PROVING HILBERT'S AXIOMS IN TARSKI GEOMETRY

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Abstract. Hilbert's geometry and Tarski's geometry have the same models, since both are able to define addition and multiplication. But has anyone given explicit proofs of Hilbert's axioms in Tarski geometry? The purpose of this note is to show that this has been done by Wanda Szmielew, by citing the specific theorems of her development that are needed. We define an explicit translation of Hilbert's language into Tarski's language and use Szmielew's theorems to verify that this translation is an interpretation, i.e. takes theorems into theorems. Of course this is not an original result, but it doesn't seem to appear explicitly in the literature and one geometer wanted to see it written out.

1. Introduction

Hilbert's geometry is the system given in his famous book [1]. Tarski's geometry is the system of axioms A10-A11 used in [2] and explained in English in [3]. Part I of the book [2] is, with "inessential changes", the manuscript developed by Wanda Szmielew at Berkeley in 1965. Our purpose here is to cite chapter and verse from her manuscript to verify the axioms of Hilbert's geometry.

We need to be slightly more precise about the axioms to be used. We first consider systems whose models are "Hilbert planes", i.e. on the Tarski side we use only A1-A9. That is, we omit the parallel axiom and all continuity axioms. On the Hilbert side, we similarly omit all continuity axioms. We verify that A1-A9 suffices to interpret Hilbert's axioms for Hilbert planes.

Then we separately verify that (i) Tarski's parallel axiom A10 suffices to interpret Hilbert's parallel axiom, (ii) Tarski's versions of line-circle and circle-circle continuity suffice to interpret Hilbert's versions of those axioms.

Finally, in the last section, we discuss the full continuity axioms of both theories.

Hilbert's language has three sorts, for points, lines, and planes, and two incidence relations, for points lying on lines or on planes. Of course, it uses betweenness $B(x, y, z)$ and equidistance $E(a, b, c, d)$, which Hilbert writes as $ab \equiv cd$. Literally, Hilbert says on page 10 that $\equiv$ is a relation between segments, which are sets of two points (not the set of all points lying on the segment); so if we read Hilbert literally, he is not using E as a relation between four points; and he says (also on page 10) that it is not necessary to have $ab \equiv ba$ as an axiom, since (in effect) that follows from the extensionality axiom for sets of points. This point doesn't matter for our purposes, as we will interpret $ab \equiv cd$ as $E(a, b, c, d)$ in Tarski's theory; whether one should formalize Hilbert directly using E or using second or third order logic does not even need discussion.

Hilbert's betweenness is strict, which means that $B(x, y, z)$ implies that x, y, and z are distinct points.

Tarski has only one sort, for points, and uses non-strict betweenness. We write $T(a, b, c)$ for Tarski's betweenness relation, to avoid confusion. Thus $T(x, x, y)$ holds, for example. Tarski uses the same equidistance relation as Hilbert, but since there is only one sort, no incidence relations are required. Since there is only one sort, in Tarski's geometry one represents lines by two distinct points, and angles by three distinct points, and circles by two distinct points (a center and a point on the circle).

In 1899, the modern concept of first-order theory had not yet been developed. Although Hilbert uses three sorts for points, lines, and planes, he defines segments and rays as sets of points, and angles as sets of two rays. In modern terms, he is literally using a third-order theory. Nevertheless the sets required are far from arbitrary sets of points, and his theory can easily be interpreted in a first-order way, using two points for a segment and three points for an angle. Rather than introduce a first-order version of Hilbert's axioms, we just go all the way in one step, interpreting Hilbert's third-order theory in Tarski's first-order theory.

Two important concepts in geometry are "points a and b are on the same side of line L" and "points a and b are on opposite sides of line L." Hilbert defines these only when a, b, and L all lie in the same plane, and his definition is that a and b are on opposite sides of L if they are not on L and segment ab meets L, while they are on the same side of L if they are not on L and segment ab does not meet L. Szmielew defines "opposite sides" in the same way, but she defines "same side" differently: a and b are on the same side of L, if there is a point c such that a and c are on opposite sides of L, and a and b are on opposite sides of L. One advantage of this definition is that it works in more than two dimensions, and can be used to define what it means for four points to be coplanar: a, b, c, and d are coplanar if c and d lie either on the same side or on opposite sides of line ab, or c lies on line ab, or d lies on line ab. (Note that it is far from immediate that the order of the four points does not matter. The definition requires a specific order.)

2. Defining an Interpretation from $\mathbb{H}$ to $\mathbb{T}$

Let $\mathbb{H}$ be Hilbert's axioms for Hilbert planes, and $\mathbb{T}$ be Tarski's A1-A9. Then we will define for each formula ϕ of $\mathbb{H}$, its interpretation ϕ^* in $\mathbb{T}$, and show that for each axiom ϕ of $\mathbb{H}$, ϕ^* is provable in $\mathbb{T}$. We define the interpretation to commute with the propositional connectives and to behave naturally with respect to quantifiers, so that it will follow easily that for each theorem ϕ of $\mathbb{H}$, ϕ^* is provable in $\mathbb{T}$.

We first divide the variables of $\mathbb{T}$ into several subsets. We may assume these variables are $v_1, v_2, \ldots, v_n, \ldots$ and by considering the subscripts mod 12 (for example) we can select disjoint subsets. Variables whose subscripts are congruent to 0 mod 12 will be used to interpret point variables; pairs of variables whose subscripts are congruent to 1 and 2 will be used to interpret line variables; triples of variables whose subscripts are congruent to 3, 4, and 5 will be used to interpret plane variables; similarly for segments, rays, and angles. Thus we avoid any problems of collision of variables in the interpretation; each variable

x of $\mathbb{H}$ is assigned to a variable (or pair or triple of variables) x^* of $\mathbb{T}$. In case x^* is a pair or triple, we use the notation x^*_k for $k = 1, 2$ and possibly 3, to denote the individual variables.

The collinearity relation $Col(a, b, c)$ in $\mathbb{T}$ is defined by $T(a, b, c) \vee T(b, c, a) \vee T(c, a, b)$. The coplanarity relation $Coplanar(a, b, c, d)$ has been defined above.

We then define the interpretation on atomic formulas as follows:

$$\begin{aligned}
E(a,b,c,d)^* &= E(a^*,b^*,c^*,d^*) \\
B(a,b,c)^* &= T(a^*,b^*,c^*) \wedge a^* \neq b^* \wedge b^* \neq c^* \wedge a^* \neq c^* \\
(\text{line } L \text{ contains point } x)^* &= Col(L_1^*, L_2^*, x^*) \\
(\text{plane } P \text{ contains point } x)^* &= Coplanar(L_1^*, L_2^*, L_3^*, x^*) \\
(x \in ab)^* &= T(a,x,b) \\
(x \in Ray(a,c))^* &= T(a,x,c) \vee T(a,c,x)
\end{aligned}$$

If angle A is determined by $Ray(a, b)$ and $Ray(a, c)$, then according to Hilbert, the three points a, b, c are not collinear. We interpret the statement "x lies in the interior of angle A" to mean "x and c lie on the same side of line ab, and x and a lie on the same side of line ac." If circle K has center a and contains point b, then the interpretation of "x lies on K" or "K contains x" is $E(a, b, a, x)^*$.

If x^* is a single variable, then $Distinct(x^*)$ is just $x^* = x^*$. If x^* is a list of two variables, $Distinct(x^*)$ is $x^*_1 \neq x^*_2$. If x^* is a list of three variables, $Distinct(x^*)$ is

$$x^*_1 \neq x^*_2 \wedge x^*_1 \neq x^*_3 \wedge x^*_3 \neq x^*_3.$$

We define the interpretation to commute with the propositional connectives, and we define it on quantified formulas as follows:

$$(\forall x\, \phi)^* = \forall x^*\, (Distinct(x^*) \rightarrow \phi^*)$$
$$(\exists x\, \phi)^* = \exists x^*\, (Distinct(x^*) \wedge \phi^*)$$

This notation covers at one blow the cases of quantifiers over points, lines, segments, angles, and circles, since x^* has been defined appropriately in each of those cases. It is thus easy to verify that inferences are preserved. Technically, we should choose a particular formulation of the predicate calculus (with specific axioms and inference rules), and then verify that inference steps are preserved by the interpretation; but we omit these details as obvious.

3. The interpretations of Hilbert's Group I axioms

In the following, when we cite "Satz so-and-so", we are referring to the theorems of Szmielew's manuscript, Part I of [2].

Axiom I,1. For every two points x, y, there exists a line that contains x and y.

The interpretation of this axiom is

$$\forall x, y \exists p, q (p \neq q \wedge Col(p, q, x) \wedge Col(p, q, y))$$

Proof. If $x \neq y$ take $p = q$ and $q = y$. If $x = y$ take $p = x$ and take y to be any other point; by Satz 3.13 there exist at least two points.

Axiom I,2. If $x \neq y$ then there exists at most one line containing both x and y.

This becomes, for all p, q, u, v, if $p \neq q$ and $u \neq v$ and $Col(p, q, x)$ and $Col(p, q, y)$ and $Col(u, v, x)$ and $Col(u, v, y)$ then for all z, $Col(p, q, z)$ if and only if $Col(u, v, z)$. This is Satz 6.16 (which is not as trivial as it may seem!)

Axiom I,3. There exist at least two points on a line.

Since lines are interpreted as pairs of distinct points, the interpretation of this axiom becomes, for every pair of distinct points p, q, there exist at least two points collinear with p and q; and of course p and q themselves will do.

Axiom I,3 has a second part, namely, there exist at three noncollinear points. That is Axiom A8 of $\mathbb{T}$.

Axiom I,4. Let p, q, r be non-collinear points; then there exists a plane containing those three points, and every plane contains at least one point.

The second part of the axiom is immediate from the interpretation of incidence. Consider the first part. We first verify that the interpretation of "p, q, and r are collinear" is $Col(p, q, r)$. Suppose p, q, and r are collinear in the sense of $\mathbb{H}$, i.e., they all lie on the same line L. Let $(a, b) = L^*$. Then $a \neq b$ and $Col(a, b, p)$ and $Col(a, b, q)$ and $Col(a, b, r)$. We must prove $Col(p, q, r)$. This is Satz 6.18; that completes the proof that the interpretation of "p, q, and r are collinear" is $Col(p, q, r)$.

Next we must prove that of three noncollinear points, no two are equal (since to prove there exists a plane, we need three distinct points). Suppose a, b, and c are non-collinear, but $a = b$. Then the line through a and c contains b, contradiction.

Finally we consider the interpretation of "every plane contains at least one point." Since a plane P is given, in the interpretation, by three points a, b, and c, it suffices to verify the interpretation of "P contains c", which is $Coplanar(a, b, c, c)$. By the definition of $Coplanar$, that means either $Col(a, b, c)$ or c is on the same side of ab as itself, since by the identity axiom for betweenness c cannot be on the opposite side of ab from itself. If c lies on ab we are done, so we may assume $\neg Col(a, b, c)$. Then let p be the reflection of c in a, denoted by $S_a(c)$ in Szmielew; that is, the extension of segment ca by an amount ca, guaranteed by Axiom A2. Then p is on the opposite side of ab from c; hence c is on the same side of ab as itself.

Axiom I, 5. Let p, q, r be non-collinear points and let planes A and B both contain p, q, and r; then planes A and B contain the same points.

Proof. We first discuss the treatment of planes in Szmielew, *cf.* Definition 9.20, page 74. On the face of it, planes are defined as sets of points; i.e. Szmielew here (as in her treatment of lines) superficially deviates from a strictly first-order treatment. However, in practice her proofs are first-order, and her definition of $Pl(pqr)$ in Definition 9.20 is such that $x \in Pl(pqr)$ is exactly $Coplanar(p, q, r, x)$ as I defined $Coplanar$ above. She uses $Pl(A, r)$ for the plane determined by line A and a point r not on A. Definition 9.20 says

$Pl(E)$ means that $E = Pl(p,q,r)$ for some non-collinear points p, q, r; so for Szmielew, planes are given by three non-collinear points. In other words: Szmielew's planes are exactly the interpretation we have defined here of Hilbert's planes.

Now we turn to the proof of Axiom I,5. Let $A^* = (a, b, c)$, and let $B^* = (e, f, g)$. Suppose the interpretation of "planes A and B both contain p, q, and r" holds; that is, assume $Coplanar(a, b, c, p)$, $Coplanar(a, b, c, q)$, $Coplanar(a, b, c, r)$ and similarly with (e, f, g) in place of (a, b, c). Then we must show that $Coplanar(a, b, c, x)$ implies $Coplanar(e, f, g, x)$. Expressed in Szmielew's notation, we have to show that $Pl(a, b, c) = Pl(e, f, g)$. (Equality between planes means that the same points lie on both planes.) By Satz 9.26, we have $Pl(a, b, c) = Pl(p, q, r)$, and similarly $p(e, f, g) = Pl(p, q, r)$. Hence $Pl(a, b, c) = Pl(e, f, g)$. That completes the proof.

Axiom I,6. If two points p, q of line A lie in the plane P, and x lies on A, then x lies in P.

Proof. This is the first part of Satz 9.25 in Szmielew.

4. The interpretations of Hilbert's Group II axioms

Axiom II,1: $B(a, b, c) \to B(c, b, a)$.

Proof. Suppose $B(a, b, c)^*$. Then $T(a, b, c)$ and a, b, c are three distinct points. By Satz 3.2 we have $T(c, b, a)$. Hence $B(c, b, a)^*$.

Axiom II,2: $a \neq b \to \exists c\, B(a, c, b)$.

Proof. This certainly follows from Satz 8.22, which says every segment has a midpoint. Whether it has a simpler proof I do not know! I don't see an earlier theorem in Szmielew from which it follows immediately.

Axiom II,3: $\neg(a \neq b \wedge b \neq d \wedge a \neq d \wedge B(a, b, d) \wedge B(b, a, d))$.

Proof. Actually we can prove $\neg(B(a, b, d) \wedge B(b, a, d))^*$ without the distinctness hypothesis, since $B(a, b, d)^* \to a^* \neq b^* \wedge b^* \neq d^* \wedge a^* \neq d^*$. Dropping the stars for convenience, then a, b, and d are distinct points so T coincides with B on these points. Then take $c = a$ in Satz 3.5. Then Satz 3.5 says $T(a, b, d)$ and $T(b, a, d)$ implies $T(a, b, a)$ and $T(a, a, d)$. By Axiom A6, from $T(a, b, a)$ we infer $a = b$, contrary to hypothesis. That completes the proof.

Axiom II,4: Pasch's axiom. Let a, b, c be three non collinear points and let L be a line in plane abc such that none of a, b, c lie on L. Let p lie on L and $T(a, p, b)$. Then there exists q on L with $T(a, q, c) \vee T(b, q, c)$.

Proof. Since b and c do not lie on L, then either c lies on the same side of L as b, or on the opposite side from b (by the definition of coplanar). Case 1, c is on the opposite side of L from b. By the definition of "opposite side" the desired point q exists, where L meets segment bc. Case 2, c is on the same side of L as b. Since segment ab meets L in p, b and a are on opposite sides of L. Then by Satz 9.8, c is on the opposite side of L from a. Hence segment ac meets L, as desired. That completes the proof of Pasch's axiom.

Remark. Possibly Pasch's axiom itself is a Satz somewhere in [2], but I don't see it. Anyway (a) it is well-known that outer Pasch (Satz 9.6) and inner Pasch (Axiom A7) together imply Pasch's theorem, and (b) it has a very short proof from Satz 9.8, exhibited above.

5. The interpretations of Hilbert's Group III axioms

Hilbert expressed his Axiom III, 1 using the words "on a given side of". Here we unwind this definition and express the axiom in terms of the primitive concepts. (To do that the correct procedure is to use Hilbert's definition of "on the same side of" rather than Szmielew's.)

Axiom III, 1. Given a point p on line L, and two points ab, and a point c not on L, then there is a point x such that $E(p, x, a, b)$ and xc meets L; and there is another point y such that $E(p, y, a, b)$ and yc does not meet L.

Proof. According to Satz 8.21, there is a point q such that qp is perpendicular to L (at p) and q is on the opposite side of L from c. Let q' be the reflection of q in p. Then q' is on the opposite side of L from q. By Axiom A4 we can extend segment $q'p$ to point x with $E(p, x, a, b)$. By Satz 7.6, p is between q and q'. By Satz 3.1, x lies on the ray pq. By Satz 3.1 again, $T(q', p, x)$. Hence x is on the opposite side of L from q'. Hence x is (by definition) on the same side of L as q. But q is on the opposite side of L from c. Hence, by Satz 9.8, x is also on the opposite side of L from c. That completes the proof of the desired properties of x. Now let y be the reflection of x in p. Then y is on the opposite side of L from x and q and y is on the same side of L as q'. But c is on the opposite side of L from x. By definition of "same side", y is on the same side of L as c. In order to show that yc does not meet L, then, we only need the theorem that two points cannot be both on the same side of L and on opposite sides of L. That is Satz 9.9. That completes the proof of Axiom III,1.

Axiom III, 2. (transitivity of congruence). If two segments are congruent to a third segment, they are congruent to each other.

Proof. This is Satz 2.3.

Since Hilbert treats E as a relation on segments (which are pairs of points), we also need to verify that the same points belong to segment ab as to segment ba. That is, $T(a, x, b) \leftrightarrow T(b, x, a)$, which is Satz 3.2. Similarly, since Hilbert uses E as a relation on segments, we need to verify that it is an extensional relation when considered as a relation on points, i.e.

$$\begin{array}{lcll} E(a,b,c,d) & \leftrightarrow & E(b,a,c,d) & \text{Satz 2.4} \\ E(a,b,c,d) & \leftrightarrow & E(a,b,d,c) & \text{Satz 2.5} \\ E(a,b,c,d) & \leftrightarrow & E(b,a,d,c) & \text{follows from the two above} \end{array}$$

Axiom III, 3. Suppose a, b, and c are collinear, and a', b', and c' are collinear, and $ab \equiv a'b'$ and $bc \equiv b'c'$, and ab and bc have no point in common, and $a'b'$ and $b'c'$ have no point in common. Then $ac \equiv a'c'$.

Proof. The second-order language about "no point in common" is interpreted this way: "ab and bc have no point in common" means that there is no point x such that $T(a, x, b)$ and $T(b, x, c)$. In particular we do not have $T(a, c, b)$, since we do have $T(b, c, c)$ by Satz 3.1. Since $Col(a, b, c)$, we have $T(a, b, c) \vee T(c, a, b) \vee T(a, c, b)$; but since $T(a, c, b)$ is ruled out, we have $T(a, b, c) \vee T(c, a, b)$. By Satz 3.2, the two disjuncts are equivalent, so we have $T(a, b, c)$. Similarly we have $T(a', b', c')$. We also have $E(a, a, a', a')$ by Satz 2.8 (all null segments are equal). Now taking $d = a$ in the five-segment axiom A5, the hypotheses of A5 hold, and the conclusion is $E(c, a, c', a')$. The desired conclusion $E(a, c, a', c')$ follows as noted above (from Satz 2.4 and Satz 2.5). That completes the proof of Axiom III, 3.

Before proceeding to the axioms about angles, the reader should review pp. 94-95 of [2], where Hilbert's treatment of angles is contrasted with Tarski's. We interpret an angle as a triple of points abc, and angle congruence is defined in Definition 11.2, by saying that $\angle abc \cong \angle pqr$ if there exist four points on the rays forming the sides of the angles forming two congruent triangles (two triangles are congruent if corresponding sides are congruent). The interior points of angle abc are those points x lying on the same side of ba as c and on the same side of bc as a. Hilbert requires $\neg Col(a, b, c)$ in order that abc be considered an angle.

Axiom III, 4. Given angle abc and given plane pqr, and a point s in plane pqr not on line pq, then there exists exactly one ray qt such that t is on the opposite side of line pq from s, and $\angle pet \cong \angle abc$.

Proof. This is Satz 11.15 in [2]; Szmielew explicitly points out that Satz 11.15 implies Hilbert's III, 4.

Axiom III, 5. The SAS criterion for triangle congruence.

Proof. This is Satz 11.49.

6. The Parallel and Continuity axioms

We note that only Axioms A1-A9 were used to verify the interpretations of Hilbert's axioms in Groups I, II, and III. Hilbert's parallel axiom IV is Satz 12.11 in [2], and the line-circle and circle-continuity axioms are formulated in the same way, or with trivial differences, in both theories.

It remains to discuss the full continuity axioms. Hilbert lists (as Axiom V,1) Archimedes's axiom, which of course is not a first-order axiom and cannot be interpreted in a first-order system. He also lists Axiom V,2:

> An extension of a set of points on a line with its order and congruence relations that would preserve the relations existing among the original elements as well as the fundamental properties of line order and congruence that follow from Axioms I-III, and V,1, is impossible.

Here Hilbert confuses a metamathematical statement about the possible extensions of models of the axioms with an axiom, or so it appears to the modern reader. However, this statement is not far removed from the Dedekind completeness axiom, since if there were an extension of the type mentioned in the axioms, we could throw away all but one "extra point", and obtain such an extension with just one "extra point." Then the points "left" and "right" of that point would be an unfilled Dedekind cut. Hence, what Hilbert had in mind is essentially equivalent to the second order axiom of Dedekind completeness. Tarski's theory with the full first-order Axiom A11 can thus interpret the first-order part of Hilbert's V,2, if we take V,2 as Dedekind completeness.

7. Conclusion

The proof that Tarski's A1–A9 suffice to interpret Hilbert's axioms I–III, and A1–A10 suffice for I–IV, is essentially contained in [2].

References

[1] David Hilbert. *Foundations of Geometry (Grundlagen der Geometrie)*. Open Court, La Salle, Illinois, 1960. Second English edition, translated from the tenth German edition by Leo Unger. Original publication date, 1899.

[2] W. Schwabhäuser, Wanda Szmielew, and Alfred Tarski. *Metamathematische Methoden in der Geometrie: Teil I: Ein axiomatischer Aufbau der euklidischen Geometrie. Teil II: Metamathematische Betrachtungen (Hochschultext)*. Springer–Verlag, 1983. Reprinted 2012 by Ishi Press, with a new foreword by Michael Beeson.

[3] Alfred Tarski and Steven Givant. Tarski's system of geometry. *The Bulletin of Symbolic Logic*, 5(2):175–214, June 1999.