this line starts. Let T be the first point above S_k on this line, such that T is a vertex both of a tile on the left and a tile on the right of FT. Then there must be equal numbers of a edges on each side of FT, and equal numbers of c edges on each side,

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Figure 18: Tile 5 has a vertex at S_{k+1}, so along $R_k S_k$ there are an a edge and a c edge.
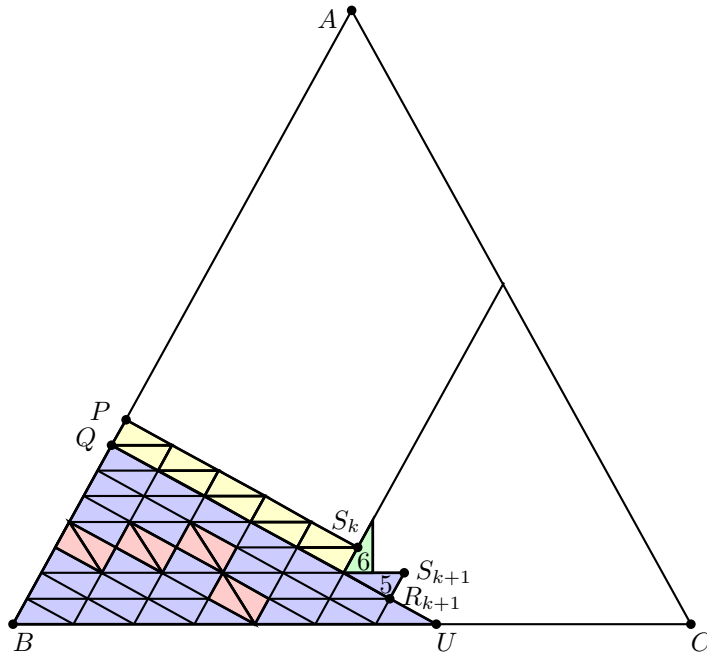
because otherwise there would be a nontrivial integer linear relation between a, b, and c. Let Q' and R' be points on QA and FT respectively such that $Q'R'$ is parallel to QR and rectangle $Q'QR_kR'$ is nicely tiled, and $Q'R'$ is as far northeast as possible. There must be such points Q' and R', with R' on the interior of segment FT, since otherwise FT has all a edges on its left side (if FT is parallel to AB), or FT has all c edges on its left side (otherwise), either of which contradicts Lemma 21. Then there are b edges all along $Q'R'$. Now we repeat the argument that we made above, with QU replaced by $Q'R'$, and we find another line $R_j S_j$ with $j < k$ such that there is one row of nicely tiled rectangles above $Q'R'$ between AB and line $R_j S_j$. Let R'' be the intersection point of $Q'R'$ and $R_j S_j$. Then there is a tile, say Tile 7, with its b edge on $Q'R'$ southwest of R''. As before there are two possibilities for the orientation of Tile 7. One of them is shown in Fig. 20, namely when Tile 7 has its α angle at R_j. Then Tile 8 has its β angle at R_j and its c edge against the a edge of the tile northwest of $R_j S_j$, so $R_j S_j$ must extend northeast until the numbers of a edges on the left and right are equal and the number of c edges on the left and right are equal and the number of b edges on the left and right are equal. If Tile 7 has the other orientation, then the same is true of the line $R_{j+1} S_j$.

Figure 19: Tile 5 has a vertex at S_k, so along $S_k R_{k+1}$ there are an a edge and a c edge.
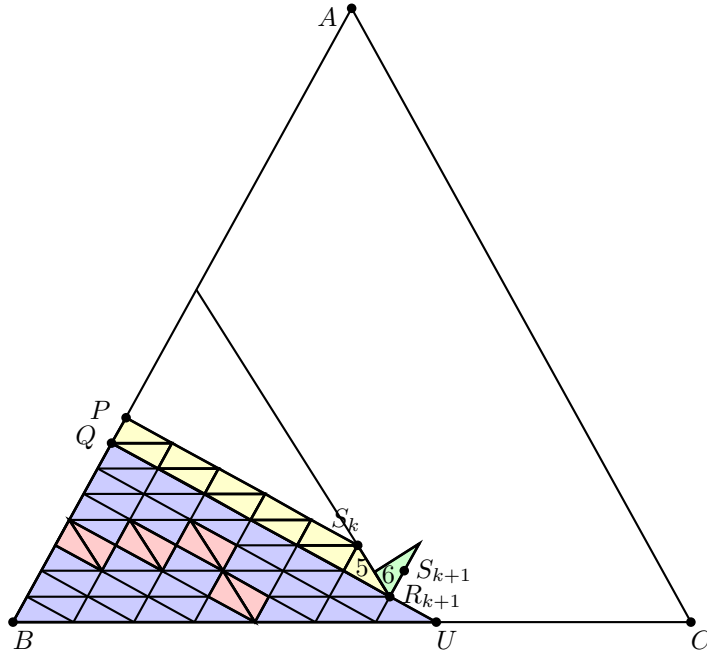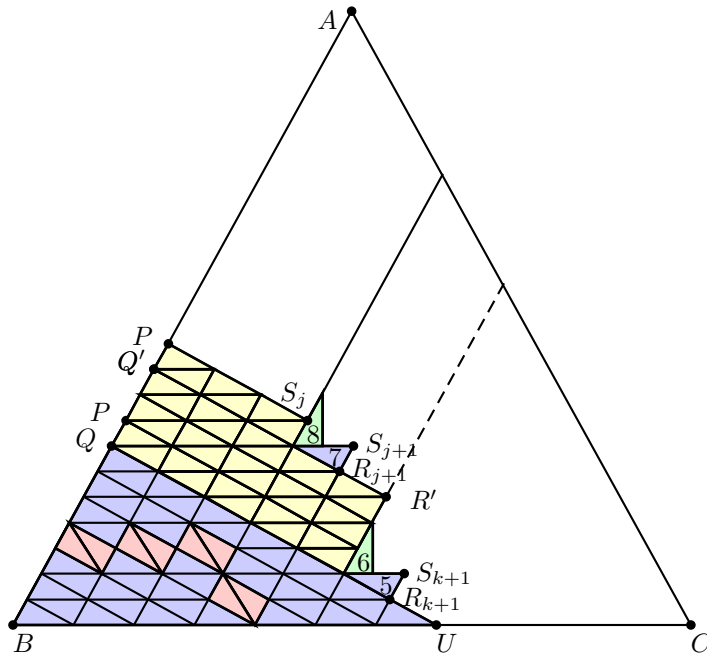
Figure 20: The second stage of the construction; the strip under $Q'R'$ is nicely tiled.

We continue in this fashion, defining narrower strips of nicely-tiled rectangles bounded on the east by lines with a edges on the left and c edges on the right. Eventually one of two things happens. Either we run out of a edges on AB, or we reach a point where one of the boundary lines $R_{j+1}S_j$ that run towards the northwest intersects AB. For example, in Fig. 19, the nicely-tiled area could fill up the region under the line $R_{j+1}S_j$ before we run out of a edges on BC. We will show that both possibilities are impossible. The second one is immediately contradictory: if $R_{j+1}S_j$ reaches AB at a point G with all the area underneath it nicely tiled, then there are only c edges underneath R_jG, but above it there is at least one a edge, giving rise to an edge relation, which contradicts Lemma 21. Therefore this possibility does not happen. We cannot reach the vertex A without running out of a edges on AB, since by Lemma 21, $d \neq 0$, so there is at least one b edge on AB. Let E be the highest vertex on AB such that below E, there are only a edges on AB. Then let Tile 9 be the tile with an edge on AB north of E and a vertex at E. Since Tile 9 does not have an a edge on AB, and by Lemma 21, there are no c edges on AB, Tile 9 must have its b edge on AB. Let EF be the line perpendicular to AB at E, where EF is a maximal segment. Then F lies on the southwest boundary L of the last strip of nicely-tiled rectangles. That boundary L does not terminate at F since there are only a edges on it south of F. (L may run southwest to northeast, or southeast to northwest; as we have seen, at each stage of the construction there are two cases, so we just say "south" here.) Thus every tile southwest of EF with an edge on EF has its b edge on EF, and there is a tile northeast of EF with a vertex at F. Hence all the tiles north of EF with an edge on EF have their b edges on EF. If Tile 9 has its right angle at E then it has its a edge on EF, contradiction. This situation is shown in Fig. 21.

Figure 21: The final contradiction when we encounter a b edge above E on AB.
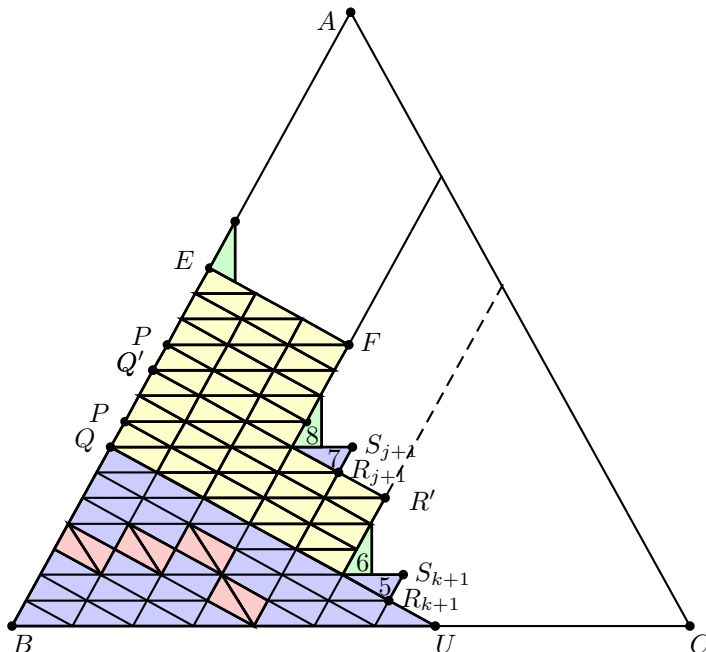
Hence Tile 9 has its α angle at E, leaving an angle of β unfilled at E. This β angle must be filled by Tile 10, which then cannot have its b edge on EF. That is a contradiction (not illustrated in a separate figure, but imagine the last tile at E in Fig. 21 placed with its α angle at E instead of its right angle as shown). That contradiction completes the proof of the theorem.

Theorem 4 *Suppose triangle ABC is isosceles with AB = AC. Let M be the midpoint of base BC, and suppose ABC is N-tiled by the right triangle T, and suppose T is similar to triangle ABM (half of triangle ABC). Then N is even and one of the following cases holds:*

(i) N/2 is a square.

(ii) N is a square, and $\alpha = \pi/4$.

(iii) ABC is equilateral and N is six times a square (so $\alpha = \pi/6$).

(iv) ABC has base angles $\pi/6$ and N is six times a square.

(v) N/2 is a sum of two squares, and α is not a rational multiple of π, and a and b are rational.

Remark. All the cases mentioned in the theorem do actually occur.

Proof. By Lemma 19, the vertex angles at B and C do not split. As mentioned above, we make the convention that β is the base angle; then by the lemma, 2α is the vertex angle. For this section only, we do not assume $\alpha \leq \beta$. If α is a rational multiple of π, then by Lemma 20, one of the first four conclusions of the theorem holds. We can therefore assume that α is not a rational multiple of π. Then by Theorem 3, $N/2$ is a square or a sum of two squares, so conclusion (i) or conclusion (v) holds. That completes the proof of the theorem.

8   Tilings by a non-isosceles right triangle T

Many of the tilings exhibited in the introduction have the tile T similar to the tiled triangle ABC. Indeed that is the case for the quadratic tilings and the biquadratic tilings, but not the case for the 3-tiling of the equilateral triangle, and various composite tilings involving that tiling as a subtiling (such as the exhibited 12-tilings, and a 6-tiling obtained from the equilateral 3-tiling). There is also, of course, the 2-tiling of an equilateral triangle, which can in turn be used with a quadratic tiling of an equilateral triangle to produce various 8-tilings. The 27-tiling shown in Fig. 10 is an example of a prime tiling with $N > 3$ in which the tile is not similar to ABC. In this section, we deal with the special case in which the tile is a right triangle. We begin with a special case of this special case.

Lemma 22 *Suppose T is a right triangle with $\alpha = \pi/8$, and suppose ABC is a right triangle and ABC is N-tiled by T for some N. Then ABC is similar to T.*

Proof. Let $\zeta = e^{i\pi/8}$. Then the degree of $\mathbb{Q}(\zeta)$ over $\mathbb{Q}$ is $\varphi(16) = 8$, by Lemma 3. Let $\sigma = \sigma_9$ be the automorphism taking ζ to $\zeta^9 = -\zeta$. Then σ fixes i, fixes $\sin J\alpha$ for J even and changes the sign of $\sin J\alpha$ for J odd, since

$$\begin{aligned}
(2i\sin J\alpha)\sigma &= (\zeta^J - \zeta^{-J})\sigma \\
&= (\zeta\sigma)^J - (\zeta\sigma)^{-J} \\
&= (-\zeta)^J - (-\zeta)^{-J} \\
&= (-1)^J 2i\sin J\alpha
\end{aligned}$$

Hence σ changes the signs of $a = \sin\alpha$ and $b = \sin\beta = \sin 3\alpha$. Since T is a right angle we have $c = 1$. Let $X = pa + qb + r$ and $Y = ma + nb + \ell$, where p, q, r, m, n, and ℓ give the numbers of tiles with sides a, b, and c along X and Y.

The area of the tile $\mathcal{A}_T$ is given by $ab/2$, since γ is a right angle. Since triangle ABC is also a right triangle its area $\mathcal{A}_{ABC}$ is given by $XY/2$. Since there is a tiling, we have the area equation $N\mathcal{A}_T = \mathcal{A}_{ABC}$, which now becomes $Nab = XY$. Writing out X and Y in terms of the edges that compose them, we have

$$Nab = (pa + db + e)(ga + mb + f)$$

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Applying σ we have
$$Nab = (-pa - db + e)(-ga - mb + f)$$

Subtracting the two equations we have
$$\begin{aligned}
0 &= (pa + db + e)(ga + mb + f) - (-pa - db + e)(-ga - mb + f) \\
&= 2e(ga + mb) + 2f(pa + db) \quad \text{since the other terms cancel}
\end{aligned}$$

Dividing by 2 we have
$$0 = gea + emb + fpa + fdb$$

Since all these quantities are nonnegative, and a and b are not zero, we have ge, em, fp and fd each equal to zero. Assume, for proof by contradiction, that $f \neq 0$. Then $p = d = 0$. Hence $e \neq 0$, since $X = pa + db + ec$ and $X \neq 0$. Hence $m = g = 0$. Hence both X and Y are composed entirely of c sides, and hence are rational, since $c = 1$. Hence $ab = XY/N$ is also rational. But it can be expressed as
$$\begin{aligned}
ab &= \sin\alpha \sin 3\alpha \\
&= (\zeta - \zeta^{-1})(\zeta^3 - \zeta^{-3}) \\
&= \zeta^4 - \zeta^2 - \zeta^{-2} + \zeta^{-4}
\end{aligned}$$

Multiplying by ζ^4 we have
$$\begin{aligned}
\zeta^4 ab &= \zeta z^8 - \zeta^6 + \zeta^2 + 1 \\
&= -\zeta^6 + \zeta^2 \quad \text{since } \zeta^8 = -1 \\
0 &= -\zeta^6 - ab\zeta^4 + \zeta^2
\end{aligned}$$

Since ab is rational, this is a polynomial with rational coefficients satisfied by ζ, and it is not identically zero; but its degree is less than the degree of the minimal polynomial of ζ, which we have seen is 8. That contradiction completes the proof of the lemma.

Lemma 23 *Suppose that T is a right triangle with $\alpha = \pi/8$, and that ABC has angle $C = 5\pi/8$ and angle $A = \alpha$. Let N be a positive integer. Then there is no N-tiling of ABC by T.*

Proof. We use the same cyclotomic field and automorphism σ as in the previous proof. The area equation can be written as
$$Nabc = XY \sin 5\alpha$$

since angle $ABC = 5\alpha$. Then σ changes the signs of a and b, since these are respectively the sines of α and 3α. Since 5 is odd, σ changes the sign of $\sin 5\alpha$. On the other hand σ fixes $c = \sin 4\alpha$. Hence, when we apply σ to the area equation, we get
$$Nabc = -(X\sigma)(Y\sigma)\sin 5\alpha.$$

Now it will not help to subtract the equations, but we can divide them instead. We obtain
$$-1 = \frac{(X\sigma)(Y\sigma)}{XY}$$

Multiplying by the denominator and adding XY to both sides, we have
$$\begin{aligned}
-XY &= (X\sigma)(Y\sigma) \\
0 &= (X\sigma)(Y\sigma) + XY
\end{aligned}$$

Writing X and Y out in terms of the $\mathbf{d}$ matrix, and remembering that σ changes the signs of a and b but not c, we have

$$
\begin{aligned}
0 &= (-pa - db + e)(-ga - mb + f) + (pa + db + e)(ga + mb + f) \\
&= 2(pa + db)^2 + 2(ef)^2 \qquad \text{since the other terms cancel out} \\
0 &= (pa + db)^2 + (ef)^2
\end{aligned}
$$

Since all the letters denote nonnegative quantities, we have $ef = 0$ and $pa + db = 0$; since $a > 0$ and $b > 0$ we have $p = d = 0$. Since $X \neq 0$, $p = d = 0$ implies $e \neq 0$; then $ef = 0$ implies

$$
f = 0. \tag{26}
$$

The area equation can also be written as

$$
Nabc = YZ \sin \alpha
$$

since angle A of triangle ABC is α. Since $\sin \alpha = a$ we have

$$
Nbc = YZ.
$$

Applying σ we have

$$
-Nbc = Y\sigma Z\sigma
$$

Adding that to the previous equation we have

$$
0 = (Y\sigma)(Z\sigma) + YZ
$$

Writing this out in terms of the $\mathbf{d}$ matrix we have

$$
\begin{aligned}
0 &= (-ga - mb + fc)(-ha - \ell b + rc) + (ga + mb + fc)(ha + \ell b + rc) \\
&= 2(ga + mb)^2 + 2(fr)^2 \\
0 &= (ga + mb)^2 + (fr)^2
\end{aligned}
$$

Hence $g = m = 0$. But by (26) we have $f = 0$. Hence $Y = ga + mb + fc = 0$. But that is a contradiction, since Y is the length of side AC of triangle ABC. That completes the proof of the lemma.

Remark. The proofs given above of the previous two lemmas are not the only proofs. In both the case of a right angle and the case when angle $C = 5\alpha$, the area of ABC is given by $\mathcal{A}_{ABC} = X^2/2$. That is obvious for the right triangle, and not very difficult in the other case. It is possible to show that this formula leads to a contradiction. We sketch the idea rather than give the complete proof. One writes X in terms of the $\mathbf{d}$ matrix as $X = pe + db + e$ (since $c = 1$, c does not appear here). Since $b = \sin 3\alpha$, we have $b = 3a - 4a^3$ by simple trigonometry. Substituting that in the expression for X we have

$$
X = pa + d(3a - 4a^3) + e.
$$

Since the area of the tile is $ab/2$, the area equation

$$
2\mathcal{A}_{ABC} = 2N\mathcal{A}_T
$$

becomes $X^2 - Nab = 0$. That is,

$$
(pa + d(3a - 4a^3) + e)^2 - Na(3a - 4a^3) = 0. \tag{27}
$$

The left side is a polynomial in a of degree 6. One shows that this polynomial cannot be zero. To do that one has to first show

$$a^4 - a^2 + \frac{1}{8} = 0, \tag{28}$$

which follows from the formula

$$\sin\left(\frac{\pi}{8}\right) = \sqrt{\frac{1}{2} - \frac{1}{4}\sqrt{2}}.$$

Applying this formula one can reduce a polynomial in a of degree 6 to a polynomial of degree 4. One cannot entirely avoid cyclotomic fields, however, as one must prove that a really has degree 4 over $\mathbb{Q}$. Given that, the polynomial (27), after reduction to degree 4, must be a multiple of (28), and that soon leads to a contradiction. The calculations are somewhat lengthier than the proofs with automorphisms given above.

Lemma 24 *Suppose the tile T is a right triangle with $\alpha = \pi/10$, and suppose that triangle ABC is N-tiled by T, and angle C is at least a right angle. Then either T is similar to ABC, or ABC is isoceles and T is similar to half of ABC.*

Proof. The possible shapes of ABC are as follows:

(i) C is a right angle and angle $A = \alpha$ and angle $B = 4\alpha$. In that case, T is similar to ABC, so we are done.

(ii) Angle C is $\pi/2 + 3\alpha$ and angles A and B are both α. In that case ABC is isosceles and T is similar to half of ABC, so we are done.

(iii) Angle C is $\pi/2 + \alpha = 3\pi/5$ and angles A and B are both 2α.

(iv) C is a right angle and angle $A = 2\alpha$ and angle $B = 3\alpha$.

(v) Angle C is $\pi/2 + \alpha = 3\pi/5$, and angle $A = \alpha$ and angle $B = 3\alpha$.

(vi) Angle C is $\pi/2 + 2\alpha = 7\pi/10$, and angle A is α and angle $B = 2\alpha$.

Your favorite computer algebra system will tell you that

$$a = \sin\frac{\pi}{10} = \frac{1}{4}(\sqrt{5} - 1)$$

$$b = \cos\frac{\pi}{10} = \frac{1}{2}\sqrt{\frac{1}{2}(5 + \sqrt{5})}$$

Therefore $\mathbb{Q}(a)$ has degree 2 over $\mathbb{Q}$ and a^2 can be expressed linearly in a:

$$(4a + 1)^2 = 5$$
$$16a^2 + 8a + 1 = 5$$
$$4a^2 + 2a - 1 = 0$$

and the result is

$$a^2 = \frac{1}{4} - \frac{1}{2}a \tag{29}$$

Similarly $\mathbb{Q}(b)$ has degree 4 over $\mathbb{Q}$, so b^4 can be expressed in terms of lower powers of b:

$$4b^2 = \frac{1}{2}(5 + \sqrt{5})$$
$$8b^2 - 5 = \sqrt{5}$$
$$(8b^2 - 5)^2 = 5$$
$$64b^4 - 80b^2 + 25 = 5$$
$$16b^4 - 20b^2 + 5 = 0$$

and the result is

$$b^4 = \frac{5}{4}b^2 - \frac{5}{16} \tag{30}$$

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We can also see that a belongs to $\mathbb{Q}(b)$: a few lines above we showed $\sqrt{5} = 8b^2 - 5$, and hence

$$
\begin{aligned}
a &= \frac{1}{4}(\sqrt{5} - 1) \\
&= \frac{1}{4}(8b^2 - 5 - 1)
\end{aligned}
$$

with the result

$$
a = 2b^2 - \frac{3}{2} \tag{31}
$$

Since a belongs to $\mathbb{Q}(b)$, a basis for $\mathbb{Q}(a, b)$ over $\mathbb{Q}$ is 1, b, b^2, and b^3. We will need to express other quantities in that basis. For example

$$
\begin{aligned}
ab^2 &= (2b^2 - \frac{3}{2})b^2 \\
&= 2b^4 - \frac{3}{2}b^2 \\
&= 2\left(\frac{5}{4}b^2 - \frac{5}{16}\right) - \frac{3}{2}b^2
\end{aligned}
$$

with the result

$$
ab^2 = b^2 - \frac{5}{8} \tag{32}
$$

We also have

$$
\begin{aligned}
a^3 &= a^2 a \\
&= (\frac{1}{4} - \frac{1}{2}a)a \qquad \text{by (29)} \\
&= \frac{1}{4}a - \frac{1}{2}a^2 \\
&= \frac{1}{4}a - \frac{1}{2}\left(\frac{1}{4} - \frac{1}{2}a\right) \\
&= \frac{1}{2}a - \frac{1}{8} \\
&= \frac{1}{2}\left(2b^2 - \frac{3}{2}\right) - \frac{1}{8}
\end{aligned}
$$

with the result

$$
a^3 = b^2 - \frac{7}{8} \tag{33}
$$

We will also need an expression for $\sin 3\alpha$. We work that out as follows:

$$
\begin{aligned}
\sin 3\alpha &= 3\sin \alpha - 4\sin^2 \alpha \\
&= 3a - 4a^3 \\
&= 3(2b^2 - \frac{3}{2}) - 4(b^2 - \frac{7}{8}) \qquad \text{by (31) and (33)} \\
\sin 3\alpha &= 2b^2 - 1 \tag{34}
\end{aligned}
$$

With these expressions in hand, we consider side AC of triangle ABC, whose length we call Y. Let g, m, and f (respectively) be the numbers of a sides, b sides, and c sides of tiles that lie along AC in the given tiling. Since $c = \sin \gamma = 1$, we have

$$
Y = ga + mb + f
$$

We next work out Y^2 in the basis 1, b, b^2, b^3.

$$
\begin{aligned}
Y^2 &= (ga + mb + f)^2 \\
&= \left(g\left(2b^2 - \frac{3}{2}\right) + mb + f\right)^2 \qquad \text{by (31)} \\
&= \left(2gb^2 + mb + \left(f - \frac{3g}{2}\right)\right)^2 \\
&= 4g^2 b^4 + 4gmb^3 + \left[m^2 + 4g\left(f - \frac{3g}{2}\right)\right]b^2 + 2m\left(f - \frac{3g}{2}\right)b + \left(f - \frac{3g}{2}\right)^2 \\
&= 4g^2 b^4 + 4gmb^3 + \left[m^2 - 6g^2 + 4gf\right]b^2 + 2m\left(f - \frac{3g}{2}\right)b + \left(f^2 - 3gf + \frac{9}{4}g^2\right)
\end{aligned}
$$

Substituting in for b^4 from (30) we have

$$
Y^2 = 4g^2\left(\frac{5}{4}b^2 - \frac{5}{16}\right) + 4gmb^3 + \left[m^2 - 6g^2 + 4gf\right]b^2 + 2m\left(f - \frac{3g}{2}\right)b + \left(f^2 - 3gf + \frac{9}{4}g^2\right)
$$

and writing it as a polynomial in b we find

$$
Y^2 = 4gmb^3 + (m^2 - g^2 + 4gf)b^2 + 2m\left(f - \frac{3g}{2}\right)b + (f^2 - 3gf + g^2) \tag{35}
$$

We use the letters A and C to stand for the angles of ABC at vertices A and C, as well as for the vertices themselves. Let X be the length of side BC, opposite angle A. Then by the law of sines we have

$$
\frac{Y}{\sin B} = \frac{X}{\sin A}
$$

Therefore

$$
X = Y\frac{\sin A}{\sin B}
$$

The area of triangle ABC is given, according to the usual cross product formula, as

$$
\mathcal{A}_{ABC} = \frac{1}{2}XY\sin C
$$

Substituting for X we have

$$
\mathcal{A}_{ABC} = \frac{1}{2}Y^2\frac{\sin A \sin C}{\sin B}
$$

Let $\mathcal{A}_T = ab/2$ be the area of the tile T. Our fundamental equation is

$$
N\mathcal{A}_T = \mathcal{A}_{ABC}
$$

That is

$$
Nab = Y^2\frac{\sin A \sin C}{\sin B}
$$

(Since $\mathcal{A}_T = ab/2$, the factors of $1/2$ cancel and disappear.) Now we take up the argument by cases according as the possible shapes of ABC, as enumerated at the beginning of the proof. First we take up case (iii), where angles A and B are each 2α. Then $\sin B$ and $\sin A$ are equal, and cancel out, so we have

$$
Nab = Y^2\sin C
$$

Since in this case angle $C = \pi/2 + \alpha$, we have $\sin C = \cos\alpha = b$, so we can cancel b on the left with $\sin C$ on the right, obtaining

$$
Na = Y^2.
$$

Putting in on the right the expression for Y^2 found in (35), and using (31) on the left, we have

$$
N\left(2b^2 - \frac{3}{2}\right) = 4gmb^3 + (m^2 - g^2 + 4gf)b^2 + 2m\left(f - \frac{3g}{2}\right)b + (f^2 - 3gf + g^2)
$$

Since both sides are expressed in the basis $\{1, b, b^2, b^3\}$, the coefficients of like powers of b are equal. Equating the coefficients of b^3 we find $gm = 0$. If $g = 0$ then from the coefficients of b^2 we have $2N = m^2$. Then from the coeficients of b we have $mf = 0$, and since $m \neq 0$ we have $f = 0$. Then the constant coefficient on the right is zero, but on the left it is $-3N/2$. Hence $g \neq 0$. But since $gm = 0$ that implies $m = 0$. Then our equation becomes

$$N\left(2b^2 - \frac{3}{2}\right) = (4gf - g^2)b^2 + (f^2 - 3gf + g^2)$$

Equating coefficients of b^2 we have

$$2N \;\; = \;\; 4gf - g^2 \tag{36}$$

Equating the constant terms we have

$$
\begin{aligned}
-\frac{3N}{2} &= f^2 - 3gf + g^2 \\
3N &= 2(3gf - g^2 - f^2) \\
&= 2(4gf - g^2 - gf - f^2)
\end{aligned}
$$

Using (36) on the right hand side we find

$$
\begin{aligned}
3N &= 2(2N - gf - f^2) \\
&= 4N - 2(gf + f^2) \\
N &= 2(gf + f^2) \\
2N &= 4gf + 4f^2
\end{aligned}
$$

But also $2N = 4gf - f^2$, by (36). Subtracting this from the last equation we have $0 = 4f^2 + g^2$. Hence $g = f = 0$. But this contradicts $m = 0$, since $g + m + f$ is the number of tiles along side AC. That contradiction completes the proof in case (iii), when ABC is isosceles with base angles 2α.

Next we take up case (iv), when angle C is a right angle, angle A is 2α, and angle B is 3α. Then we have

$$
\begin{aligned}
2N\mathcal{A}_T &= Y^2 \frac{\sin A \sin C}{\sin B} \\
Nab &= Y^2 \frac{\sin 2\alpha}{\sin 3\alpha}
\end{aligned}
$$

We have $\sin 2\alpha = 2 \sin \alpha \cos \alpha = 2ab$, so the ab on the left cancels with $\sin 2\alpha$ on the right:

$$N = \frac{2Y^2}{\sin 3\alpha}$$

We have $\sin 3\alpha = 3 \sin \alpha \cos^2 \alpha - \sin^2 \alpha = 3ab^2 - a^3$, so

$$N(3ab^2 - a^3) \;\; = \;\; 2Y^2$$

We must express the left side in terms of the basis. We have already done the work; substituting the values for ab^2 and a^3 found in (32) and (33) we have

$$
\begin{aligned}
N(3(b^2 - \frac{5}{8}) - (b^2 - \frac{7}{8})) &= 2Y^2 \\
N(2b^2 - 1) &= 2Y^2 \\
N(b^2 - \frac{1}{2}) &= Y^2
\end{aligned}
$$

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and substituting the value for Y^2 found in (35) we have

$$N(b^2 - \frac{1}{2}) \quad = \quad 4gmb^3 + (m^2 - g^2 + 4gf)b^2 + 2m\left(f - \frac{3g}{2}\right)b + (f^2 - 3gf + g^2)$$

The right hand side is the same as in the previous case, but the left hand side is slightly different. As before, the coefficient of b^3 must be zero on the right, so $gm = 0$, and as before, if $g = 0$ then we compare coefficients of b^2. This time we find $N = m^2$. Comparing the coefficients of b we find $mf = 0$, but since $m \neq 0$ we have $f = 0$, and hence the constant term on the right is zero, while on the left it is $-N/2$. Therefore $g \neq 0$; but since $gm = 0$, we have $m = 0$. Then from equating the coefficients of b^2 we have

$$N = 4gf - g^2 \tag{37}$$

and by equating the constant terms we find

$$-\frac{N}{2} \quad = \quad f^2 - 3gf + g^2$$
$$\frac{N}{2} \quad = \quad 3gf - g^2 - f^2$$
$$\quad = \quad 4gf - g^2 - gf - f^2$$

Using (37) on the right hand side we find

$$\frac{N}{2} \quad = \quad N - gf - f^2$$

Solving for N we have

$$N = 2gf + 2f^2 \tag{38}$$

Now we have two expressions equal to N, namely (37) and (38). Equating them, we have

$$2gf + 2f^2 \quad = \quad 4gf - g^2$$
$$2f^2 \quad = \quad 2gf - g^2$$
$$f^2 + (f^2 - 2gf + g^2) \quad = \quad 0$$
$$f^2 + (f - g)^2 \quad = \quad 0$$

It follows that f and $f - g$ are both zero. Hence f and g are both zero, which contradicts $m = 0$ since $g + m + f$ is the number of tiles along side AC. That completes the proof in case (iv).

Next we take up case (v), in which angle C is $\pi/2 + \alpha$, angle A is α, and angle B is 3α. Then we have

$$Nab \quad = \quad Y^2 \frac{\sin A \sin C}{\sin B}$$
$$\quad = \quad Y^2 \frac{\sin \alpha \cos \alpha}{\sin 3\alpha}$$
$$\quad = \quad Y^2 \frac{ab}{\sin 3\alpha}$$

and canceling ab we have

$$N \quad = \quad \frac{Y^2}{\sin 3\alpha}$$

This equation differs from case (iv) only by a factor of 2 on the right side, but it is not exactly the same, so we must carry out the calculations. Expressing $\sin 3\alpha$ in terms of b by means of (34), we find

$$N(2b^2 - 1) \quad = \quad Y^2$$

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Substituting for Y^2 as before, we find

$$N(2b^2 - 1) \quad = \quad 4gmb^3 + (m^2 - g^2 + 4gf)b^2 + 2m\left(f - \frac{3g}{2}\right)b + (f^2 - 3gf + g^2)$$

Comparing the coefficients of b^3 we again find $gm = 0$. Assume, for proof by contradiction, that $m \neq 0$. Then from the coefficients of b, if $g = 0$ then $f = 0$ and vice-versa. If $f = g = 0$, then the constant term is zero on the right but $-N$ on the left. Hence $g \neq 0$. But since $gm = 0$ that contradicts the assumption $m \neq 0$. That contradiction shows that $m = 0$.

Equating the coefficients of b^2 we have

$$2N = 4gf - g^2 \tag{39}$$

and from the constant terms we have

$$\begin{aligned} N &= 3gf - g^2 - f^2 \\ &= 4gf - g^2 - gf - f^2 \\ &= 2N - gf - f^2 \qquad \text{by (39)} \end{aligned}$$

Solving for N we have

$$\begin{aligned} N &= gf + f^2 \\ 2N &= 2gf + 2f^2 \end{aligned}$$

and equating this expression for $2N$ with the one in (39), we have

$$\begin{aligned} 4gf - g^2 &= 2gf + 2f^2 \\ 0 &= g^2 - 2gf + 2f^2 \\ &= (g - f)^2 + f^2 \end{aligned}$$

Hence $f = 0$ and $g = 0$; and since we already have $m = 0$, this is a contradiction. That completes case (v).

The last case is case (vi), in which angle C is $\pi/2 + 2\alpha = 7\pi/10$, and angle A is α and angle $B = 2\alpha$. Then we have

$$\begin{aligned} Nab &= Y^2 \frac{\sin A \sin C}{\sin B} \\ &= Y^2 \frac{\sin \alpha \cos 2\alpha}{\sin 2\alpha} \\ &= Y^2 \frac{\sin \alpha (\cos^2 \alpha - \sin^2 \alpha)}{2 \sin \alpha \cos \alpha} \\ &= Y^2 \frac{a(b^2 - a^2)}{2ab} \end{aligned}$$

Canceling a and multiplying by the denominator, we have

$$2Nab^2 \quad = \quad Y^2(b^2 - a^2)$$

Putting in the value for a^2 from (29) on the right, and the value of ab^2 from (32) on the left, we find

$$\begin{aligned} 2N\left(b^2 - \frac{5}{8}\right) &= Y^2\left(b^2 - \left(\frac{1}{4} - \frac{1}{2}a\right)\right) \\ 2Nb^2 - \frac{5}{4}N &= Y^2\left(b^2 - \frac{1}{4} + \frac{1}{2}\left(2b^2 - \frac{3}{2}\right)\right) \qquad \text{by (31)} \\ &= Y^2(2b^2 - 1) \end{aligned}$$

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Substituting for Y^2 from (35) we have

$$2Nb^2 - \frac{5}{4}N = (2b^2 - 1)\left\{ 4gmb^3 + (m^2 - g^2 + 4gf)b^2 + 2m\left(f - \frac{3g}{2}\right)b + (f^2 - 3fg + g^2) \right\}$$

$$= 8gmb^5 + (m^2 - g^2 + 4gf)2b^4 + 4m\left(f - \frac{3g}{2}\right)b^3 - 4gmb^3$$
$$+ (-m^2 + g^2 - 4gf + 2(f^2 - 3fg + g^2))b^2$$
$$- 2m\left(f - \frac{3g}{2}\right)b - (f^2 - 3fg + g^2)$$

Multiplying (30) by b we have

$$b^5 = \frac{5}{4}b^3 - \frac{5}{16}b$$

We use this to eliminate the b^5 term, and also we use (30) to eliminate the b^4 term:

$$2Nb^2 - \frac{5}{4}N = 8gm\left(\frac{5}{4}b^3 - \frac{5}{16}b\right) + (m^2 - g^2 + 4fg)2\left(\frac{5}{4}b^2 - \frac{5}{16}\right)$$
$$+ 4m\left(f - \frac{3g}{2}\right)b^3 - 4gmb^3$$
$$+ (2f^2 - 10gf + 3g^2 - m^2)b^2$$
$$- 2m\left(f - \frac{3g}{2}\right)b - (f^2 - 3fg + g^2)$$

Expressing the right hand side as a polynomial in b, we have

$$2Nb^2 - \frac{5}{4}N = 4fmb^3 + (g^2/2 + 2f^2 + 3m^2/2)b^2$$
$$+ (gm/2 - 2fm)b$$
$$- 3g^2/8 + fg/2 - f^2 - 5m^2/8$$

Equating the coefficients of b^3 we find $fm = 0$. Hence either $m = 0$ or $f = 0$. Suppose, for proof by contradiction, that $f = 0$. Then the coefficient of b on the right is $gm/2$, but on the left it is zero. Hence $g = 0$ or $m = 0$. If $g = 0$ then equating the constant terms we have

$$\frac{5}{4}N = \frac{5m^2}{8}.$$

Then $2N = m^2$. Equating the quadratic terms we have

$$2N = \frac{3m^2}{2}$$
$$= 3N \qquad \text{since } m^2 = 2N$$

This is a contradiction, so $g \neq 0$. Still assuming $f = 0$, we now have $m = 0$. Equating the constant terms we have

$$\frac{5N}{4} = \frac{3g^2}{8}$$
$$10N = 3g^2$$

Equating the quadratic terms we have

$$2N = \frac{g^2}{2}$$
$$10N = \frac{5}{2}g^2$$

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Now $10N$ is equal to both $3g^2$ and $(5/2)g^2$, which is impossible. This contradiction shows that $f \neq 0$. Then from the cubic terms we have $m = 0$. From the quadratic and constant terms we have

$$2N = g^2/2 + 2f^2$$
$$\frac{5}{4}N = f^2 - \frac{fg}{2} + \frac{3g^2}{8}$$

Multiplying the first of these equations by 5 and the second by 8 we have

$$10N = \frac{5}{2}g^2 + 10f^2$$
$$10N = 8f^2 - 4fg + 3g^2$$

Subtracting the second equation from the first we have

$$0 = 2f^2 + 4fg - \frac{1}{2}g^2$$

Let $\xi = g/f$, which is legal since $f \neq 0$. Dividing the previous equation by f^2 we have

$$0 = 2 + 4\xi - \frac{1}{2}\xi^2$$

The discriminant of this quadratic equation is $\sqrt{16 - 4} = \sqrt{12} = 2\sqrt{3}$, which is irrational. Hence there are no integer solutions g and f of the equations above. That completes the proof of the lemma.

Theorem 5 *Suppose that the triangle ABC is N-tiled by a non-isosceles right triangle T. Then either*

(i) ABC is similar to T, or

(ii) ABC is isosceles, and T is similar to half of ABC. In this case exactly one vertex angle splits, and it splits into exactly two pieces.

(iii) ABC is equilateral, and T is a 30-60-90 triangle, and N is six times a square.

Proof. Let P, Q, and R be the total number of α, β, and γ angles of tiles at the vertices of ABC. Since $\gamma = \pi/2$ we have $R \leq 1$. We begin by showing that if $R = 1$, then the specific value $\alpha = \pi/8$ implies conclusion (i) or (ii) of the theorem. Assume $R = 1$. Then by the definition of R, one tile has a right angle at a vertex of ABC; it must be at angle C, the largest angle of triangle ABC. Assume that $\alpha = \pi/8$. If vertex C is split, and one α angle occurs at C in addition to the right angle, then angle C is $5\pi/8$, and one of the two remaining angles is α. That case has been ruled out in Lemma 23. If the angle at vertex C is split, and two additional α angles occur there, then triangle ABC is isosceles with base angles α. So T is similar to half of ABC, which is conclusion (ii) of the theorem. Therefore we may assume that the angle at vertex C is not split (so only the right angle occurs there). If each of the other two vertices is split into two α angles, then angles A and B are equal, and ABC is a right isosceles triangle. This case has been ruled out in Lemma 22. If the four α angles are distributed one and three, then ABC is similar to T, which is conclusion (i) of the theorem. That disposes of the case $\alpha = \pi/8$ and $R = 1$.

We will prove that either one of the conclusions of the theorem holds, or $P + Q + R = 4$. First we take up the case $R = 0$. Then we have

$$P\alpha + Q\beta = \pi$$
$$\alpha + \beta = \pi/2$$

We note that $P = Q = 2$ does provide a solution of those two equations. We shall show that in every other case, the conclusion of the theorem holds (or a contradiction ensues). We begin by

considering the possibilities for Q. We have $\beta < \pi/2$ and $\alpha \le \beta$, and $\alpha + \beta = \pi/2$. Therefore $\beta \ge \pi/4$. From $P\alpha + Q\beta = \pi$ we then obtain $Q \le 4$. If $Q = 4$ then $\beta = \alpha = \pi/4$, contradicting the assumption that T is not isosceles. Hence $Q < 4$.

Next we consider the case $Q = 3$. Then subtracting the equation $\alpha + \beta = \pi/2$ from the equation $P\alpha + Q\beta = \pi$ we obtain

$$(P - 1)\alpha + 2\beta = \pi/2.$$

If $P > 1$ then the left side exceeds $\alpha + \beta = \pi/2$, contradiction. If $P = 1$ then $\beta = \pi/4$, but then $\alpha = \pi/4$, contradicting $\alpha < \beta$. Hence we must have $P = 0$. Then $-\alpha + 2\beta = \pi/2$. Together with $\alpha + \beta = \pi/2$ this implies that $\alpha = \pi/6$ and $\beta = \pi/3$. With $R = 0$ and $P = 0$, all the angles of ABC must be composed of three β angles, so ABC is equilateral, and there is no vertex splitting. Then case (ii) holds, since an equilateral triangle is isoceles and T is similar to half of ABC. That disposes of the case $Q = 3$.

Next we claim that if $Q = 1$, then $P \ge 4$, and if $Q = 0$ then $P > 4$. First suppose $Q = 1$. Then $P\alpha + \beta = \pi$. Subtracting $\alpha + \beta = \pi/2$ we have $(P - 1)\alpha = \pi/2$. Since $\alpha < \pi/4$, we have $P - 1 > 2$, i.e. $P \ge 4$, as claimed. Now assume $Q = 0$. Then $P\alpha + Q\beta = \pi$ implies $\alpha = \pi/P$; but since $\alpha < \pi/4$, we have $P > 4$. This completes the proof of the claim.

Now we consider the cases $Q = 0$ or $Q = 1$ further (still assuming $R = 0$). In that case $P \ge 4$, so in particular $P > Q$. That is, there are more α angles than β angles at the vertices of ABC. Therefore, there is another vertex V of the tiling at which there are more β angles than α angles (since there are N of each altogether). Fix such a vertex V, and let k be 1 if V is a boundary or a non-strict vertex, and $k = 2$ if V is a strict interior vertex; so the angle sum at V is $k\pi$. Let n, m, and ℓ be the numbers of α, β, and γ angles (respectively) at V. Then we have $m > n$ and

$$\begin{pmatrix} n & m & \ell \\ 1 & 1 & 1 \\ P & Q & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} k\pi \\ \pi \\ \pi \end{pmatrix}$$

Solving for γ by Cramer's rule (or Gaussian elimination) we find

$$[(m - \ell)P - Q(n - \ell)]\gamma = [n(1 - Q) + m(P - 1) + k(Q - P)]\pi. \tag{40}$$

As long as we don't divide both sides by the coefficient of γ, the equation is valid whether or not that coefficient is zero. Remembering that $\gamma = \pi/2$ we have

$$(m - \ell)P - Q(n - \ell) \quad = \quad 2[n(1 - Q) + m(P - 1) + k(Q - P)] \tag{41}$$

Assume $Q = 0$. Then we will prove $\beta = \pi/3$ and $\alpha = \pi/6$. Since $Q = 0$, we have $P > 4$ as proved above. We have

$$\begin{aligned}
(m - \ell)P &= 2[n + m(P - 1) - kP] \\
P(m - \ell - 2m + 2k) &= 2(n - m) \\
P(-\ell - m + 2k) &= 2(n - m) \\
P(\ell + m - 2k) &= 2(m - n) \\
P &= \frac{2(m - n)}{\ell + m - 2k} \tag{42}
\end{aligned}$$

The right side is positive since $m > n$, so the left side is also positive. Since we still are assuming $R = 0$, the assumption $Q = 0$ means that all the vertices of ABC are composed only of α angles, so $\pi = P\alpha$, i.e. $\alpha = \pi/P$. We claim k cannot be 1. Suppose that $k = 1$ and $\ell = 0$; then $\pi = n\alpha + m\beta$, and since $\beta > \pi/4$ we have $m = 0, 1, 2,$ or 3. Since $2\beta + 2\alpha = \pi$ we must have $m = 3$ in order that $m > n$. Then $n = 1$ or $n = 0$. If $n = 0$ then $\beta = \pi/3$ as desired. Hence we may suppose $n = 1$; then $n - m = 2$, so $P = 2(m - n)/(\ell + m - 2k) = 4$, contradicting

$P > 4$. On the other hand if $k = 1$ and $\ell = 1$ we have only an angle of $\pi/2$ to fill with α and β angles, and it cannot be done with $m > n$. That proves that k cannot be 1.

So $k = 2$. Then $P = 2(m - n)/(\ell + m - 4)$. We cannot have $\ell = 4$, as in that case there would be four right angles at V, and hence there would be no α and β angles at V, contradicting $m > n$. If $\ell = 3$ and $n > 0$ then there is one α and one β at V, contradicting $m > n$. If $\ell = 3$ and $n = 0$ then $\pi/2$ is made of m angles of measure β, i.e. $\beta = \pi/m$. If $m = 3$ our claim that $\beta = \pi/3$ holds; and since $\alpha < \beta$ and $\alpha + \beta = \pi/2$, we have $\beta > \pi/4$, and hence $\beta \neq \pi/m$ for any m (except possibly 3, which we are trying to prove must be the case). If $\ell = 2$ and $n > 0$ then there must be 3 β angles and one α at V, i.e. $m = 3$ and $n = 1$ (since if $n = 2$ then $m = 2$ because $2\alpha + 2\beta + 2\gamma = 2\pi$, so $n < 2$).

$$
\begin{aligned}
P &= 2(m - n)/(\ell + m - 4) \\
&= 2/(2 + 3 - 4) \\
&= 2
\end{aligned}
$$

contradicting $P > 4$. If $\ell = 2$ and $n = 0$ we have $m\beta = \pi$, and the only possibility with $\beta > \pi/4$ is $\beta = \pi/3$, which is the desired conclusion. If $\ell = 1$ then there is $3\pi/2$ to make up from α and β angles, and since $\alpha = \pi/P$, we have $\beta = \pi/2 - \alpha = \pi/2 - \pi/(2P)$. Then $n\alpha + m\beta = 3\pi/2$ becomes

$$
n\frac{\pi}{P} + m\left(\frac{\pi}{2} - \frac{\pi}{P}\right) = \frac{3\pi}{2}.
$$

Solving this equation for n we find

$$
n = 3P/2 - m(P - 2)/P.
$$

Then

$$
\begin{aligned}
P &= \frac{2(m - n)}{\ell + m - 4} \\
&= \frac{2(m - (3P/2 - m(P - 2)/P))}{1 + m - 4} \\
&= \frac{2m - 3P + 2m(P - 2)/P}{m - 3} \\
P^2(m - 3) &= 2mP - 3P^2 + 2m(P - 2) \\
mP^2 - 4mP + 4m &= 0 \\
P^2 - 4P + 4 &= 0 \qquad \text{since } m > n, \text{ so } m \neq 0 \\
(P - 2)^2 &= 0 \\
P &= 2
\end{aligned}
$$

But that contradicts $P > 4$, so $\ell = 1$ is impossible.

Hence $\ell = 0$. Then the angle of 2π at V is made up entirely of α and β angles. Since we still have $R = Q = 0$, $\alpha = \pi/P$, so $n\alpha + m\beta = 2\pi$ becomes

$$
n\left(\frac{\pi}{P}\right) + m\left(\frac{\pi}{2} - \frac{\pi}{P}\right) = 2\pi.
$$

Solving for n we find

$$
n = 2P - m\frac{2 - P}{P}.
$$

Then

$$
\begin{aligned}
P &= \frac{2(m - n)}{\ell + m - 4} \\
&= \frac{2(m - n)}{m - 4} \qquad \text{since } \ell = 0
\end{aligned}
$$

$$= \frac{2(m - (2P - m(P-2)/P))}{m-4}$$

$$(m-4)P^2 = 2mP - 4P^2 + 2m(P-2)$$

$$mP^2 = 4mP - 4m$$

$$P^2 - 4P + 4 = 0 \qquad \text{since } m \neq 0$$

$$(P-2)^2 = 0$$

$$P = 2$$

But that contradicts $P > 4$, so $\ell = 0$ is also impossible. That exhausts all the possibilities for ℓ and k. Hence $Q = 0$ is possible only when $\beta = \pi/3$, as claimed.

Now assume, for proof by contradiction, that $Q = 1$. Then we have by (40)

$$(m - \ell)P - (n - \ell) = 2[m(P-1) + k(1-P)]$$

$$\ell(1-P) + mP - n = 2(m-k)(P-1)$$

$$(P-1)(m-\ell) + (m-n) = (P-1)(2m - 2k)$$

$$m - n = (P-1)(m - 2k + \ell)$$

$$2k - \ell = \frac{n + (P-2)m}{P-1}$$

$$= \frac{n - m + (P-1)m}{P-1}$$

$$= m - \frac{m-n}{P-1}$$

The numerator $m - n$ is positive, and the denominator $P - 1$ is at least 3, so $P - 1$ divides $m - n$. Since $m - n$ is positive, we have $m - n \geq P - 1 \geq 3$, so $m \geq 4$. Since $\beta > \pi/4$ and $m \geq 4$, we have $m\beta > \pi$, so the β angles at vertex V occupy more than a straight angle. Hence $k \neq 1$. Hence $k = 2$, and also $\ell = 0$ or $\ell = 1$, since more than one right angle cannot be accommodated at V (since $m \geq 4$, and $4\beta > \pi$). Substituting 2 for k we have

$$4 - \ell = m - \frac{m-n}{P-1}$$

$$(P-1)(4-\ell) = (P-1)m - (m-n)$$

$$m(P-2) = (P-1)(4-\ell) - n$$

$$m = \frac{P-1}{P-2}(4-\ell) - \frac{n}{P-2}$$

Since $P \geq 4$, the fraction $n/(P-2)$ is positive, and the fraction $(P-1)/(P-2)$ is between 0 and 1, so

$$m < \frac{P-1}{P-2}(4-\ell)$$

$$< 4 - \ell \leq 4$$

But this contradicts the result $m \geq 4$ that we proved above. This contradiction completes the proof that $Q \neq 1$.

Now assume $Q = 2$. We claim that $P = 2$. Subtracting $\alpha + \beta = \pi/2$ from $P\alpha + Q\beta = \pi$, and using $Q = 2$, we have

$$(P-1)\alpha + \beta = \pi/2$$

But since $\alpha + \beta = \pi/2$, it follows that $P = 2$ as claimed. That completes the proof that $P = 2$, in case $R = 0$ and $Q = 2$.

We now consider the case $R = 1$. We claim $P + Q + R = 4$. Assume first that no vertex splitting occurs, i.e. each angle of ABC is equal to α, β, or γ. Since $R = 1$, angle $C = \gamma = \pi/2$.

Then angles A and B are each equal to α or β; but since T is not isosceles, we must have $B = \beta$ and $A = \alpha$, so ABC is similar to T, and case (i) of the theorem holds. Hence we may assume from now on (under the assumption $R = 1$) that some vertex splitting does occur; and we proved already that when $R = 0$, some vertex splitting occurs. Therefore $P + Q \geq 3$.

We have $P\alpha + Q\beta = \pi/2$. Subtracting the equation $\alpha + \beta = \pi/2$, we have $(P-1)\alpha + (Q-1)\beta = 0$. Since $P + Q \geq 3$, we cannot have $P = Q = 1$; hence either $P = 0$ or $Q = 0$. If $P = 0$ then $\alpha = (Q-1)\beta$, which is impossible since $\alpha < \beta$. Hence $Q = 0$ and $\beta = (P-1)\alpha$. Since $\alpha + \beta = \pi/2$, we have $\alpha = \pi/(2P)$. Since $Q = 0$, there are more α angles than β angles at the vertices of ABC. Therefore there exists a vertex V at which there are more β angles than α angles, i.e. at which $m > n$. We have

$$n\alpha + m\beta + \ell\gamma = k\pi \qquad \text{where } k = 1 \text{ or } k = 2$$
$$n\frac{\pi}{2P} + m\left(\frac{\pi}{2} - \frac{\pi}{2P}\right) + \ell\frac{\pi}{2} = k\pi \qquad \text{since } \alpha = \frac{\pi}{2P}$$

Dividing by $\pi/2$ we have

$$\frac{n}{P} + m\left(1 - \frac{1}{P}\right) + \ell = 2k \tag{43}$$

Now we consider the possible values of ℓ. We have $\ell \leq 3$ since $\ell\gamma < 2\pi$ and $\gamma = \pi/2$. Since $m > n$ there is at least one β angle at V. If $\ell = 3$ then we must have $m = n = 1$, since T is not isosceles and $\alpha + \beta = \pi/2$; but since $m > n$, that is ruled out.

Consider the case $\ell = 2$. That means π of the angle at V is used up by two γ angles. Since $m > 0$, there is at least one β angle at V, so the total angle at V is more than π. Therefore $k = 2$ and we must have α and β angles adding up to π. It cannot be two of each since $m > n$. Hence there are at least three β angles. Since $\beta > \pi/4$, there cannot be as many as four β angles, as that would make more than 2π total at V. Hence there are exactly three β angles, i.e. $m = 3$. Equation (43) becomes

$$\frac{n}{P} + 3\left(1 - \frac{1}{P}\right) + 2 = 4$$

Solving for P we find $P = 3 - n$. Since $P + Q \geq 3$ and $Q = 0$, we have $P \geq 3$. Hence $n = 0$ and $P = 3$. Since $R = 1$ and $Q = 0$, we now have $P + Q + R = 4$, which is what we are trying to prove. That disposes of the case $\ell = 2$.

Now consider the case $\ell = 1$. Then we have $k = 2$, since if $k = 1$ we must have α and β angles adding up to $\pi/2$, with more β than α angles, but that is not possible, since one β angle leaves room for only one α angle. With $k = 2$ and $\ell = 1$, we must have α and β angles adding up to $3\pi/2$, with more β than α angles. We cannot have $m > 4$ since $5\beta = 5\pi/2 - 5\alpha = 5\pi/2 - 5\pi/(2P) > 3\pi/2$, since $P \geq 3$. By (43) we have

$$\frac{n}{P} + m\left(1 - \frac{1}{P}\right) + 1 = 4$$
$$\frac{n}{P} + m\left(1 - \frac{1}{P}\right) = 3$$
$$n = (3 - m)P + m$$

With $m \leq 3$ we have $n = (3 - m)P + m \geq m$, contradicting $m > n$. With $m = 4$ we have $n = 4 - P$. Since $n \geq 0$ and $P \geq 3$, the only possibilities are $n = 0$ and $P = 4$, or $n = 1$ and $P = 3$. If $P = 3$ then $P + Q + R = 4$, which is what we are trying to prove. Hence we may assume $n = 0$, $m = 4$, and $P = 4$. Then $\alpha = \pi/8$ and $\beta = 3\pi/8$. But at the beginning of the proof we already disposed of the case $R = 1$ and $\alpha = \pi/8$.

Finally we consider the case $\ell = 0$. First consider $k = 1$; then the total angle at V is π and none of it is used by right angles, so $n\alpha + m\beta = \pi$. Equation (43) becomes

$$\frac{n}{P} + m\left(1 - \frac{1}{P}\right) = 2$$

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which implies

$$m\left(1 - \frac{1}{P}\right) \leq 2$$

$$m \leq \frac{2P}{P-1}$$

$$m < 3$$

That implies $m \leq 2$, since m is an integer. But since $m > n \geq 0$, we must have $m = 1$ and $n = 0$, or $m = 2$ and $n = 1$, or $m = 2$ and $n = 0$. If $m = 1$ and $n = 0$ we have

$$2 = \frac{n}{P} + m\left(1 - \frac{1}{P}\right)$$

$$= \left(1 - \frac{1}{P}\right)$$

$$< 1$$

so the case $m = 1$ and $n = 0$ is impossible. If $m = 2$ we have

$$2 = \frac{n}{P} + m\left(1 - \frac{1}{P}\right)$$

$$= \frac{n}{P} + 2\left(1 - \frac{1}{P}\right)$$

$$0 = \frac{n}{P} - \frac{2}{P}$$

$$= \frac{n-2}{P}$$

$$n = 0$$

Then the equation $n\alpha + m\beta = \pi$ becomes $2\beta = \pi$; but since $\beta < \pi/2$, that is impossible. This contradiction eliminates the case $\ell = 0$, $k = 1$.

That leaves the case $\ell = 0$, $k = 2$. Then we have from (43)

$$\frac{n}{P} + m\left(1 - \frac{1}{P}\right) = 4$$

$$m - n = (m - 4)P$$

Since $m - n > 0$, we have $m \geq 5$. Since $P \geq 3$ and $P = (m - n)/(m - 4)$, we have

$$3 \leq \frac{m-n}{m-4}$$

$$3m - 12 \leq m - n$$

$$m \leq 6 - n/2$$

Since $5 \leq m$ we have only the possibilities $m = 5$, $n = 0$, 1, or 2, and $m = 6$, $n = 0$. Since $P = (m - n)/(m - 4)$, if $m = 6$ we have $P = 3$, and in that case $P + Q + R = 4$ as claimed. So we may assume that $m = 5$, in which case we have $P = 5$, 4, or 3, according as $n = 0$, 1, or 2. Then $\alpha = \pi/(2P)$ is $\pi/10$, $\pi/8$, or $\pi/6$. In case $P = 3$ we have $P + Q + R = 4$ as desired. In case $P = 4$ we have $\alpha = \pi/8$, and $R = 1$, and this case was already treated at the beginning of the proof. In case $m = P = 5$, we have $\alpha = \pi/10$. This case cannot occur, by Lemma 24.

We have now proved that in every case in which there is vertex splitting, either $P+Q+R = 4$, or $\alpha = \pi/6$ and $\beta = \pi/3$, or conclusion (i) or conclusion (ii) of the theorem holds. Now assume $P + Q + R = 4$. Then exactly one vertex angle of ABC is split, and that angle is split into only two pieces. The other two angles must therefore each be equal to an angle of the tile T. If two of them are equal to the same angle of T, then ABC is isosceles. We may assume T is not equilateral, since in that case no vertex splitting could occur. The two equal angles cannot

be β since $2\beta + \gamma > \alpha + \beta + \gamma = \pi$. Therefore the two equal angles are α. Since an α angle cannot split, the vertex angle must split. It could split into $\alpha + \beta$, or $\alpha + \gamma$, or 2β, or $\beta + \gamma$. The latter is impossible, since then $2\alpha + \beta + \gamma = \pi$, which contradicts $\alpha + \beta + \gamma = \pi$ since $\alpha > 0$. If the vertex angle splits into $\alpha + \beta$, then the angle sum of ABC is $\pi = 3\alpha + \beta$; subtracting $\alpha + \beta = \pi/2$ we have $\alpha = \pi/4 = \beta$, contradicting the hypothesis that T is not isosceles. If the vertex angle splits into $\alpha + \gamma$, then then angle sum of ABC is $3\alpha + \gamma = \pi$. Since $\gamma = \pi/2$, we have $\alpha = \pi/6$. Hence $\beta = \pi/3$ and $\alpha + \gamma = 2\pi/3 = 2\beta$. Hence the vertex angle of ABC is 2β. If the vertex angle splits into 2β, then conclusion (ii) of the theorem holds, i.e., the tile is similar to half of ABC.

We have proved that if the two unsplit angles of ABC are equal, conclusion (ii) of the theorem holds. Therefore we may assume that the two unsplit angles of ABC are not equal; hence they are equal to two different angles of T. Therefore triangle ABC is similar to T, which is conclusion (i) of the theorem.

We had proved that either $P + Q + R = 4$, or $\alpha = \pi/6$, or one of the conclusions of the theorem holds. We have now disposed of the case $P + Q + R = 4$. That leaves only the case when $\alpha = \pi/6$ to consider. But in that case, by Theorem 4, conclusion (iii) of the theorem holds. That completes the proof.

9   Conclusions

We have classified the tilings of equilateral and isosceles triangles ABC, and also tilings in which the tile is a right triangle. Our main results are summarized in these theorems:

Theorem 6 *If ABC is a right triangle and T is similar to ABC, then*

(i) If $\tan\alpha$ is rational, then ABC can be N-tiled by T if and only if N is a square, or $N = e^2 + f^2$ for integers e and f such that $\tan\alpha = e/f$.

(ii) if $\alpha = \pi/6$, then ABC can be N-tiled by T if N is a square, or three times a square, and for no other values of N.

(ii) If $\tan\alpha$ is not rational and $\alpha \neq \pi/6$, then ABC can be N-tiled by T if and only if N is a square.

Proof. All the tilings mentioned in the theorem were exhibited and illustrated, so it only remains to prove that there are no more. Suppose ABC is a right triangle and T is similar to ABC, and ABC is N-tiled by T.

Ad (i): Suppose $\tan\alpha$ is rational. Then T is not a 30-60-90 triangle, since $\tan(\pi/6) = 1/\sqrt{3}$ is not rational. Then by Theorem 2, N is either a square, or a sum of squares $e^2 + f^2$ where $\tan\alpha = e/f$.

Ad (ii). Suppose $\alpha = \pi/6$. Then by Theorem 2, N is either a square, or three times a square, as claimed.

Ad (iii). Suppose $\tan\alpha$ is not rational and $\alpha \neq \pi/6$. Then by Theorem 2, N is a square. That completes the proof.

Theorem 7 *If ABC is equilateral, then ABC can be N-tiled if $N = 3m^2$ and the tile is isoceles with base angles $\pi/6$, or if $N = 6m^2$ or $2m^2$ and the tile is a right angle with $\alpha = \pi/6$, or if N is a square and the tile is similar to ABC. If T is a right triangle with $\alpha \neq \pi/6$, then there are no N-tilings of ABC by T for any N.*

Remark. In [1], we prove that there are no tilings of an equilateral ABC by *any* tile other than those mentioned here, except possibly by a non-isosceles tile with a $120°$ angle. The case of a tile with a $120°$ angle is taken up in [3]. But in *this* paper, the only tilings of an equilateral ABC that are ruled out are those by a right-angled tile.

Proof. The $3m^2$ tilings are formed by dividing ABC into three tiles as in Fig. 1, that is, by three line segments from the center to the vertices, and then quadratically tiling each of those triangles. The $2m^2$ tilings are formed by dividing ABC in half along an altitude, and then quadratically tiling each half. The $6m^2$ tilings are formed by dividing ABC in half along an altitude, and then tiling each half into $3m^2$ tiles, as illustrated in Fig. 11. If N is a square, there is the quadratic tiling. If the tile T is a non-isosceles right triangle then, according to Theorem 5, T is similar to half of ABC. Then T is a 30-60-90 triangle, so $\alpha = \pi/6$. By Theorem 4, the only possibile tilings by this tile are when $N = 2m^2$ or $N = 6m^2$. That completes the proof of the theorem.

Theorem 8 *If ABC is isosceles but not equilateral, and T is a right triangle or is similar to ABC, then ABC can be N-tiled by tile T if and only if one of the following conditions holds:*

(i) N is a square and T is similar to ABC, or

(ii) N is a square and T is similar to half of ABC, and $\alpha = \beta = \pi/4$, or

(iii) $N/2$ is a square, and T is similar to half of ABC, or

(iv) $N/6$ is a square, T is a 30-60-90 triangle, and the base angles of ABC are $\pi/6$.

Proof. Since T is assumed to be a right triangle, if T is not isosceles, then either T is similar to ABC or T is similar to half of ABC, by Theorem 5. If T is similar to half of ABC, then by Theorem 4, and the hypothesis that ABC is not equilateral, one of the listed alternatives holds. If T is a right isosceles triangle, then $\alpha = \beta = \pi/4$, so all the angles of ABC are multiples of $\pi/4$. Since ABC is isosceles, then two of its angles are equal, so these two must be $\pi/4$, making ABC and half of ABC both similar to the tile T, so conclusion (ii) holds. (We can either tile ABC quadratically, or tile each half of ABC quadratically, using a smaller tile of the same shape.)

Now suppose that T is not a right triangle. If T is similar to ABC then by Lemma 11, either N is a square, or $\gamma = \pi/2$. Hence, under the assumption that T is a not a right triangle, either (i) holds or T is not similar to ABC. That completes the proof of the theorem.

Theorem 9 *If ABC is similar to T, and ABC is not a right triangle, and not isosceles or equilateral, then ABC can be N-tiled by T only when N is a square.*

Proof. This follows immediately from Theorem 2.

Taken together these theorems provide a complete characterization of the triples (ABC, N, T) such that ABC can be N-tiled by T, when ABC is similar to T, or T is a right triangle. The remaining possibilities for ABC and T, except for the case when T has a $120°$ angle and is not isosceles, are successfully treated in [1] and [2], where it is shown that is exactly one more family of tilings not mentioned here. The case when T has a $120°$ angle and is not isosceles is taken up in [3].

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