No triangle can be decomposed into seven congruent triangles

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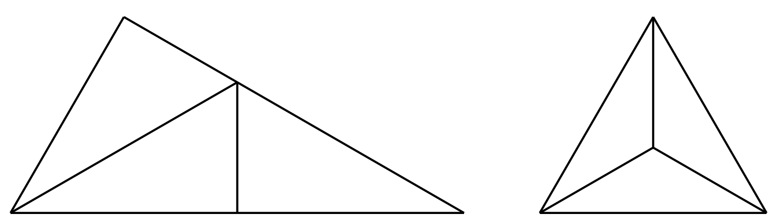
Abstract

We investigate the problem of cutting a triangle into N congruent triangles. While this can be done for certain values of N, we prove that it cannot be done for $N = 7$. This result is a special case of much more general results obtained in [1], but the proof in this paper may still be of some interest, because only methods of Euclidean geometry are used (including simple trigonometry that can in principle be done by geometric arguments).

1 Introduction

We consider the problem of cutting a triangle into N congruent triangles. Figures 1 through 1 show that, at least for certain triangles, this can be done with $N = 3$, 4, 5, 6, 9, and 16. Such a configuration is called an N-tiling.

Figure 1: Two 3-tilings



The method illustrated for $N = 4$,9, and 16 clearly generalizes to any perfect square N. While the exhibited 3-tiling, 6-tiling, and 5-tiling clearly depend on the exact angles of the triangle, *any* triangle can be decomposed into n^2 congruent triangles by drawing $n - 1$ lines, parallel to each edge and dividing the other two edges into n equal parts. Moreover, the large (tiled) triangle is similar to the small triangle (the "tile"). It follows that if we have a tiling of a triangle ABC into N congruent triangles, and m is any integer, we can tile ABC into Nm^2 triangles by subdividing the

Figure 2: A 4-tiling, a 9-tiling, and a 16-tiling

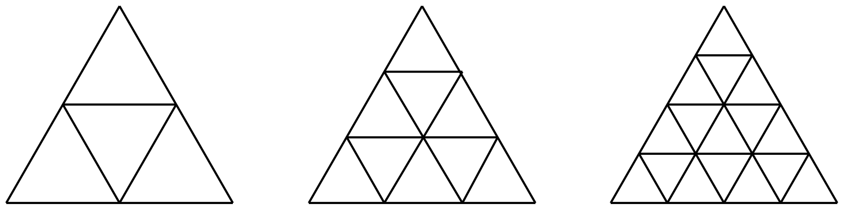
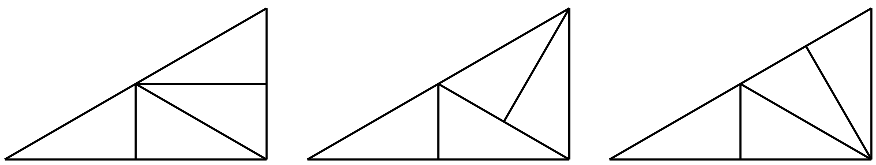


Figure 3: Three 4-tilings



first tiling, replacing each of the N triangles by m^2 smaller ones. Hence the set of N for which an N-tiling of some triangle exists is closed under multiplication by squares.

Let N be of the form $n^2 + m^2$. Let triangle T be a right triangle with perpendicular sides n and m, say with $n \geq m$. Let ABD be a right triangle with base AD of length m^2, the right angle at D and altitude mn, so side BD has length mn. Then ABD can be decomposed into m triangles congruent to T, arranged with their short sides (of length m) parallel to the base AD. Now, extend AD to point C, located n^2 past D. Triangle ADC can be tiled with n^2 copies of T, arranged with their long sides parallel to the base. The result is a tiling of triangle ABC by n^2+m^2 copies of T. This is a rigid tiling. The 5-tiling exhibited in Fig. 3 is the simplest example, where $n = 2$ and $m = 1$. The case $N = 13 = 3^2 + 2^2$ is illustrated in Fig. 4.

If the original triangle ABC is chosen to be isosceles, then each of the n^2 triangles can be divided in half by an altitude; hence any isosceles triangle can be decomposed into $2n^2$ congruent triangles. If the original triangle is equilateral, then it can be first decomposed into n^2 equilateral triangles, and then these triangles can be decomposed into 3 or 6 triangles each, showing that any equilateral triangle can be decomposed into $3n^2$ or $6n^2$ congruent triangles. Note that these are different tilings than those obtained by the method of the first paragraph of this section. For example we can 12-tile an equilateral triangle in two different ways, starting with a 3-tiling and then subdividing each triangle into 4 triangles ("subdividing by 4"), or starting with a 4-tiling and then subdividing by 3.

The elementary constructions just described suffice to produce N-tilings when N has one of the forms n^2, $n^2 + m^2$, $2n^2$, $3n^2$, or $6n^2$. The smallest N not of one of these forms is $N = 7$. The main theorem of this paper is that there is no 7-tiling. In [1], we have completely solved

Figure 4: A 5-tiling

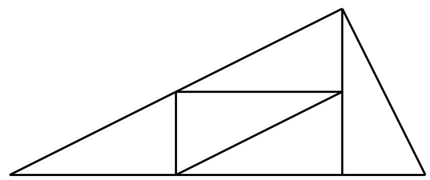
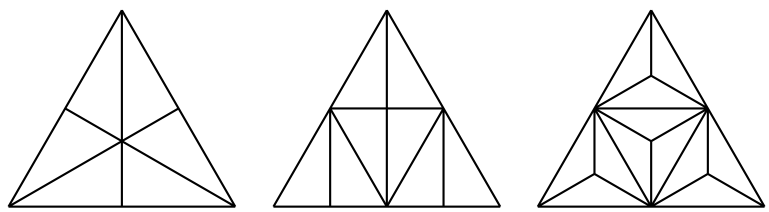


Figure 5: A 6-tiling, an 8-tiling, and a 12-tiling



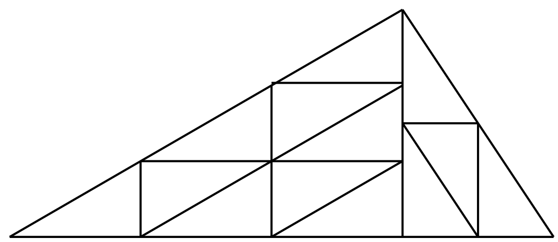
the problem of determining the values of N for which there exists some N-tiling. The proof given here for the special case $N = 7$ may still be of some interest, since it uses only elementary methods of of Euclidean geometry (including some elementary trigonometry, which could be done by geometric methods). To tackle the next interesting case, $N = 11$ by these methods would require hundreds, if not thousands, of pages. Luckily, we found a more abstract approach in [1].

The examples of N-tilings given above are well-known. They have been discussed, in particular, in connection with "rep-tiles" [5]. A "rep-tile" is a set of points X in the plane (not necessarily just a triangle) that can be dissected into N congruent sets, each of which is similar to S. An N-tiling in which the tiled triangle ABC is similar to the triangle T used as the tile is a special case of this situation. That is the case, for example, for the n^2 family and the $n^2 + m^2$ family, but not for the 3-tiling, 6-tiling, or the 12-tiling exhibited above. Thus the concepts of an N-tiling and rep-tiles overlap, but neither subsumes the other. As far as I have so far been able to discover, there is (until now) not a single publication mentioning the concept of an N-tiling in general. The paper [4] also contains a diagram showing the n^2 family of tilings, but the problem considered there is different: one is allowed to cut N copies of the tile first, before assembling the pieces into a large figure, but the large figure must be similar to the original tile. The two books [2] and [3] have tantalizing titles, but deal with other problems.

2 Definitions and Notation

We give a mathematically precise definition of "tiling" and fix some terminology and notation. Given a triangle T and a larger triangle ABC, a

Figure 6: A 13-tiling



"tiling" of triangle ABC by triangle T is a list of triangles $T_1, \ldots, T_n$ congruent to T, whose interiors are disjoint, and the closure of whose union is triangle ABC. A "strict vertex" of the tiling is a vertex of one of the T_i that does not lie on the interior of an edge of another T_j. A "strict tiling" is one in which no T_i has a vertex lying on the interior of an edge of another T_j, i.e. every vertex is strict. For example, the tilings shown above for $N = 5$ and $N = 13$ are not strict, but all the other tilings shown above are strict. The letter "N" will always be used for the number of triangles used in the tiling. An N-tiling of ABC is a tiling that uses N copies of some triangle T.

Let a, b, and c be the sides of triangle ABC, and angles α, β, and γ be the angles opposite sides a, b, and c, i.e. the interior angles at vertices A, B, and C. An *interior vertex* in a tiling of ABC is a vertex of one of T_i that does not lie on the boundary of ABC. A *boundary vertex* is a vertex of one of the T_i that lies on the boundary of ABC.

In the case of a non-strict tiling, there will be a non-strict vertex V; so V lies on an edge of T_j, with T_j on one side of the edge and T_i (having vertex V) on the other side. Consider the maximal line segment S extending this edge which is contained in the union of the edges of the tiling. Since there are triangles on each side of S, there are triangles on each side of S at every point of S (since S cannot extend beyond the boundary of ABC). Hence the length of S is a sum of lengths of sides of triangles T_i in two different ways (though the summands may possibly be the same numbers in a different order). Let us assume for the moment that the summands are not the same numbers. Then it follows that some linear relation of the form

$$pa + qb + rc = 0$$

holds, with p, q, and r integers not all zero (one of which must of course be negative), and the sum of the absolute values of p, q, and r is less than or equal to N, since there are no more than N triangles.

By the law of sines we have

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

Up to similarity then we may assume

$$a = \sin\alpha$$

$$\begin{aligned} b &= \sin\beta \\ c &= \sin\gamma \end{aligned}$$

Since $\gamma = \pi - (\alpha + \beta)$ we have $\sin(\gamma) = \sin(\alpha + \beta)$, so

$$p \sin\alpha + q \sin\beta + r \sin(\alpha + \beta) = 0.$$

If S is a maximal segment containing a non-strict vertex, then there will be integers n and m such that n triangles have a side contained in S and lie on one side of S, and m triangles have a side in S and lie on the other side of S. In that case we say S is of type $m : n$. For example, Fig. 3 shows a 5-tiling with a maximal segment of type $1 : 2$. This definition does not require that the lengths of the subdivisions of the maximal segment all be the same (as they are in Fig. 3).

3 2-tilings, 3-tilings and 4-tilings

In this section, we warm up by characterizing 2-tilings, 3-tilings and 4-tilings. Not only will these results be used later, but the ideas introduced in the proofs will also be used later.

Lemma 1 *If, in a tiling, P is a boundary vertex (or a non-strict interior vertex) and only one interior edge emanates from P, then both angles at P are right angles and $\gamma = \pi/2$.*

Proof. If either the two angles at P are different, then their sum is less than π, since the sum of all three angles is π. Therefore the two angles are the same. But $2\alpha < \alpha + \beta < \pi$ and $2\beta < \beta + \gamma < \pi$. Therefore both angles are γ. But then $2\gamma = \pi$, so $\gamma = \pi/2$.

Theorem 1 *If triangle ABC is 2-tiled by T, then ABC is isosceles and the tiling divides it into two right triangles by means of an altitude.*

Proof. The two triangles T_1 and T_2 have a total of 2π angles, of which π are accounted for by the vertices of ABC. An interior vertex (strict or not) would require there to be three triangles. Hence there is exactly one more vertex, and it is a boundary vertex. Call that vertex P. Since there are only two triangles, only one interior edge emanates from P, and its other end must be at the opposite corner of triangle ABC. Relabeling if necessary, we can assume this corner is B and P lies on AB. By the lemma, $\gamma = \pi/2$ and the angles at P are right angles. Then $AB = BC = c$ since these sides are opposite the right angles of T_1 and T_2 respectively. Hence triangle ABP is congruent to triangle CBP and the tiling is as described in the theorem. That completes the proof.

We could have reached the conclusion that just one interior edge emanates from P in another way, which seems overly complicated in this example, but will be useful below. The triangles T_1 and T_2 have together six boundary segments. Four of these occur on the boundary of ABC, and the other remaining boundary segments must suffice to count each interior edge from both sides. In this case there are just two remaining (because $6 - 4 = 2$) and hence there is exactly one interior edge, whose two sides account for these two boundary segments. In general in an N-tiling there are $N\pi$ radians to account for, of which π are in the corners of triangle ABC, and the rest are distributed between boundary and interior vertices. If there are k boundary vertices then there are $k+3$ boundary segments on the boundary of ABC, leaving $3N-k-3$ to be accounted for by counting each side of each interior edge. In a strict tiling, the number of interior edges will thus be half of $3N-k-3$, but in a non-strict tiling, a more detailed accounting must be made. In the next proof, we will apply this technique to the case $N=3$.

Theorem 2 *If triangle ABC is 3-tiled by T, then either (i) ABC is equilateral and the tiling consists in connecting the center of ABC to its vertices, or (ii) ABC is a 30-60-90 triangle, and there is no interior vertex of the tiling; the shared side of two of the T_i is perpendicular to the hypotenuse of ABC at its midpoint P, and meets side b at Q, say, and the other interior edge connects Q to the vertex B (where the angle of ABC is $\pi/3$). See Fig. 1.*

Proof. Suppose ABC is 3-tiled by T_1, T_2, and T_3. First we suppose the tiling is strict. The total of the angles in the tiling is 3π, since there are three copies of T. The total angle accounted for by the vertices of ABC is π. Each strict interior vertex accounts for 2π and each boundary vertex for π. Thus there are only two possibilities: one interior vertex and no boundary vertices, or two boundary vertices and no interior vertex.

First assume that there is one interior vertex and no boundary vertices. Since there are no boundary vertices, three of the nine boundary segments of the T_i are on the boundary of ABC, and the other six are double-counted as the two sides of three interior edges. Since at least three edges must emanate from an interior vertex, all three edges do emanate from the one interior vertex P. Since there are no boundary vertices, they must terminate in the three vertices A, B, and C. That is, at least the tiling has the topology of the tiling in (i). The three angles at P must all be γ, since any other sum of three angles chosen from α, β, and γ is at most $\beta+2\gamma$, which is less than 2π because

$$\begin{aligned}\beta+2\gamma &= \beta+\alpha+(\gamma-\alpha)+\gamma\\ &= \pi+\gamma-\alpha\\ &< 2\pi\end{aligned}$$

Hence $3\gamma=2\pi$, so $\gamma=2\pi/3$. Hence the c sides of all three T_i are the faces of triangle ABC, which is thus equilateral. Now let $AP=a$; then in triangle APC, we have $PC=b$; hence in triangle CPB, we have $PB=a$; hence in triangle PBA we have $AP=b$. Hence AP is equal to both a and b, so $a=b$ and the T_i are isosceles. Hence the tiling is the one described in (i) of the theorem.

Next assume that there are two boundary vertices P and Q (and hence no interior vertex). Then there are five boundary segments, i.e. sides of copies of T lying on the boundary of ABC. Since there are only three triangles, two of the triangles must account for two sides each, i.e. two of the angles of ABC are not "split", i.e. are not shared by more than one T_i. Hence each T_i is similar to triangle ABC. Of the 9 sides of the three T_i, five occur on the boundary of ABC, and the other 4 occur in the interior. Since each interior side is counted twice, as a boundary of the triangles on either side, there must be exactly two interior edges. One of these interior edges must connect P and Q, because if not, then both interior edges would have to connect P or Q to the opposite vertex. But if one edge connects (say) Q to the opposite vertex, then the edge from P is blocked from reaching the opposite vertex, and vice-versa, if one edge connects P to the opposite vertex, the other edge cannot connect Q to the opposite vertex. Hence it cannot be that both interior edges connect P or Q to the opposite vertex. The only other possibility is that one of these edges connects P to Q. The other interior edge must connect one of P or Q to the opposite vertex of ABC, which must be split. That means that one of P or Q (by relabeling we can assume it is P) has only one interior edge emanating from it. That implies that γ is a right angle, by Lemma 1. Changing the labels A, B, and C if necessary, we can assume that P lies on AB, Q lies on BC, and QA and QP are the interior edges. Angles QPB and QPA are right angles, and triangle AQP is congruent to triangle QPB. Sides AQ and QB are opposite the right angle and hence are equal. Hence $AP = PB$ and P is the midpoint of AB. The angle at B is not split. Since triangle ABC is similar to each triangle T_i, but its area is 3 times larger, the similarity factor is $\sqrt{3}$. Let T_1 and T_2 be the two triangles sharing side PQ, with $T_1 = QPA$ and $T_2 = QPB$. Then T_3 shares side AQ, which is side c in triangle T_1, so the third vertex C of ABC, which is also the vertex of T_3 opposite AQ, must be the right-angled vertex of ABC. Now triangle CAB is similar to triangle PQB, since they have the same angle at B and right angles at C and P respectively. Hence AB and QB are corresponding sides. Their ratio is therefore $\sqrt{3}$, i.e. $AB = \sqrt{3}QB$. But since $AB = AP + PB = 2AV$, we have $2PB = \sqrt{3}QB$. Hence angle $B = \pi/6$ and angle $PQB = \pi/3$, and the tiling is as described in (ii) of the theorem. That completes the proof in case of a strict tiling.

Now suppose the tiling is non-strict. Since only three triangles are involved, the only possible type of non-strict vertex is the type we shall call $2:1$ below, where one side of (say) T_1 is matched by two sides, one of T_2 and one of T_3. There cannot be two such vertices as the three triangles will have only this one side of T_1 in common, and if the sides of T_2 and T_3 that touch do not have the same length, a triangle ABC will not be formed. Hence, of the 3π in total angles, π is accounted for at the interior vertex P, and π is accounted for by the vertices of ABC, leaving π to be accounted for by a single boundary vertex Q. With one boundary vertex there are 4 boundary segments on the boundary of ABC, leaving $3 \cdot 3 - 4 = 5$ in the interior (counting each side of each interior segment). Three of those are the three sides that lie on the maximal segment of the non-strict vertex P. The other two are the two sides of one more interior segment with an endpoint at P. Since there is only one boundary vertex, the three endpoints of the interior segments must end at Q and at two corners of the triangle. The maximal segment must end at Q and one corner, which we may label A, and the other interior segment runs from P to another corner, say B. Since only two triangles share vertex Q, we have $\gamma = \pi/2$ by Lemma 1. But now triangle QPB has two right angles, at Q and P. That contradiction completes the proof.

Theorem 3 *If triangle ABC is 4-tiled by T, then (a) there is no interior vertex, and (b) T is a 30-60-90 triangle, and the tiling is one of those shown in Fig. 3 (or a reflection of these), or T can be any triangle and the tiling is one of the n^2 family as illustrated in Fig. 2.*

Proof. First suppose the tiling is strict. The four triangles have angles totaling 4π. The vertices of ABC account for π of this, and the remaining 3π must be accounted for. There are just two possibilities: one interior vertex and one boundary vertex, or no interior vertices and three boundary vertices.

First assume there is one interior vertex P and one boundary vertex Q. Then there are four boundary segments and $(12-4)/2 = 4$ interior edges. Then these four edges must emanate from P and go to A, B, C, and P. By the lemma, the angle at Q is a right angle and $\gamma = \pi/2$. Hence all four angles at P must be right angles. Then triangle APQ has two right angles, contradiction. That disposes of the case of one interior vertex and one boundary vertex.

Next assume there are three boundary vertices and no interior vertex. Then there are six boundary segments and $(12-6)/2 = 3$ interior edges.

First assume that one of the interior edges terminates in A, B, or C (splitting the angle there). Then there are not enough edges to provide two edges at each boundary vertex, so one boundary vertex has only one edge terminating there. Hence by the lemma, $\gamma = \pi/2$. Label the split vertex B and let Q be the interior vertex at the other end of the interior edge emanating from B. Then either triangles ABQ and CBQ are both 2-tiled, or one is 3-tiled and the other is congruent to the tile T. First assume ABQ is 3-tiled. There are two possible 3-tilings; in one case, angle AQB is $\pi/3$, and the fourth triangle T_3 can contribute only an angle of $\pi/6$ at each vertex, not enough to make π and remove a vertex. So this case is impossible. In the other possible 3-tiling, ABQ is a 30-60-90 triangle similar to T, and the three sides of ABC are $a+c$, $2b$, and b. Neither $a+c$ nor $2b$ can be a side of T, so we must have $QB = b$, and angle $C = \pi/3$. We then necessarily have the third tiling show in Fig. 3 (or its reflection). Next assume that ABQ is 2-tiled. Then $AQ = QB$ and QP is an altitude of triangle AQB. The third interior vertex R must lie on QC or on BC. First assume R lies on BC. Then the third interior edge is QR, and both angles at R are right angles by the lemma. We therefore have the first tiling shown in Fig. 3. Next assume R lies on AC. Then the third interior edge must be BR, and angle QRB must be a right angle by the lemma. Then triangle PQB is congruent to triangle RQB and we have the third tiling in Fig. 3. This disposes of the case in which one of the interior edges terminates in a vertex of ABC.

Now the three interior edges terminate only in the three boundary vertices. It follows that triangle ABC is similar to triangle T; since there are four triangles, the similarity factor is 2: triangle ABC is twice the size of T, and the same shape. If two boundary vertices lie on the same side of ABC, three interior edges cannot exist. Therefore one boundary vertex lies on each side of ABC. Label them so that P lies on AB, Q lies on BC, and R lies on AC. Then the three interior edges form triangle

PQR. Let $RQ = a$, $PQ = b$, and $RQ = c$. Then if $CQ = a$, it follows that $QB = a$ and $AR = RC = b$, and we have an n^2-family tiling. If, on the other hand, $CQ \neq a$, then since $RQ = c$ we must have $CQ = b$ and $b \neq a$. But then $RC = a$ and hence $AR = a$. But by definition $RP = a$, and by the similarity of ABC to T, we have $AB = 2c$ and hence $AP = c$. Hence $a = b$, contradiction. This completes the proof in the case of a strict tiling.

Now assume that there is a single non-strict vertex P. If this vertex is of type $3:1$ then every one of the four triangles shares a side with the maximal segment S. There are no more triangles that can share the two interior vertices on S, so only one edge emanates from each of these vertices on the side bounding three triangles. By the lemma the angles at these vertices are right angles. But then the middle of the three triangles has two right angles, contradiction. Hence the non-strict vertex does not have type $3:1$. If it has type $2:2$, then similarly the angles at the interior vertices are right angles, so the union of the four copies of T has four vertices and cannot be a triangle ABC. Therefore the type of the non-strict vertex must be $2:1$.

The non-strict vertex accounts for π of the 4π angles of the T_i, and since π is accounted for by the corners of ABC, that leaves either one strict interior vertex and no boundary vertices, or two boundary vertices and no strict interior vertices.

First assume there is one strict interior vertex Q and no boundary vertices. The maximal segment must run from a vertex (say B) to Q, since it cannot run to another vertex of ABC. The other edges emanating from Q must be AQ and BQ. There are 3 boundary segments and 9 double-counted interior edges. The maximal segment, since it is of type $(2:1)$, contains 3 of these edges, and the other 6 correspond to three additional interior edges. These are AQ, CQ, and the other edge emanating from P. The endpoint of that edge must be a vertex of ABC, which by relabeling we can assume is C. Then by the lemma, the angles at P are right angles, and $\gamma = \pi/2$. But only three angles meet at Q, so one of them must be greater than $\pi/2$. This contradiction disposes of the case of one strict interior vertex.

Therefore the second case must hold: there are two boundary vertices and no strict interior vertices. Then there are five boundary segments and seven double-counted interior segments, of which three lie on the maximal segment, so there are two additional interior segments, one of which has one end at the non-strict vertex P. That makes five ends of interior segments (two of the maximal segment, and three of the four ends of the two additional interior segments) that must terminate at two boundary vertices and/or the three corners of ABC. The maximal segment cannot connect two vertices of ABC, so it must have one end at a boundary vertex Q. The other end of the maximal segment is either at a vertex of ABC or at the other boundary vertex R.

Assume first that the maximal segment connects Q to a vertex of ABC. Relabeling, we can assume it is vertex B, and the Q lies on AC, and triangle CQB is congruent to T while triangle AQB is 3-tiled. By Theorem 2, it follows that AQB is a right triangle or is equilateral. First assume AQB is equilateral. Then angle CQB is $\pi/6$ and angle $AQC =$

$5\pi/6$, not a straight angle, contradiction. Hence AQB is not equilateral. Hence the other case holds, namely that AQB is a 30-60-90 triangle, tiled as in Fig. 1. Since QB is a single side of T in triangle QBC, the tiling of AQB must be oriented so that QB is the smallest side of QBA. The tiling then must be the third one shown in Fig. 3, or its reflection.

The only remaining case is that the maximal segment connects Q to the other boundary vertex R. Relabeling, we can assume that Q lies on AC and P lies on AB. Assume first that an interior segment emanates from P on the same side of PQ as A. Then its other endpoint must be A. Hence the angles it makes at P are right angles and $QP = PR$. The second interior segment cannot have endpoints at both Q and R, so at one of those points, only the maximal segment meets the boundary of ABC. By the lemma then, angle AQR or angle BQR is a right angle, contradiction, since that would make two right angles in one of those triangles. Hence no interior segment emanates from P on the same side of PQ as A. Then triangle T is congruent to AQR. Since not both angle CQR and angle BRQ can be right angles (else QC and RB would be parallel), an interior segment must emanate from one of them; relabeling, we can assume it is R. The other endpoint of this segment must be C. The other interior segment has one endpoint at P, and its other endpoint must also be at C, since there is no interior vertex. Then by the lemma, CP is perpendicular to QR. Also by the lemma, PQ must be perpendicular to AC. But then triangle CQP has right angles at both P and Q, contradiction. That completes the proof.

4 Strict 7-tilings

Theorem 4 *There is no strict 7-tiling.*

Proof. Consider a strict 7-tiling. Since it is composed of 7 triangles, the angles make a total of 7π. Of that 7π, there is π in the corners of the large triangle, and 2π for each interior vertex, and π for each boundary vertex (i.e. vertex lying on an edge of the large triangle but not in a corner). Therefore we have either: zero interior vertices and 6 boundary vertices, or one interior vertex and 4 boundary vertices, or two interior vertices and two boundary vertices. We consider these three cases one by one.

Case 1: Zero interior vertices and 6 boundary vertices. Since there are 6 boundary vertices, there are 9 sides of triangles on the boundary and $(21-9)/2 = 6$ interior edges. If at any boundary vertex, only one interior side terminates there, then γ must be a right angle. Assume that γ is not a right angle. Then consider a boundary vertex P on side AB, the next vertex to A. It must connect to vertex Q on side AC, the next vertex to A (else no side can escape from Q). The other edge from P must not connect to side AC, or else no edge can escape from Q. So P connects to a vertex R on side BC. The second edge from Q must connect either to R or to a vertex S on BC between C and R. Assume the latter. But then, a second edge from S must intersect either PR or PQ, so that case is impossible and Q connects to R. Consider another boundary vertex U. Relabeling, we can assume U lies on PB or RB. Assume first that U is on RB. The other end of both edges leaving U must be at P or at another vertex V on PB, since there are no interior vertices. Since two edges leave U, there is such a vertex V on PB and UV is an edge. Then either UP or RV must be an edge, to tile quadrilateral $PRUV$. Assume it is RV. Then the second edge leaving U must terminate at another boundary vertex W on VB, and there are no more edges to leave W, so the angles at W are right angles and $\gamma = \pi/2$. Similarly if UP is an edge instead of RV, the second edge leaving V must terminate at another boundary vertex W on UB, and there are no more edges to leave W. Hence the assumption that U is on RB has led to a contradiction (under the assumption that γ is not a right angle). The remaining alternative is that U is on PB. But then the argument proceeds similarly: the other end of both edges leaving U must be at R or at another vertex V on RB. Since two edges leave U, there is such a vertex V on RB and UV is an edge. From that point the argument is exactly the same as in the case U is on RB. Hence the assumption that γ is a right angle has led to a contradiction. Hence γ is a right angle.

Six boundary vertices means that 9 edges of triangles lie on the boundary of the large triangle. Since there are only seven triangles, that means that there exist two triangles with two edges on the boundary, i.e. there are two triangles located at vertices of triangle ABC which do not share that vertex with any other triangle. Hence the final triangle ABC is similar to triangle T. Since there are supposedly seven copies of T tiling ABC, the similarity factor is $\sqrt{7}$. Let us suppose that the final triangle ABC has angle α and A, angle β at B, and a right angle γ at C. Consider the triangle T_1 of the tiling that has a vertex at A. It must have angle α at that vertex. Let P be its vertex on AC and Q its vertex on AB. Then $PQ = a$. Let T_2 be the triangle that shares side PQ with T_1, and let R be its third vertex. Since there are no interior vertices, R must lie on AB, or on AC, or on BC. Case 1a, R lies on AC. Then both T_1 and T_2 have their right angle at P. R is not equal to C since it is $2b$ from A, while C is $\sqrt{7}b$ from A. Consider the triangle T_3 that shares side QR with T_2. Let S be its third vertex. What is angle QRS? It cannot be γ, since side c of triangle QRS is QR. It cannot be β, since that would make angle ARS equal to $\alpha + \beta$, a right angle, and side RS equal to a, which would leave S in the interior of ABC. Hence angle QRS must be α, and hence angle $ARS = 2\alpha < \pi/2$, so S must lie on AB. That makes the total angle π at Q equal to 3β, so $\beta = \pi/3$, $\alpha = \pi/6$, so $a = \sin\alpha = 1/2$ and $b = \cos\alpha = \sqrt{3}/2$. Any linear combination of a, b, and c is thus of the form $p + q\sqrt{3}$; but $AC = \sqrt{7}$ is the sum of several sides of the basic triangle, contradiction. This contradiction disposes of Case 1a. Case 1b, R lies on AB. Then the right angles of T_1 and T_2 are both at Q. Let T_3 be the triangle sharing side PR with T_2, and let S be its third vertex. If S lies on AC, then angle RPS must be either β or α; if it is α then $2\beta + \alpha = \pi$, which is impossible since $\alpha + \beta = \pi/2$ and $\beta < \pi$. If it is β then $3\beta = \pi$, which is impossible as in the previous case. Hence S does not lie on AC. S cannot lie on AB as that would make angle PRS more than $\pi/2$. Now T_1, T_2, and T_3 all share vertex P, but since S does not lie on AC, a fourth triangle T_4 shares that vertex too, and has angle at least α there. If T_3 has angle β at P then the total angle at P is at least $3\beta + \alpha > 2\beta + 2\alpha = \pi$, contradiction. Hence T_3 has angle α at P, and $PQRS$ is a paralleogram. Let triangle T_4 be the triangle sharing side $PS = b$ with T_3. The angle of T_4 at P must be either α or a right angle, but a right angle is too large, since the other angles at P total $\pi/2 + \beta$. The right angle of T_4 must therefore be at S, making S an interior vertex, contradiction. That disposes of Case 1b, R lies on AB. But since the right angle of T_1 must lie at either P or Q, Case 1a and Case 1b are exhaustive, so Case 1 has been shown to be impossible.

Case 2: one interior vertex, and four boundary vertices. Then there are $(21 - 7)/2 = 7$ interior edges. There can be at most one triangle T_i that has one vertex on each side of ABC. (Call such a triangle an "interior triangle".) Hence six or seven triangles have one or more sides on the boundary of ABC. With four boundary vertices, there are seven boundary segments to be accounted for. If there is an interior triangle, then one of the remaining six must account for two boundary segments, so one of the vertices A, B, or C is not "split", i.e. shared by two or more triangle T_i. If there is no interior triangle, then each of the seven triangles must account for exactly one boundary segment, which means that *all* of the vertices A, B, and C are "split".

Case 2a: There is an interior triangle. Let T_1 be that triangle, having vertex P on AB, vertex Q on BC, and vertex R on AC. That creates three triangles BPQ, APR, and QRC. The single interior vertex must occur in the interior of one of these triangles (since this is a strict tiling, it cannot occur on the boundary of T_1). Relabeling the vertices if necessary we can assume that it occurs in triangle APR. At least three edges leave that interior vertex, so triangle APR is divided into at least three triangles congruent to T. In fact it must be divided into exactly three triangles, since at least three are needed for T_1, BPQ, and QRC, so the possibilities are three or four, but there is no 4-tiling with an interior vertex, by Theorem 3 Hence APR is 3-tiled, and there is only one 3-tiling with an interior vertex, by Theorem 2. Therefore T is the triangle with $\alpha = \beta = \pi/6$ and $\gamma = 2\pi/3$, and triangle APR is equilateral. Consider the angles at R: angle ARP is $\pi/3$, and PRQ is $\pi/6$, so QR is perpendicular to AC and therefore angle QRC must be composed of three angles α. That means that triangle QRC contains three smaller triangles, which makes seven counting the three in APR and T_1, leaving none to cover BPQ. This disposes of case 2a.

Case 2b: No interior triangle, and all vertices A, B, and C are split. Then there is an interior edge emanating from each of A, B, and C. Any pair of these must intersect in an interior vertex. But there is only one interior vertex, so they all intersect in a common point P, the interior vertex, forming three triangles ABP, ACP, and BCP. These triangles are tiled by T, without interior vertices, since there is only one interior vertex. Hence none of them is 3-tiled. A 4-tiling requires a boundary vertex on each side, which would mean another interior vertex, so none of them is 4-tiled either. They cannot all be 2-tiled, as that would use only six triangles. That leaves only the possibility that two of them are congruent to T and the other is 5-tiled; relabeling vertices if necessary, we can assume it is ABP that is 5-tiled. Since there are four boundary vertices they must all be on AB and the five triangles all share vertex P, with one side contained in AB. This is impossible for several reasons, for example, since just two triangles share each boundary vertex, all those angles must be right angles, contradicting the fact that all those edges meet at P. This disposes of Case 2b, and hence of Case 2.

Case 3: two interior vertices and two boundary vertices. Then there are $(21-5)/2 = 8$ interior edges. Suppose first that two boundary vertices P and Q occur on AB, with P adjacent to A and Q adjacent to B. Let U and V be the interior vertices. If γ is not a right angle, then two edges must leave P and two edges must leave Q. One of the edges from P, and one from Q, can go to an interior vertex, and one to vertex C. One more edge can go from P or from Q to an interior vertex, but after that we are blocked–there is no place to put the rest of the 8 edges. Hence the two boundary vertices do not occur on the same side of the large triangle. Say P occurs on AB and Q occurs on BC. Then each of P and Q can connect to both interior vertices, and one of them can connect to an opposite vertex of the large triangle, but that is not enough edges. Therefore P and Q do not connect to C. To use up 8 edges, we must have an edge connecting U and V, and one of U and V, say V, connects to both B and C, while U connects to A. But then, there are exactly four angles at V, totaling 2π. That means two of them must add to at least π, which means γ is a right angle.

If just three edges emanate from V then there are exactly three angles at V. If three angles add to 2π, they must all be γ, since $2\gamma+\beta < 2\pi$. But then $\gamma = 2\pi/3$, contradicting our conclusion that γ is a right angle. Hence at least four edge emanate from U and four from V, for the required total of 8. Every one of the eight interior edges then has one end at U or one end at V. The four edges emanating from U go to the edge vertices P and Q and to two vertices of the large triangle, say A and B. V must lie in one of the four regions formed by the angles at U; three of those are triangles, which leave only room for three edges to emanate from V. Hence V must lie in the quadrilateral $UPCQ$, and must connect to all four corners of that quadrilateral. Now we have five edges emanating from V and four from U, and as above, all the angles at U are right angles. Now consider the angles at P. Angle APU and angle UPV are either α or β, since those triangles have their right angle at U. Hence angle VPB is γ, a right angle. But then $PC = c$, since it is opposite the right angle at V, and on the other hand it is less than c, since it is opposite angle VBP, which is not a right angle. This contradiction eliminates Case 3.

That completes the proof of the theorem.

5 Non-strict 7-tilings

Lemma 2 *Suppose that a 7-tiling contains a non-strict vertex of type $3:1$. Then the tile is a right triangle, $c = 3a$, and the smallest angle α satisfies the equation $\sin\alpha = 1/3$.*

Proof. Let the maximal segment be PQ, running from P in the "north" to Q in the "south." Suppose three triangles T_1, T_2, and T_3 occur on the west side of PQ, meeting PQ at vertices U and V, and T_4 lies on the east side of PQ. Let c be the longest side of T; then $c = 3a$ or $c = 2a + b$. It is impossible that the three congruent triangles have a common vertex

S, so that SPU, SUV, and SVQ are congruent triangles. Hence there are two distinct points S and R such that $T_1 = SPU$ and $T_3 = RVQ$ are two of the triangles in the tiling. T_2 may have a side SU in common with T_1 or a side RV in common with T_3. In either case the common side is perpendicular to PQ and T is a right triangle, so $\gamma = \pi/2$ and $\alpha + \beta = \pi/2$. We have $PU = UV = a$, and $SV = c = 3a$ or $2a + b$. If $SV = 3a$ then $\sin\alpha = UV/SV = 1/3$ as claimed in the lemma. The case $c = 2a + b$ is impossible, since no right triangle has sides a, b, and $2a + b$.

If, on the other hand, T_2 does not have a side in common with T_1 or T_3 then there will be altogether five triangles on the left of PQ sharing vertices U and V. Let W be the west vertex of T_2. Then W cannot lie on SP, since if it does, WU is longer than both SP and SU, but one of those sides must be the longest side of triangle T, since the third side of SPU is less than PQ, which is one side of the copy of T on the right of PQ. Similarly, W does not lie on RQ. Let T_5 and T_6 be the other copies of T in M sharing vertices U and V, respectively. Let M be this six-triangle configuration. We claim that the boundary of M contains at least five non-straight angles. At R there is either another non-strict vertex with a non-straight angle, or at least (if R is a vertex of T_2) the boundary of M is not straight. Similarly at S. There is an angle at the east vertex of T_4 (the triangle on the right of PQ). We claim there are also non-straight angles at P and Q. For those to be straight angles, we would have to have the sum of two angles of T equal to π. But at (one of) P or Q, the angle of T_4 is the small angle α, the angle from T_1 or T_3 is not α, since α is the angle at S or R; so the sum of the two angles is less than π (in fact it is $\pi - \beta$). At the other of P or Q we would need β to be a right angle to create a straight angle. In that case T_2 would have a side in common with T_1 or with T_3, contradiction. Hence there are vertices of M (at least) at P, Q, R, S, and the third vertex of T_4–five in total. Suppose it were possible to place one more copy of T next to M so as to form a triangle. If the triangle is placed to the right of PQ against one of the sides of T_4, that may eliminate a vertex at P or Q, but will leave a vertex at the other of P or Q, as well as creating one more new vertex and leaving three old ones–too many vertices for a triangle. If the triangle is placed to the left of PQ, against T_1 or T_3, again it may eliminate a vertex at P or Q, and possibly at S or R, but it will create a new vertex and leave at least three old ones. Placing it anywhere else will leave vertices at all three vertices of T_4; but since part of M exists outside the convex hull of those vertices, those cannot be the vertices of a triangle containing M. Hence, a tiling with a vertex of type $3:1$ in which T_2 does not have a side in common with T_1 or T_3 cannot occur in a 7-tiling. That completes the proof of the lemma.

Lemma 3 *A 7-tiling cannot contain a maximal segment of type* $3:2$.

Proof. Consider a non-strict tiling containing a maximal segment of type $3:2$. Let T_1, T_2, and T_3 occur on the left of the (vertical) maximal segment PQ, with vertices U and V on PQ. Already five triangles will have vertices on the maximal segment PQ. If they do not have sides in common then three more triangles will be required to fill in the gaps–more than seven altogether. Say then that T_1 and T_2 have a side in common.

Let U be the vertex that T_1 and T_2 share on PQ. By Lemma 1, $\gamma = \pi/2$ and T_1 and T_2 both have a right angle γ at R. Suppose, for proof by contradiction, that the sides of T_1 and T_2 on PQ are equal to a, and that T_1 and T_2 share a common side (but not necessarily a common vertex west of PQ). But then they *do* share a common vertex S west of PQ, since they have their angles at U equal (both right angles) and their west angles both equal to α, hence the angles SPU and SVU are both β, and the sides opposite are both b, so the common vertex S is b away from U. Then triangle SPU is congruent to SVU. Let T_4 and T_5 be the triangles on the east side of PQ, sharing vertex R on PQ, and let T_6 be another triangle west of PQ sharing vertex V (there must be one since T_2 does not have a right angle there.) Let E be the east vertex of T_4. If T_4 and T_5, the two triangles on the east of PQ, do not share a side (and hence have right angles at R), then a seventh triangle must occur between them, and we have too many vertices: P, Q, W, and S at least. Hence T_4 and T_5 do share a side, and their angles at P and Q are acute, and their angles at R are right angles. Hence EP and EQ are equal to c and PQ is $2b$, since it cannot be $2a$ as it is the sum of three sides of T_1, T_2, and T_3.

Then this six-triangle configuration M has vertices at P, E, Q, S (the shared west vertex of T_1 and T_2), and the west vertex W of T_3. Triangle T_6 must then fill the angle at V, or else the seventh triangle would need to touch V, leaving more than three exterior vertices, namely P, Q, E, and at least one vertex west of PQ.

Suppose, for proof by contradiction, that W lies on line SQ. Then angles VWQ and VWS are right angles, so VQ, the side opposite angle VWQ in triangle T_3, equals c, and triangle SVW is congruent to QVW. M then forms a quadrilateral, and along PQ we see $2a + c = 2b$. Angle $WSV = \alpha$, by the congruence of triangles WSV and WQV. The angles of triangle SPQ are β at P, α at Q, and hence $\gamma = \pi/2$ at S. This angle at S is also equal to 3α, since each of triangles T_1, T_2, and T_6 has angle α there. Hence $\alpha = \pi/6$, so $a = \frac{1}{2}$, $b = \sqrt{3}/2$, and $c = 1$. But then the equation $2a + c = 2b$ does not hold, contradiction. Hence W is does not lie on line SQ.

Hence there are two vertices of M west of PQ (either S and W or vertices of T_6). M thus has at least five vertices. To reduce this to three vertices by placing one more triangle east of PQ is impossible. Similarly it is impossible to reduce the number of vertices to three by placing a new triangle along PS or WQ. But then, no matter where else we place T_7, P, E, and Q will remain vertices, and there must be a fourth vertex west of PQ, so the result cannot be a triangle. This contradiction shows that we cannot have $VU = PU = a$ as we assumed above.

Now we drop the contradictory assumption $VU = PU = a$ and begin anew. Again we have right angles in T_1 and T_2 at U, but this time one of them has side b along PQ (not c, since that must be opposite the right angle at U). The other one has side a along PQ, since $2b \geq a + b > c$. Hence they do not share a common west vertex. Let S be the west vertex of T_1 and X the west vertex of T_2. Again T_6 will have to be placed between T_2 and T_3 with a vertex at V. That will give us a six-triangle configuration M with vertices P, E, Q, S, X, and W (the west vertex of T_3). We only have to show that placing one more triangle T_7 cannot possibly produce a triangular configuration. (That is not *prima facie* impossible just because there are six vertices–it could happen if M had two collinear sides separated by two sides forming a "notch" into which T_7 would just fit–so some further argument is required.) If the two triangles T_4 and T_5 east of PQ do not share a side, then T_7 would have to be placed east of PQ between the two, leaving a vertex at P (since T_1 has an acute angle at P), as well as vertices at X and S, which are distinct, and of course at least one vertex east of PQ, totaling more than three. Hence T_4 and T_5 do share a side; hence their right angles are both on PQ at vertex R. Hence their angles at P and Q are acute, and triangle PER is congruent to triangle QER. Hence the two sides PR and RQ are equal, and either equal to a or to b. The length of PQ is thus either $2a$ or $2b$ (measured from the right side), and also either $2a + b$ or $2b + a$ (measured from the left side). Three of the four possible equations here are immediately impossible, leaving only the possibility $2b = 2a+b$; hence $b = 2a$. Now if the angle of T_1 at P is α, then the southwest vertex S of T_1 lies on the north side of T_2, and S is thus not on the convex hull of M, and triangle T_7 cannot fill the obtuse angle $\pi - \beta$ exterior to M at S. Hence the angle of T_1 at P is not α; so it must be β. Then the northwest vertex X of T_2 lies on the south side of T_1, and thus not on the convex hull of M, but there is an obtuse exterior angle $\pi - \beta$ at X that cannot be filled by T_7. We conclude that it is not possible to place T_7 to create a triangle. That completes the proof of the lemma.

Lemma 4 *A 7-tiling cannot contain a maximal segment of type* $4:1$.

Proof. Let PQ be a (north-south) maximal segment with four triangles on the left and one on the right. Of the four triangles on the left, we cannot have three sharing a vertex not on PQ, so the minimum "tile" (consisting of all the triangles touching PQ) contains at least six triangles, and contains seven without making a triangle, unless the four triangles on the left occur in two pairs, each pair having a common side. Let the northernmost of these pairs be T_2 and T_3, and let T_4 and T_5 be the southern pair. Let V be the vertex on PQ shared by T_3 and T_4. Let T_1 be the triangle on the right of PQ. Let E be the eastern vertex of T_1. let R be the vertex between P and V (shared by T_2 and T_3) and S the vertex between V and Q (shared by T_4 and T_5). Let M be the figure formed by these five triangles. Note that we have not proved that triangles T_2 and T_3 have a common "west" vertex, i.e. their shared sides may be of different length, and the same goes for T_4 and T_5.

By Lemma 1, $\gamma = \pi/2$ and T_2 and T_3 have right angles at R, and T_4 and T_5 have right angles at S. Then P and Q are vertices of M, as are E and the western vertices of T_3 and T_5. Thus we cannot afford to insert two triangles between T_3 and T_4 (that is, anywhere inside the angle at V between T_3 and T_4), as that will make seven altogether and the result will not be a triangle, as it must contain a westernmost vertex in addition to P, Q, and E. Hence there is just one triangle T_6 between T_3 and T_4. The longest side c of T_3 (the hypotenuse) is shared with T_6, and since no other triangle can be inserted between T_3 and T_4, the side of T_6 shared with T_3 must also have length c. Similarly, T_6 must share side c with T_4; but then

T_6 has two two sides equal to the hypotenuse, which is a contradiction. This contradiction completes the proof of the lemma.

Lemma 5 *No 7-tiling contains a non-strict vertex of type other than* $3:1$ *or* $2:1$ *or* $2:2$.

Proof. Let V be a non-strict vertex in a 7-tiling. Then for some integers m and n, there is a maximal segment S containing V of type $m:n$. We have $m+n \le 7$ since there are only 7 triangles in the tiling. Visualize S as oriented in the north-south direction, with n triangles west of S and m triangles east of S. Let M be the configuration of triangles in the final tiling that touch S. No more than two triangles on the same side of S can share a common vertex that is not on S. Hence if n (or m) is three, then at least four triangles must occur on the west (or east) of S; and if n (or m) is four, then at least five triangles must occur on the west (or east); and neither n nor m can be as much as five, since then at least seven triangles would be required on one side of S.

We may change "east" and "west" if necessary to ensure $m \le n$. Suppose $n = 4$. Since we have proved above that S cannot be of type $4:1$, there are at least two triangles east of S, and as remarked above, at least five west of S. For this to be the case, the triangles T_1 and T_2 on the northwest must share a side and both have right angles where they meet S, and the same for triangles T_3 and T_4 on the southwest, and for triangles T_5 and T_6 on the east, and then T_7 must share vertex V on S with T_2 and T_3. That means that the largest angle γ is a right angle, and the seven-triangle configuration has vertices at the endpoints of S and at least one vertex east of S and at least one vertex west of S, and hence is not a triangle. Hence $n = 4$ is not possible.

Suppose $n = 3$. We have proved above that type $3:2$ is impossible. We now consider type $3:3$. But as remarked above, this would require four triangles east of S and four triangles west of S, making more than seven, so $3:3$ is impossible. That completes the proof of the lemma.

The following figures show some possible configurations in which a maximal segment of type $2:1$ could occur. In these figures, α has to be as shown, either $\pi/6$ or $\arctan \frac{1}{2}$, so that $2a = c$ or $2a = b$. In the first two figures, γ has to be a right angle. In the third figure, γ and β have one degree of freedom; the figure illustrates the case $\gamma = 80$ degrees.

Figure 7: Two three-triangle configurations

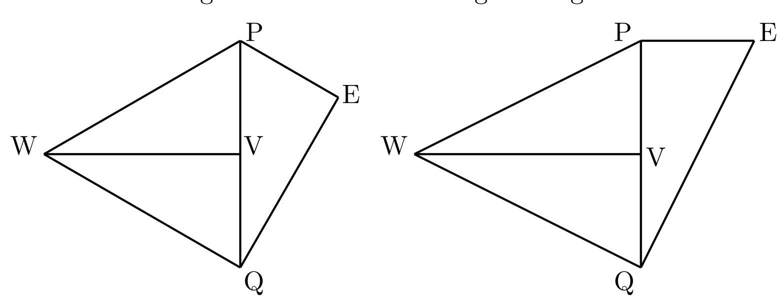


Figure 8: Two five-triangle configurations

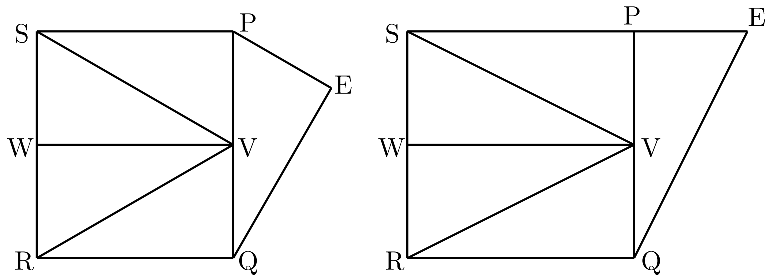
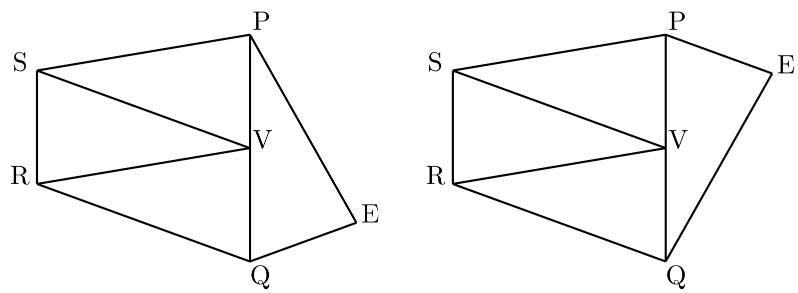


Figure 9: Two four-triangle configurations



Lemma 6 *Suppose that a 7-tiling contains a maximal segment of type* $2:1$*. Then the tiling contains one of the six configurations shown in the preceding figures. To state the conclusion without reference to a figure: the smallest angle* α *of the tile is* $\pi/6$ *or* $\arcsin\frac{1}{2}$*, so* $2a = c$ *or* $2a = b$*; the maximal segment has length* $2a$*, with the non-strict vertex* V *at its midpoint, and one of the following holds.*

(i) 3 triangles meet at the non-strict vertex, two of them having a right angle there. See Fig. 8.

(ii) 5 triangles meet at the non-strict vertex. Denoting the maximal segment by PQ*, with midpoint* V*, the triangles "west" of* PQ *with vertices at* P *and* Q *have right angles at* P *and* Q*, and angle* β *at* V*, and the other two triangles west of* PQ *have angle* α *at* V*. See Fig. 9.*

(iii) 3 triangles on one side of the maximal segment share the vertex V*, with the middle one (the one that does not share a side with the maximal segment) having angle* α *at* V*, and sharing another vertex with each of the other two triangles with which it shares vertex* V*. Moreover, each two adjacent triangles of the three on one side of the maximal segment form a parallelogram. See Fig. 10.*

Note that in cases (i) and (ii), the tile is a right triangle, while in case (iii) that is not asserted.

Proof. The previous lemmas have ruled out all possible types of maximal segments except $3:1$, $2:2$, and $2:1$. We consider the possible configurations in which a maximal segment has type $2:1$. For convenience of description, let us orient triangle ABC so the maximal segment PQ is north-south, with two triangles on the west and one on the east. Because there is just one triangle on the east, the length of PQ is either b or c. The two triangles on the left must divide the maximal segment equally, because if they did not, then the two segments would have lengths a and b and their sum (the side of the one triangle on the right) would necessarily be c, but of course $a+b<c$. It follows that the two segments have length a, and $2a=b$ or $2a=c$, since if the segments had length b instead of a, we would have $c=2b\geq a+b$, so a triangle with sides a, b, and c would be impossible.

We consider the "minimal configuration" M containing all the T_i with a vertex at V. How many triangles will M contain? It contains at least three. We will analyze the possibilities.

Let the direction of PQ be "north-south" with P at the north. Let T_1 and T_2 be the west triangles and T_3 the east triangle with sides on PQ, and V the midpoint of PQ, the shared vertex of T_1 and T_2. If T_1 and T_2 share a side, then the largest angle γ is a right angle, so T_1 and T_2 must also share their west vertex W (at distance b from PQ). Then we have $2a=b$ or $2a=c$, leading to the possibilities listed in part (i) of the conclusion and illustrated in Fig. 8.

We therefore may assume that T_1 and T_2 do not share a side, and at least one additional triangle T_4 shares their common vertex V at the midpoint of PQ. Let S be the west vertex of T_1 and R the west vertex of T_2, and E the east vertex of T_3. Then angles PSV and VRQ are both α, since they are opposite side a.

Assume first that there is exactly one triangle T_4 between T_1 and T_2 sharing vertex V. One of the sides of T_4 lying along SV or RV must be b or a and since the a sides of T_1 and T_2 lie on PQ, the sides SV or RV must each be b or c. Assume, for proof by contradiction, that triangle T_4 does not share west vertices with T_1 and T_2, i.e. it is not triangle SVR. Then one of its vertices lies on SV or on RV, since the b side of T_4 cannot be longer than the b or the c sides of T_1 and T_3. Interchanging "north" and "south" if necessary, we can assume that the north vertex X of T_4 lies on SV, and hence is a non-strict vertex. Since SV must be larger than XV, we have either $SV=c$ and $XV=b$, or $SV=c$ and $XV=a$, or $SV=b$ and $XV=a$. The maximal segment of this non-strict vertex has one end at V and extends westward along SV. By Lemma 5, its type must be $2:1$ or $3:1$ or $2:2$.

Case 1, angles SVP and RQV are equal. Since these are corresponding angles made by the transversal PQ to SV and RQ, lines SV and RQ are parallel. Then the alternate interior angles SVR and VRQ are equal. Angle $VRQ=\alpha$, since it is opposite $VQ=a$. Therefore angle $SVR=\alpha$. Therefore $XV=b$ or $XV=c$; but since X lies on SV and $X\neq S$ we have $SV=c$ and $XV=b$. If the type of the maximal segment of X is $2:1$, we have $a+b=c$, which is impossible since $a+b>c$, or $2b=c$, which is impossible since $2b>a+b>c$. If it is $3:1$ we have $3b=c$, or $a+2b=c$, or $2a+b=c$, all of which are impossible since $a+b>c$. By Lemma 5, the only remaining possibility for the type of the maximal segment of X is $2:2$. If γ is not a right triangle, then there must be three triangles on each side of the maximal segment, making six triangles, which together with T_2 and T_3 is more than seven. Hence γ is a right angle. Since $SV=c$, angle SPV is a right angle and angle $SVP=\beta$.

Hence angle $RQV = \beta$ and angle RVQ is a right angle. Since $XV = b$, the right angle of T_4 is at X. Therefore the south side of T_4 is c, and extends west of R on RV. We now have a third non-strict vertex at R. The exterior angle at R is more than $\pi/2$, so we will need at least two more triangles to sharing vertex R to be placed south of RV. But we must also place at least two more triangles with sides on line SV, since the maximal segment of X has type $2 : 2$. That makes eight triangles altogether, which is more than seven. That disposes of Case 1.

Case 2, angles SVP and RQV are not equal. Since triangles SPV and RQV are congruent and $PV = VQ = a$, we must have angles SVP and RVQ equal; and they must both be equal to β, since the angle of T_4 at V is at least α, and $2\gamma + \alpha > \alpha + \beta + \gamma = \pi$. Then the angle of T_4 at V cannot be γ, since $2\beta + \gamma > \pi$. If it is α then we have $2\beta + \alpha = \pi$, which would make $\beta = \gamma$ so angles SVP and RQV would be equal and Case 1 would apply. Therefore it is β and we have $3\beta = \pi$. Then side XV of triangle T_4 must be a, since it cannot be c, because it is less than SV. Consider the type of the maximal segment of X. Assume, for proof by contradiction, that it is $2 : 1$. Then $2a = c$ and X is the midpoint of SV. Since $\beta = \pi/3$, the equation $2a = c$ implies γ is a right angle and $\alpha = \pi/6$. Then the side of T_4 on line RV must be the c side, since if the vertex of T_4 on RV lies between R and V, the distance from X to that vertex is less than c. Hence T_4 has R for its southwest vertex. Now we have a four-triangle configuration with parallel north and south boundaries SP and RQ, a concave exterior vertex at X, and a right angle at the east vertex E. If three additional triangles are added sharing vertex X, the resulting configuration of seven triangles will not be a triangle, since it has three vertices P, Q, and E and more to the west of PQ. If two triangles are added sharing vertex X, they must not be placed so as to create new non-strict vertices, as that would require placing a seventh triangle west of PQ, leaving at least four vertices. Therefore if two more triangles are added sharing vertex X, they share a west vertex W and are triangles SXW and RXW. The resulting six-triangle configuration is convex and has six vertices. Placing one more triangle can decrease the number of vertices of a convex configuration by at most one, so this configuration cannot be completed to a 7-tiling. Hence the concave exterior vertex at X must be filled by just one triangle. If this triangle T_5 is not SXR then its west vertex W lies on SV extended, and it has side b along SV. Its south vertex U lies on RX. New concave exterior angles are created at S and U, each of which is more than $\pi/2$, and hence each will require placing at least two more triangles with vertices S and U respectively. But that will require a total of 9 triangles. Hence triangle T_5 must be triangle SXR. We now have a five-triangle configuration including rectangle $SPQR$ and triangle PQE. Either $QE = b$ or $PE = b$. Suppose, for proof by contradiction, that $QE = b$. Then it cannot be that both RS and QE lie on sides of the final triangle, since the area to be filled south of RQ would require more than two triangles. With only two more triangles available, we cannot create more non-strict vertices. Consider placing a triangle T_6 south of QE. Then it must share vertices Q and E. If we do not place the right angle at E then there will be a concave exterior vertex at Q that will require two more triangles to fill, contradiction. Hence T_6 must have its right angle at E. Now we have a six-triangle convex configuration with five vertices. This cannot be completed to a 7-tiling since adding one triangle to a convex configuration can reduce the number of vertices by at most one. Hence QE is one of the sides of the final triangle. Placing a triangle T_6 north of PE without creating a new non-strict vertex would require that T_6 have side a along PE; that would create a concave exterior vertex at P greater than $\pi/2$, which could not be filled with one more triangle. Hence PE is also one of the sides of the final triangle. Then there must be a triangle T_6 north of SP whose c side lies on PE extended. But now two triangles have sides on line RS, and we have already seen that not both RS and QE can be sides of the final triangle. So at least two more triangles will be required west of RS, but we have only one more available. This contradiction shows that $QE \neq b$.

Therefore $PE = b$. Then it cannot be that both RS and PE lie on sides of the final triangle, since the area to be filled north of SP would require more than two triangles. With only two more triangles available, we cannot create more non-strict vertices. Consider placing a triangle T_6 north of PE. Then it must share vertices P and E. If we do not place the right angle at E then there will be a concave exterior vertex at P that will require two more triangles to fill, contradiction. Hence T_6 must have its right angle at E. Now we have a six-triangle convex configuration with five vertices. This cannot be completed to a 7-tiling since adding one triangle to a convex configuration can reduce the number of vertices by at most one. Hence PE is one of the sides of the final triangle. Placing a triangle T_6 south of QE without creating a new non-strict vertex would require that T_6 have side a along QE; that would create a concave exterior vertex at Q greater than $\pi/2$, which could not be filled with one more triangle. Hence QE is also one of the sides of the final triangle. Then there must be a triangle T_6 south of RQ whose c side lies on QE extended. But now two triangles have sides on line RS, and we have already seen that not both RS and PE can be sides of the final triangle. So at least two more triangles will be required west of RS, but we have only one more available. This contradiction disposes of Case 2. That in turn completes the proof by contradiction that triangle T_4 is triangle SVR.

Now that we know T_4 is triangle SVR, we again have cases to consider. Case 1, angles SVP and RQV are equal. Since these are corresponding angles made by the transversal PQ to SV and RQ, lines SV and RQ are parallel. Then the alternate interior angles SVR and VRQ are equal. Angle $VRQ = \alpha$, since it is opposite $VQ = a$. Therefore angle $SVR = \alpha$. Since angles RSV and VQR are opposite side RV in their respective triangles, they are equal. Since VQ and SR are opposite angle α, they are equal. Hence sides SV and RQ are also equal. That makes $SVQR$ a parallelogram. This is the configuration described in part (iii) of the lemma, so we are finished with Case 1.

Case 2, angles SPV and RQV are equal. Then since angle $PSV = \alpha$, angles SPV and RQV are equal either to β or to γ. Then angles SVP and RVQ are also equal (either to γ or to β). They cannot be equal to γ, because in that case it would not be possible to place even one triangle T_4 between T_1 and T_2, since $\alpha + 2\gamma \geq \alpha + \beta + \gamma = \pi$. Hence SVP and

RVQ are equal to β. Then 2β plus angle SVR equals π. Since angles SPV and RQV are both equal to γ and SP and RQ are both equal to b, SR is parallel to PQ, and triangle SVR is isosceles, with both sides SV and RV equal to c. Hence angles SRV and RSV are both equal to γ. Angles SRV and RVQ are alternate interior angles of the transversal RV of parallel lines SR and PQ, so they are equal; but angle $RVQ = \beta$, so $SRV = \beta$ also. Then angle SRV is opposite side c and hence equals γ; hence $\beta = \gamma$. Hence the triangles T_i are isosceles with $2a = b = c$, and triangle SVR is congruent to the T_i, and hence is a fourth triangle T_4 belonging to the tiling. In this case conclusion (iii) of the theorem holds.

That completes the proof of the lemma in case there is only one triangle between T_1 and T_2.

We still have to consider the case in which there are two or more triangles, T_4 (with a side on SV) and T_5 (with a side on RV), and possibly still more triangles, between T_1 and T_2. Changing south and north if necessary, we can assume angle $PQE = \alpha$. Angle QPE is either β or γ. If γ is not a right angle, or if γ is a right angle but SPE is not a straight angle, then this 5-triangle configuration has vertices (at least) at S, P, E, Q, and R, and at least one more vertex W of T_4. If W is a concave vertex it will have to be removed by placing T_6 with a vertex at W. That will leave at least five other vertices, all convex; that cannot be reduced to three by placing one more triangle; hence W is not a concave vertex. Unless W occurs on the line SR, we then have six convex vertices, which cannot be reduced to three by placing two more triangles. The only way W can occur on SR is if there are two triangles between T_1 and T_2, and they share vertex W and have a right angle there. In that case γ is a right angle. Hence γ must be a right angle, and one of two cases holds: either $2a = c$ and angle $QPE = \beta$, so $\alpha = \pi/6$ and $\beta = \pi/3$, and there are two triangles sharing a vertex at the midpoint of SR and another vertex at V, or $QPE = \gamma$, and SPE is a straight angle. Then $2a = b$, rather than $2a = c$, and $\alpha = \arctan \frac{1}{2}$. Now consider the angles at V. The angle of T_1 and V is β. If T_2 has angle γ there, that leaves room for only one triangle between them, with angle α; hence T_2 has angle β at V. We have $\beta = \pi/2 - \arctan(1/2)$, which is about 63.43 degrees. Hence any triangles between T_1 and T_2 have angle α at V, since β is too big to fit. The angle to be filled is $\pi - 2\beta = 2\alpha$, so more than two triangles cannot fit, but two fit nicely. If these two triangles are placed "naturally" then both of their west sides will lie on SR. These are the two configurations described in part (ii) of the lemma. The last two triangles must be placed in this configuration, because any other placement would place a side of length b along a side of length c in at least two places, each of which would require placing two more triangles to make these vertices have type $2:2$, contradiction.

That completes the proof of the lemma.

Lemma 7 *Suppose that a 7-tiling contains a non-strict vertex. Then the type of that vertex is* $2:1$*, the tile is a right triangle whose smallest angle* α *is* $\pi/6$ *or* $\arcsin \frac{1}{2}$*, and 3 triangles meet at the non-strict vertex. (See Figure 8.)*

Proof. After the previous lemma, it only remains to rule out cases (ii) and (iii) of that lemma's conclusion. We first rule out case (iii). We may suppose the non-strict vertex is V, the midpoint of the maximal segment PQ, with P at the north, Q at the south; that triangles T_1, T_2, and T_4 are on the west of PQ, sharing vertex V, and that P is a vertex of T_1 and Q is a vertex of T_2, and S is the shared west vertex of T_1 and T_2 and R is the shared west vertex of T_4 and T_2. Triangle T_3 is east of PQ, and its east vertex is called E. $PV = VQ = SR = a$. We have RQ parallel to SV and SP parallel to RV. By changing "north" and "south" if necessary, we can assume that angle $RQV = \beta$ and angle $SPV = \gamma$. There are four cases to consider. Namely:

Case 1: $2a = b$, angle $E = \beta$, angle $VQE = \alpha$

Case 2: $2a = b$, angle $E = \beta$, and angle $VQE = \gamma$

Case 3: $2a = c$, angle $E = \gamma$, angle $VQE = \alpha$, and $b \neq c$

Case 4: $2a = c$, angle $E = \gamma$, and angle $VQE = \beta$, and $b \neq c$

We call this four-triangle configuration M. All we have to do is prove that it is not possible to add three more tiles to M and thereby create a triangle. This could be done by computer, but it is within reach to do it by hand.

M has five exterior vertices, all of which have angles less than π (because they are composed of two angles of the tile triangle, not both γ) except possibly P in case 1, where two angles γ share vertex P. In cases where M is convex and five-sided, placing three new triangles must leave at least two of the original five edges as part of the boundary of the final triangle. Hence two of the sides of the final triangle contain sides of M. There are 10 pairs of sides of M to consider; in each case we can ask whether it is possible to draw a third side and fill in the remaining area with copies of the tile.

Case 1 divides into Case 1a (when $\gamma = \pi/2$), Case 1b (when $\gamma > \pi/2$), and Case 1c (when $\gamma < \pi/2$). Before subdividing into these cases, we first argue that SR cannot be a side of the final triangle. Assume, for contradiction, that both EQ and SR are sides of the final triangle. Extend EQ and SR to their intersection point L. The final triangle must include triangle SLE, hence must contain triangle RQL. But the area of triangle RQL is five tiles, not three, which we see as follows: Angle $RQL = \gamma$, since the other two angles at Q are α and β. Angle $SRV = \gamma$ (because it is opposite SV which is oppositive angle SPV), and angle $VRQ = \alpha$, since it is opposite $VQ = a$. S Angle $L = \alpha$. Since the angles at R must add up to π, angle $QRL = \beta$. Then angle $QLR = \alpha$, in order that the angles of triangle QLR add to π. So triangle QLR is similar to the tile, and since $RQ = c$, the similarity factor is c/a. The area of RQL is therefore c^2/a^2 times the area of a single tile. To complete a 7-tiling this way would thus require $3a^2 = c^2$. But $c^2 = a^2 + b^2 = a^2 + (2a)^2 = 5a^2$, not $3a^2$. (Nevertheless, it does not seem that we can complete this configuration to a 9-tiling, but that is irrelevant.) This contradiction shows that not both EQ and SR are sides of the final triangle. If SR is a side, then another triangle T_5 must be placed east of EQ. If T_5 does not share vertices E and Q, then at least one concave exterior vertex is created, which will require placing T_6 south of EQ or on QE extended north of E. We will then have vertices at S, R, at P unless $\gamma = \pi/2$, and at one of Q or E, at most one of which can be removed by placing T_7, and at least two more on line QE, which is too many. Hence T_5 must share vertices E and Q. The resulting five-triangle configuration is convex. Since in Case 1, angle $E = \beta$, the vertex at E remains a vertex. Since $EQ = c$, angle γ does not occur in T_5 at Q, so the vertex at Q remains a vertex too. Then the five-triangle convex configuration has six vertices. T_6 must share the existing vertices, as if we create a new non-strict vertex we do not have enough triangles to fill the exterior angles thus created and tile a triangle. But if T_6 shares existing vertices, the resulting six-triangle configuration will be convex, and will have at least five vertices, so cannot be completed to a triangle by adding one more triangle T_7. This contradiction proves that SR is not a side of the final triangle.

Now we subdivide Case 1. First consider case 1a, when $\gamma = \pi/2$, and hence SPE is a straight line. This is the only case in which M is a quadrilateral, rather than having five sides. As proved above, we will have to add one triangle T_5 west of SR, with westernmost vertex W, with one of its sides containing segment SR (which has length a). If we place this triangle so that W lies on SE extended and on RQ extended, then we will create a 5-tiling. Placing T_5 with vertices at S and R but with angle γ at R instead of β will create a concave vertex at R with an exterior angle greater than $\pi/2$, so two tiles would be required at vertex R, leaving vertices at Q, E, S, and W at least. Placing triangle T_5 with one side along line SR but extending north of S would require placing at least two more triangles north of SE, leaving vertices R, Q, E, and W. Placing triangle T_5 with one side along SR but with northernmost vertex on SR south of S would require us to place T_6 on SR north of T_5, and since $SR = a$, T_6 would extend south of R, requiring T_7 to share vertex R. Then seven triangles would be used and more than three vertices would still exist, e.g. Q, E, W, and the south vertex of T_6. Hence the triangle west of SR must be placed with W on SE extended, forming a 5-tiling. We are then asked to add two more triangles to produce a 7-tiling.

Side WE has length $2b + a = 5a > 2c = 2\sqrt{5}a$ and hence cannot be entirely covered by placing two triangles north of WE. Hence no triangle can be placed north of WE, which is thus a side of the final triangle. Side WQ has length $2c$ and hence if one triangle is placed south of WQ, a second one must be placed there as well; if these are placed so as to cover all of WQ then the result is not a triangle. Hence no triangles can be placed south of WQ, which must be a second side of the final triangle. We are thus asked to place two triangles east of EQ and complete a 7-tiling. If these two triangles can be placed so that Q remains a vertex of the final triangle then the vertices of the final triangle will be W, Q, and a third vertex U east of E on WE extended. Triangle QEU must be composed of two copies of the tile T. These two triangles share vertices E and another vertex X on QU. The angle at X is a right angle by Lemma 1. Angle $PEQ = \beta$. Angle $EQX =$ angle EUX since both are opposite side EX. Hence angle $QEX =$ angle UEX, and these angles are not γ since the right angles occur at X. If they are β then, adding the three angles at E, we have $3\beta = \pi$; but since $\alpha = \arctan(1/2)$, we do not have $\beta = \pi/3$. If angles QEX and UEX are both α then we have $2\alpha + \beta = \pi$, but that is impossible since $\alpha + \beta + \gamma = \pi$ and $\gamma > \alpha$. Hence it is not the case that Q is a vertex of the final triangle.

Therefore we will have to extend side RQ past Q by adding another triangle south of QE. Extending side RQ past Q will require creating a right angle at Q; if that is done with one triangle, then the side it shares with QE will have length b, creating a non-strict vertex at distance b along QE, which has length c. That will create a concave vertex with exterior angle more than $\pi/2$, which cannot be filled with our one remaining triangle. Hence both remaining triangles will have to share vertex Q, using angles α and β. Hence the shared side of those two triangles cannot have length a or b in both triangles, and it cannot have length c either since the first one has its side of length c along QE. Hence these two new triangles do not even share a vertex along their shared side, and cannot form a 7-tiling. That disposes of case 1a.

Now we take up case 1b, when $\gamma > \pi/2$. Then the boundary of M is concave at P, so in addition to adding a triangle T_5 west of SR, with westernmost vertex W, we must add T_6 north of P, with a vertex at P. Suppose, for proof by contradiction, that $\gamma \neq 2\pi/3$, in which case T_6 does not fill the vertex at P, or that $\gamma = 2\pi/3$ but T_6 is not placed with angle γ at P. Then we must also add T_7 with a vertex at P. That would mean that no triangles can be added south of PE or west of W, so that W and Q must both be vertices of the final triangle, and R must not be a vertex, and hence lies on WQ. Moreover, nothing can be added touching QE, so QE must lie on one side of the final triangle. Therefore the third vertex is either E or lies northeast of E on QE extended. It follows that triangle T_7 must have side a along PE, since it has a vertex at P and cannot extend beyond E along line PE, and $PE = a$ is the shortest side of the tile. E cannot be a vertex of the final triangle, since W and Q are vertices and triangle T_7 does not lie inside triangle WQE, since it extends north of E along line QE. Therefore only two tiles meet at E. Hence by Lemma 1, $\gamma = \pi/2$. But in Case 1b, $\gamma > \pi/2$, so this is a contradiction. This contradiction proves that $\gamma = 2\pi/3$ and T_6 is placed with angle γ at P.

Triangle T_5 has to be placed west of SR, since we proved above that SR cannot be a side of the final triangle. Assume, for proof by contradiction, that it is placed with its a side along SR. Consider the three interior angles at R. They are angle $QRV = \alpha$, angle $VRS = \gamma$, and angle WRS which might be γ or β, but not α since angle $W = \alpha$, because it is opposite SR. If angle $WRS = \gamma$ then a concave exterior angle exists at R that would have to be filled by one more triangle T_7, leaving four vertices W, S, E, and Q still present. S would be a vertex since the angles there would be α from T_6, α from T_1, β from T_4, and β from T_5, and their sum is $2\alpha + 2\beta = 2\pi - 2\gamma = 2\pi/3 \neq \pi$. Four vertices remaining means a triangle is not created. Hence angle $WRS \neq \gamma$. Hence angle $WRS = \beta$. Then WRQ is a straight line. Consider the angles at S. They are α from T_6, α from T_1, β from T_4, and this time γ from T_5. Their sum is $2\alpha + \beta + \gamma = \alpha + \pi > \pi$. Hence a concave exterior vertex exists at S after the placement of T_5 and T_6. The exterior angle is $\beta + \gamma$, too large to be filled by the placement of one more triangle T_7. This contradiction shows that T_5 cannot be placed with its a side along SR.

Hence triangle T_5 must be placed west of SR in such a way that not both S and R are vertices. Suppose, for proof by contradiction, that T_5 is placed so that it does not have a vertex at R. It must have two vertices on line SR, of which say U is the southernmost. We then have a concave exterior vertex either at R (if U is south of R) or at U (if U is north of R). T_7 will have to be placed to fill this exterior concavity. Since Q and E will remain vertices after the placement of T_7, the third vertex must be W; hence S lies on line WE. That implies that T_7 has S for a vertex and the sum of the angles at S must be π. Those angles are α from T_6, α from T_1, β from T_4, and an unknown angle from T_5. The unknown angle must be $\pi - 2\alpha - \beta = \gamma - \alpha$. This cannot be γ, nor can it be α, since $2\alpha = \gamma = 2\pi/3$ implies $\alpha = \pi/3$ which in turn implies $\beta = 0$, a contradiction. Hence the angle of T_5 at S is $\beta = \gamma - \alpha$. Hence $\alpha + \beta = \gamma$; but also $\alpha + \beta = \pi - \gamma$, which contradicts $\gamma = 2\pi/3$. (So we do not even have to analyze the impossible situation near R.) This contradiction shows that R must be a vertex of T_5 and S is not a vertex. Since $SR = a$ is the shortest side of the tile, triangle T_5 extends north of S along SR. Then T_7 must be placed with a vertex at S north of SE. There will then be vertices E, Q, W, and a vertex north of S on line SE. That is four vertices at least, so no triangle is created. That disposes of Case 1b.

Now we take up case 1c, when $\gamma < \pi/2$. Triangle T_5 must be placed west of SR. Suppose, for proof by contradiction, that it is placed with angle β at R and angle γ at S. Then the first five triangles form a quadrilateral with vertices at W, Q, E, and P, and straight angles at R and S. Consider the possibility that WSP is a side of the final triangle. Then triangle T_6 must be placed with a side along PE and a vertex at P. But the angle at P will then be at least $2\gamma + \alpha$, which is more than π. Hence WSP cannot be a side of the final triangle. But the length of WP is $2b$, and we have $2b > c$ since $2b = 4a > 3a = a + b > c$. Hence at least two triangles will have to be placed north of WSP. But that will leave vertices W, Q, E, and another vertex north of WSP–more than three. This contradiction shows that T_5 cannot be placed west of SR with angle β at R and angle γ at S. If it is instead placed with angle γ at R and angle β at S, there will be a concave exterior angle at R, and convex vertices at W, S, P, E, and Q. Even if it is possible to fill the exterior angle at R with T_6, that still leaves five vertices in a convex configuration, or more than five vertices with some concave exterior angles. In either case, a triangle cannot be created by adding T_7. Hence T_5 cannot be placed in either orientation with its a side along SR. Therefore T_5 must be placed with a side extending past SR, either north or S or south of R along line SR (or both). Suppose, for proof by contradiction, that T_5 extends north of S to a vertex N on RS extended, but has R for a vertex. Then T_6 must be placed north of PS to fill the concave exterior angle at S. That leaves convex vertices at W, N, E, and Q at least. There will also be a vertex at R unless T_5 has angle β there; that must be the case since five convex vertices is too many to create a triangle by placing T_7. Then T_5 has angle α at N and γ at W. Hence NS has length $c - a$. Triangle T_6 must also eliminate vertex P, or there will again be five vertices, which is too many. To do that, T_6 must fill the entire angle $NSP = \gamma$, and must supply an angle equal to $\pi - 2\gamma$ at P. Since $SP = b$, T_6 has angle

β at N, so it must have α at P. Therefore $\alpha = \pi - 2\gamma$. Adding $\beta + \gamma$ to both sides of this equation we have $\pi = \pi + \beta - \gamma$, which implies $\beta = \gamma$. We now have a 6-tiling of quadrilateral $NEQW$. But this quadrilateral is a parallelogram: NPE is parallel to WRQ because transversal PQ makes equal alternate interior angles $RQP = \beta$ and $QPE = \gamma$, and QE is parallel to WN because transversal NE makes complementary corresponding interior angles $WNP = \alpha + \beta$ and $QEP = \beta = \gamma$. By placing one more triangle T_7, one cannot turn a parallelogram into a triangle. This contradiction shows that T_5 cannot be placed with R for a vertex.

Now suppose, for proof by contradiction, that T_5 is placed with one vertex at S and a second vertex U south of R on SR extended. Then T_6 must be placed south of RQ to fill the concave exterior angle at R. That leaves convex vertices at W, U, E, and P at least. There will also be a vertex at S unless T_5 has angle γ there; that must be the case since five convex vertices is too many to create a triangle by placing T_7. Then T_5 has angle α at U and β at W. Hence UR has length $b-a = a$, so the angle of T_6 opposite UR must be α. Triangle T_6 must also eliminate vertex Q, or there will again be five vertices, which is too many. To do that, T_6 must have a vertex at Q, and must fill the entire angle $URQ = \beta$, and must supply an angle equal to $\pi - \alpha - \gamma\beta$ at R. Since $RQ = c$, T_6 has angle γ at U, so it must have α at Q. Therefore to eliminate vertex Q we must have $2\alpha + \beta = \pi$. This implies $\alpha = \gamma$, so the tile is an equilateral triangle. Now we have a 6-tile convex configuration with vertices at W, U, E, P, and S. Placing T_7 can decrease the number of vertices by at most one, since the configuration is convex. Hence no final triangle can be created. This contradiction shows that triangle T_5 cannot be placed with a vertex at R or at S.

Hence triangle T_5 is placed west of SR, with two vertices on line SR somewhere, but not at R or at S. Then at least two concave exterior vertices will be created somewhere on line SR, which must be filled by placing triangles T_6 and T_7 with one vertex each on line SR and a side contained in line SR. That will create two new vertices on line SR, say N to the north and U to the south. We then have vertices W, E, N, and U, even if straight angles at P and Q are created by placing T_6 and T_7. Four vertices do not make a triangle, so this placement of T_5 is also contradictory. That disposes of case 1c, and with it, of case 1.

Now we take up Case 2. In that case SR is parallel to PQ since the alternate interior angles SVP and RSV are both equal to β. We ask which pairs of sides of the pentagon $SPEQR$ could be (contained in) sides of the final triangle.

Case 2a: SR and PE are sides. Let N be the intersection point of SR extended and EP extended. Then triangle NSP has angle β at S and γ at P, because the angles at S and P must add to π. Then angle N is α, and triangle NSP is similar to the tile T. But it has side $b = 2a$ opposite angle α, so its area is four times that of the tile. It therefore requires four triangles congruent to T_1 to cover triangle NSP, which is eight total, more than seven.

Case 2b: SR and QE are sides of the final triangle. Then we have to add triangle T_5 southwest of RQ, and the third side will have to be east of

PE and require at most two triangles to complete the figure, one of which, say T_6, will have to be north of SP, say SPN and one, say T_7, will have to be east of PE, say PEX. But it will not be possible for T_7, adding just one angle at E, to have a side extending QE, since then by Lemma 1, the tile would contain a right angle, so $\gamma = \pi/2$, and triangle PEQ would have two right angles, one at Q and one at E. This contradiction disposes of case 2b.

Case 2c: SR and SP are sides. Since PE and QE are not sides (by Case 2a and Case 2b), there must be two new triangles sharing vertex E, one on each side of the existing triangle PQE. These triangles must share (respectively) side PE and side QE, since otherwise additional triangles will have to be placed sharing vertex P or Q, and a triangle will not be created. Thus we have triangle $T_5 = PEF$ and triangle $T_6 = QEG$, with FEG and RQG and SPF straight lines. Then angle $PFE = \gamma$ (being opposite PE which has length c) and angle $EPF = \gamma$ (so that SPF is straight, since angle $VPS = \beta$ and angle $QPE = \alpha$), so angle $PEF = \alpha$ and the tile is isosceles with $\beta = \gamma$, since triangle PEF has two angles γ. Angle $QEG = \gamma$ since the sum of angles at E is π. Angle $G = \alpha$ since it is opposite $QE = a$. Hence angle $EQG = \beta = \gamma$. Now there are three angles γ at Q and since RQG is a straight angle, $\gamma = \pi/3$. Hence the isosceles tile is actually equilateral. But that contradicts the equation $b = 2a$. That disposes of Case 2c.

Now suppose, for proof by contradiction, that SR is (contained in) a side of the final triangle. Then since SP is not an edge, we must place a triangle, say T_5, north of SP, and since PE is not an edge, we must place T_6 east of PE, and since QE is not an edge, we must place T_7 south of QE. Then we count vertices: We have the north vertex N of T_5 north of SP, and the south vertex of T_7, south of QE, and R. In addition, S will be a vertex unless N lies on RS extended; and N must also be the third vertex triangle T_6, which has PE for one side. Consider the angles at P. Angle $QPE = \alpha$ because angle $PQE = \gamma$ and angle E is opposite $PQ = b = 2a$. Angle $SPQ = \gamma$. Angle SPN must be γ, since if it is α or β, the remaining angle NPE will be π or more, but it is a single angle of triangle T_6. Since there are four angles at P and one of them is α, we have $\gamma > \pi/2$. Since in triangle NSP, γ is used at P, angle NSP is either α or β. Then the angles at S are β, α, and angle NSP; the total is at most $\alpha + 2\beta$ which is less than π, since $\beta < \pi/2 < \gamma$. Hence N does not lie on RS extended, after all. This contradiction shows that SR is not contained in a side of the final triangle.

We therefore must place T_5 west of SR. Let W be its western vertex. Unless T_5 is placed so that at least one of vertices S and R are eliminated, i.e. WSP is a straight angle or WRQ is a straight angle, then there will be six vertices, too many to allow the creation of a triangle by placing two more copies of T_1. Suppose, for proof by contradiction, that WSP is a straight angle. Angle $VSP = \alpha$ and angle $VSR = \beta$. Hence angle $WSR = \gamma$. Since $SR = a$ because it is opposite to angle $SVR = \alpha$, angle $SWR = \alpha$. Hence angle $WRS = \beta$ and side $WS = b$. We will show that no triangle can be placed north of WSP. If T_6 is placed north of WSP with north vertex N then there will be vertices Q, E, and N. Assume, for proof by contradiction, that T_6 has WP for a side. The length of side

WSP is $2b$, so we must have $c = 2b$. But $b = 2a$, so $c = 4a$. This is impossible, since then $4a = c < a + b = 3a$. This contradiction shows that T_6 cannot have WP for a side. Hence two triangles, T_6 and T_7, must be placed north of WSP. If any side of length a is placed along WSP then we need at least three triangles (which is too many). Suppose triangles T_6 and T_7 are placed north of WSP with their b sides along WS and WP. Then there will be vertices Q and E, and in order to create a final triangle, there must be straight angles at P and at W, and triangles T_6 and T_7 must share a side. By Lemma 1, triangles T_6 and T_7 have right angles at the shared vertex on WSP. Hence $\gamma = \pi/2$. Then T_6 and T_7 have acute angles at P and W. Hence there is a vertex at W, as well as at Q, E, and the north vertex of T_6, so no triangle is created. This contradiction shows that WSP is not a straight angle after the placement of T_5.

It follows that WRQ is a straight angle after the placement of T_5. Since angle $RQV = \beta$ and angle $QRV = \alpha$, we have angle $QVR = \gamma$ and hence $RQ = c$. We have angle $VRS = \gamma$ and since angle $WRQ = \pi$ we have angle $WRS = \beta$. If angle $W = \alpha$ then T_5 has $SR = a$ for a side and WSP is a striaght angle, which have already disproved. Hence angle $W = \gamma$ and triangle T_5 has side c along RS extended, creating a concave exterior vertex at S, which must be filled by T_6. Let the north vertex of T_5 be N. Then $NS = a$, so it is possible to place triangle $T_6 = NSP$. If this is done, we have a six-triangle convex configuration with five vertices $WQEPN$. (There is a vertex at P since the angles at P are $2\alpha + \gamma < \pi$; there is a vertex at N since the angles there are $\alpha + \beta < \pi$.) Such a configuration cannot be completed to a triangle by placing T_7. Hence triangle T_6 is not NSP. But any other way of placing T_6 with a vertex at S will create another convex exterior vertex north of or on WSP and east or or on SR extended, so T_7 will have to be placed north of WSP and east of SR extended. That will leave vertices at Q, E, and W, as well as at least one vertex north of SP, so a triangle will not be created. This contradiction finally disposes of Case 2.

We now take up Case 3. Since the angles at P are γ and β, the possibility that SPE is straight does not arise; M is a convex pentagon. Adding three triangles will still leave two of the five sides on the boundary; hence at least two sides of M are (contained in) sides of the final triangle. We will consider each of the ten pairs of two sides and show that those two sides cannot be sides of the final triangle.

Case 3a: SR and QE are sides of the final triangle. Then let X be their intersection point. The transversal RQ makes alternate interior angles $RQX = \gamma$ and $QRS = \gamma + \alpha$ with QE and SR, so X lies to the south of RF. The triangle RFX can be covered exactly by four copies of T_1 (since it is similar to the tile but has side $c = 2a$ opposite angle α), but it must be contained in the final triangle, which is contradictory, since only three more triangles are available. That disposes of Case 3a.

Case 3b: SR and PE are sides of the final triangle. Then let X be their intersection point; triangle T_5 will be required to cover SXP. Then XPE must be a side of the final triangle, or else we will need to place T_6 and T_7 north of XPE, leaving vertices R, Q, and at least two north of or on XPE, contradiction. Therefore XPE is a side. Since EQ is not a side, we must place triangle T_6 with a side on line QE. Suppose, for proof by contradiction, that T_6 does not have E for a vertex. Then a convex exterior vertex is created on line QE, which must be filled by T_7. If T_6 extends north of E along QE then T_7 must be placed with a side on XE; but that would create another exterior vertex on XE, because XE has length greater than c, and so no triangle would be created. If, on the other hand, T_6 has its north vertex on QE south of E, then T_7 must be placed east of QE, leaving parallel sides RQ and XE, so no triangle is created. Hence T_6 does have E for a vertex. Now suppose, for proof by contradiction, that T_6 does not have Q for a vertex. If the south vertex of T_6 is north of Q, then T_7 must be placed east of QE, leaving parallel sides RQ and XE, so no triangle is created. If the southwest vertex U of T_6 lies on QE south of Q, then since $QE = b$ and T_6 has E for a vertex, T_6 must have side c along QE, and T_7 will have to be placed south of RQ and west of QE, in order to fill the concave exterior angle at Q. That will leave vertices at X, R, U, and the southeast vertex of T_6, so no triangle is created. This contradiction proves that T_6 has Q for a vertex, as well as E. In other words, T_6 has QE for a side.

Let Y be the third vertex of T_6. Since $QE = b$, angle $EYQ = \beta$. If angle $YQE = \gamma$, then EY is parallel to XR, and we have two pairs of parallel sides in the six-triangle configuration, a problem that cannot be fixed by placing T_7. Hence angle $YQE = \alpha$. Assume, for proof by contradiction, that γ is not a right angle. Then we have five vertices X, E, Y, Q, and R. If $\gamma < \pi/2$ then this is a convex pentagon, and cannot be completed to a triangle by placing T_7. If $\gamma > \pi/2$, then there is a concave exterior angle at E, which possibly could be filled by T_7 if $\gamma = 2\pi/3$, but then another concave exterior angle would be created on XPE somewhere, since the length of XPE is more than c. This contradiction proves that γ is a right angle. Therefore PEY is a straight angle. Now, however, we have parallel sides $XPEY$ and RQ; since $XPEY$ has length more than c, we must place T_7 south of RQ with its c side along RQ. Suppose, for proof by contradiction, that T_7 has angle α at R. Then there is a vertex at R, since the sum of the angles at R is $2\alpha + \gamma < \pi$, since we know $\gamma = \pi/2$ and hence, in view of $c = 2a$, $\alpha = \pi/6$. We then have vertices at X, R, Y, and the third vertex of T_7, so no triangle is created. This contradiction proves that T_7 does not have angle α at R. Hence it has angle β at R, making a straight angle there. Then T_7 has angle α at Q. The sum of the angles at Q is then $\beta + 3\alpha = \pi/2 + 2\alpha < \pi$. Hence there is a vertex at Q. Since there are also vertices at X, Y, and the third vertex of T_7, making at least four vertices, no triangle is formed. That disposes of Case 3b.

Case 3c: SR and RQ are both sides of the final triangle. Since the intersection of lines PS and QR lies west of SR, PS is not a side of the final triangle. That requires the placement of a triangle T_5 north of PS. Since PE is parallel to RQ, we must place a triangle T_6 north of PE. This triangle T_6 cannot reach all the way to SR extended, requiring the placement of T_7 north of T_5. This cannot leave a triangle, as we have vertices at Q, R, and a north vertex X on RS extended, and in addition a fourth vertex either at E, or if T_6 created a straight angle at E, then at another vertex of T_6 on QE extended. That vertex lies east of QP extended and hence cannot coincide with X. This disposes of Case 3c.

Case 3d: SR and SP are both sides of the final triangle. Then by Case 3b we would have to place T_5 north of PE. It would have to have angle α at P in order not to extend north of PS. Then E would become a non-strict vertex, with T_5 extending past E. Since by Case 3c, we must place one triangle T_6 south of RQ, we must fill the concave exterior angle at E with a single triangle T_7. Hence by Lemma 1, $\gamma = \pi/2$. T_7 and T_5 must share a common east vertex Y, or else further concave exterior angles are formed. At Y the sum of angles is $2\beta = 2\pi/3 < \pi$, so there are vertices at S, Y, and the northeast vertex Z of T_5. Since Q does not lie inside triangle SYZ, there is a fourth vertex and no triangle is formed. This disposes of Case 3d.

Case 3e: RQ and PE are both sides of the final triangle. This is impossible since they are parallel, because transversal PQ makes equal alternate interior angles β with RQ and PE. That disposes of Case 3e.

Case 3f: RQ and SP are both sides of the final triangle. The intersection point of lines RQ and SP, say W, lies to the west of SP and triangle SPW is congruent to the tile. Since by Cases 3a to 3d, SR is not a side of the final triangle, we must place T_5 as triangle SPW. Since PE is parallel to RQ, we must place T_6 north of PE and south of SP extended. This is only possible if T_6 has P for a vertex and has angle α there, so the north side of T_6 extends segment SP, i.e. there is a straight angle at P. That creates a concave exterior angle at E, since $PE = a < b$ (we cannot have $a = b$ since $c = 2a$). Let N be the vertex of T_6 on SP extended and let X be the south vertex of T_6. We must therefore place T_7 with a vertex at E and a side along QE and a side along PX. Unless T_7 is triangle QEX, more exterior concave angles (and hence no triangle) will be formed; hence T_7 is triangle QEX. Then considering the triangles meeting at E, by Lemma 1, $\gamma = \pi/2$. Triangle QEX has angle α at Q and β at X. The sum of the angles at Q is $\beta + 2\alpha < \pi$, so there is a vertex at Q. The sum of the angles at X is $2\beta < \pi$, so there is a vertex at X. There are also vertices at N and W, so no triangle is formed. That disposes of Case 3f.

Case 3g: RQ and EQ are both sides of the final triangle. By Cases 3a to 3d, SR is not a side of the final triangle, so we must place T_5 west of SR but touching segment SR. Assume, for proof by contradiction, that R is not a vertex of T_5. Then the southern vertex of T_5 on SR lies north of R and south of S. But $SR = a$, so the portion of SR south of T_5 has length less than a and cannot be covered by a triangle lying north of RQ extended; that will leave a concave exterior angle that cannot be filled. This contradiction proves that R is a vertex of T_5. There are two possible orientations of T_5: either it has its a side along SR, so that W lies on SP extended, or it has its c side along SR, so that S is the midpoint of the east side of T_5. Assume, for proof by contradiction, that S is the midpoint of the east side of T_5. Let N be the north vertex of T_5, and W its west vertex. Then WN is parallel to QE, since the corresponding interior angles made by transversal WQ are angle $NWQ = \gamma$ and angle $EQR = \alpha + \beta$, which are supplementary. Since EQ is (in Case 3g) a side of the final triangle, NW cannot be, so triangle T_6 must be placed northwest of NW. Then T_7 has to have a vertex at S to fill the concave exterior angle there. But by Case 3e, PE is not a side, so some triangle has to be placed north of PE; but we have no more triangles, so a contradiction has been reached. That disposes of Case 3g.

Case 3h: SP and QE are both sides of the final triangle. We note that the intersection point X of those two sides lies to the northeast of PE, because of the alternate interior angles made by the transversal PQ. Hence triangle T_6 will have to be placed with its a side along PE. But then its α angle is not at P and it cannot lie south of SP extended, contradiction. That disposes of Case 3h.

Case 3i: SP and PE are both sides of the final triangle. By Case 3e, RQ is not a side of the final triangle, and by Case 3h, QE is not a side. By Cases 3a to 3d, SR is not a side. Therefore we will have to place T_5 west of SR, T_6 south of RQ, and T_7 southeast of QE. T_7 will have to be placed with a vertex at E in order to avoid creating a concave exterior vertex on QE. Let W be the west vertex of T_5; this must be the west vertex of the final triangle. It must therefore lie on SP extended. Hence T_5 has WS for its north side. If T_5 has R for a vertex, then there is a concave exterior angle at the western vertex of T_6 on RQ or RQ extended. Hence T_5 has its southern vertex Y south of R on SR extended. T_5 has angle γ at S, since the sum of angles there must be π, and angle β at W, since the side opposite W is greater than $SR = a$. Hence T_5 has angle α at Y. But WR is parallel to PE, so the intersection point of lines WY and PE will lie north of SP. Hence SP, PE, and WY cannot be sides of a triangle including any points (such as V) south of SP. But in Case 3i, by hypothesis SP and PE are sides of the final triangle, and WY must be a side since it is on the boundary of the seven-triangle configuration. This contradiction disposes of Case 3i.

Case 3j: PE and QE are both sides of the final triangle, then as we have already shown, none of SP, SR, and RQ can be sides. Hence we will have to place the remaining three triangles on those sides, say T_5 on SR, T_6 on RQ, and T_7 on SP. T_6 must be placed with its c side on RQ (or else a concave exterior angle will be created), so its angle at Q will not be γ, and the seven-triangle configuration has a vertex at Q, since the angle sum there is less than π. Assume, for proof by contradiction, that T_5 and T_7 do not have the same third vertex. Then we have four vertices–those two plus E and Q. That contradiction shows that T_5 and T_7 must share their third vertex, say N. Let X be the southern vertex of T_6. For a triangle to be formed, the sides must be NPE, XQE, and NRX. Now $RS = a$ so in T_5, NS must be b or c. In T_7, $SP = b$ so NS must be a or c. Therefore $NS = c$, and angles NRS and NPS are both γ. The angle sum at P is then $2\gamma + \beta = \pi$. But since $\alpha + \gamma + \beta = \pi$, this yields $\gamma = \alpha$, so the tile is equilateral, contradicting $2a = c$. That contradiction disposes of Case 3j.

We have now shown that no pair of sides of M can be sides of the final triangle. This completes Case 3.

Now we take up Case 4. As in Case 3, M is convex.

We first prove that the three sides of the final triangle cannot be SP, PE, and RQ. Suppose, for proof by contradiction, that those are the three sides. The intersection point W of lines SP and RQ lies to the west of SR, because the transversal SR makes corresponding interior angles $RSP = \beta + \alpha$ and $SRQ = \gamma + \alpha$, and in Case 4 we have $c > b$, which implies $\gamma > \beta$. Triangle WSP has angle β at R (since the angle sum at R is π) and angle γ at S (since the angle sum at S is π); hence it has angle α at W, opposite side $SR = a$. Hence triangle WSP is congruent to the tile and we can call it T_5. Let X be the intersection point of lines PE and RQ, which exists because we have assumed the third side is RQ. Then triangle QEX is 2-tiled by triangles T_6 and T_7. Hence the tile is a right triangle, $\gamma = \pi/2$. Since there is a straight angle at Q, we have $\beta = \pi/3$ and hence $\alpha = \pi/6$. Consider the angle sum of triangle WPX: the angle at W is α, that at P is $\gamma + \alpha$, so that at X is $\pi - \gamma - 2\alpha = \pi/6 = \alpha$. Hence triangle QEX is congruent to the tile; it cannot be 2-tiled. This contradiction proves that the three sides of the final triangle cannot be SP, PE, and RQ.

We shall now argue that none of the ten pairs of sides of M can be (contained in) sides of the final triangle.

Case 4a: SP and PE are both sides of the final triangle. Then QE does not lie on the third side, because of the alternate interior angles made by transversal PQ to SP and PE. Hence triangle T_5 is required south of QE. T_5 must be placed along QE with a vertex at E, since PE is a side of the final triangle. Let X be the east vertex of T_5. SR cannot lie on a side of the final triangle, since the intersection point of SR and PE lies north of SP, because the transversal SP of RS and PE makes corresponding interior angles $RSP = \beta + \alpha$ and $SPE = \gamma + \alpha$, whose sum is $\pi + \alpha > \pi$. Hence triangle T_6 must be placed west of SR. Let W be the west vertex of T_6. Since RQ cannot be a side of the final triangle when SP and PE are sides (as shown above), T_7 must be placed south of RQ. Now, what can be the vertices of the final triangle? P is one of them; the others must be X (the east vertex of T_5), and W (the west vertex of T_6). Then S is not a vertex; hence triangle T_6 has angle γ at S. Since RQ is not a side, W lies to the west of the intersection point Y of lines SR and RQ. But $YS = b$ since angle $YRS = \beta$. Hence $WS = c$. Then the γ angle of T_6 must be opposite WS; but we already proved T_6 has angle γ at S. Since in Case 4, we have $b \neq c$, we have $\beta < \gamma$ so T_6 has only one angle equal to γ. This contradiction disposes of Case 4a.

Case 4b: SP and QE are sides of the final triangle. The intersection point X of these two sides lies to the east, because of the alternate interior angles made by the transversal PQ. Suppose, for proof by contradiction, that γ is a right angle. Since $2a = c$ we have $\alpha = \pi/6$ and $\beta = \pi/3$. Considering the angle sum of triangle PQX, we have $\alpha + \beta$ at P and β at Q, so we have $\pi - \alpha - 2\beta = \alpha$ at X. Hence triangle PXE is similar to the tile, but with α opposite its b side PE. Hence the similarity factor is $b/a = \sin(\pi/3)/\sin(\pi/6) = \sqrt{3}$. Hence the area of triangle PXE is three times that of the tile, and after it is tiled there will be no more triangles, but there will still be vertices at X, S, R, and Q. This contradiction shows that γ is not a right angle.

Let W be the intersection point of lines RQ and SP. Suppose, for proof by contradiction, that RQ is the third side. Then triangle WSR is congruent to the tile, having angle β at R, angle γ at S, and angle α at R. Give triangle WSR the name T_5. Then triangle XPE is 2-tiled by T_6 and T_7; hence the tile is a right triangle, which we have shown is not the case. This contradiction shows that RQ is not the third side. Since triangle

WSR is congruent to the tile, but RQ is not the third side, at least two triangles will be required west of line SR, and at least one south of RQ. That will use seven triangles, and leave none to tile triangle PEX. That disposes of case 4b.

Case 4c: SP and RQ are sides of the final triangle. These lines intersect in the west vertex W. Triangle WSR is congruent to the tile, and we call it T_5. Then, by cases 4a and 4b, triangles T_6, and T_7 will have to have sides PE and QE, respectively. (If these triangles do not have vertices matching already existing vertices, no triangle will be formed.) Let X be the third vertex of T_7, and N the third vertex of T_6. Then the final triangle is WXN. A straight angle is formed at P, and since angle $SPV = \gamma$ and angle $QPE = \alpha$, we must have angle $EPN = \beta$. But $PE = b$ so the angle at N must also be β. But in Case 4, we have $b \neq c$ so $\beta \neq \gamma$. Hence $\beta = \alpha$, since T_6 has two angles equal to β. Then angle $PEN = \gamma$. Now suppose, for proof by contradiction, that angle $QEX = \alpha = \beta$. Then considering the angle sum at E we have $2\gamma + \alpha = \pi$. But

$$\begin{aligned} 2\gamma + \alpha &= 2(\pi - \alpha - \beta) + \alpha \\ &= 2\pi - 3\alpha \end{aligned}$$

which implies $\alpha = \pi/3$. But if α and β are both $\pi/3$ then so is γ, contradiction, since $b \neq c$. This contradiction proves that angle QEX is not β. Therefore it is γ, and the angle sum at E tells us $3\gamma = \pi$, or $\gamma = \pi/3$. But again that implies $\alpha = \beta = \gamma$, contradiction. That disposes of Case 4c.

Case 4d: SP and SR are sides of the final triangle. Then by Case 4a, triangle T_5 will be required east of PE; by Case 4b, triangle T_6 will be required south of QE; and by Case 4c, triangle T_7 will be required south of RQ. The three vertices of the final triangle must be S, a point X east of P on SP extended, which must be the east vertex of T_5, and a point Y south of R on SR extended, which must be the south vertex of T_7. Since triangle SXY must include T_6, located south of QE, triangles T_7, T_6, and T_5 must all share a vertex on line YX, with an angle sum of π there. But then the side of T_7 along RQ must be longer than $RQ = c$, contradiction. That disposes of Case 4d.

Case 4e: SR and PE are sides of the final triangle. Let N be the intersection of PE and SR, which lies to the north of SP. Then triangle NSP is similar to triangle T_1, since it has angle β at P and angle γ at S (and hence α at N), but the side opposite angle α is b, not a. So triangle NSP could be tiled by an integral number K of copies of T_1 only if $K = (b/a)^2$ is an integer. If $K = 3$ then a triangle is not formed, since we have vertices at N, E, Q, and R. If $K = 2$ then NSP is 2-tiled, so $\gamma = \pi/2$. Since $c = 2a$ in Case 4, we have $\alpha = \pi/6$ and $\beta = \pi/3$. Then $b/a = \sin\beta/\sin\alpha = \sqrt{3}$, so $K = (b/a)^2 = 3$, not 2. Hence $K \neq 2$. But these are all the possibilities for K. That disposes of Case 4e.

Case 4f: PE and QE are sides of the final triangle. Then we require T_5 south of QR by Case 4c, T_6 north of SP by Case 4a, and T_7 west of SR by Case 4e. In order that a triangle be formed, the new triangles must share existing vertices along the sides mentioned, and we must have straight angles at Q and P. The vertices of the final triangle are E, the north vertex N of T_6, on PE extended, and the south vertex X of T_5, on EQ extended. Then the three new triangles must share a vertex W west of SR and the angle sum there must be π. But then the north side of T_5 will be WQ, which is longer than c, since $RQ = c$. That contradiction disposes of Case 4f.

Case 4g: PE and RQ are sides of the final triangle. Then we must place T_5 south of QE with its a side along QE. In case $\gamma = \pi/2$ this can be done, with the third vertex X at the intersection point of PE and RQ. Otherwise more triangles will be required south of QE. Since neither SP nor SR is a side when PE is, by Case 4a and Case 4d, triangles T_6 and T_7 will be required north of SP and west of SR, so there can be no second triangle south of QE. Hence $\gamma = \pi/2$. Since $c = 2a$, we then have $\alpha = \pi/6$ and $\beta = \pi/3$. The three vertices of the final triangle are X, and the north vertex N of T_6, which lies on PE extended, and the west vertex W of T_7, which lies on QR extended and is also a vertex of T_6. If T_6 does not have vertices W and P or T_7 does not have vertices S and R, no final triangle is formed. We have $SP = b$; angle $RSV = \beta$; angle $WRS = \beta$ because the angle sum at R is π; hence $WS = b$. Hence WP, the south side of T_6, is $2b$. Hence $c = 2b$. But $c = 2a$. Hence $a = b$ and $\alpha = \beta$. This contradicts $\alpha = \pi/6$ and $\beta = \pi/3$. This contradiction disposes of Case 4g.

Case 4h: SR and QE are sides of the final triangle. Because PQ is parallel to SR, the intersection point X of QE and SR lies to the south of R. At least one triangle T_5 will have to be placed inside triangle RQX. Since SR is a side, PE and SP are not sides, by Case 4e and Case 4d, so triangles T_6 and T_7 are required north of SP and northeast of PE, areas which cannot intersect triangle RQX. Hence triangle T_5 must be exactly triangle RQX. Angle $RXQ = \beta$ because angle $PQE = \beta$ and PQ is parallel to SRX. But angle RXQ is opposite side $RQ = c$, so angle $RQX = \gamma$. Hence $\beta = \gamma$. Hence $b = c$. But in Case 4 we have $b \neq c$. This contradiction disposes of Case 4h.

Case 4i: Suppose QE and RQ are sides of the final triangle. Since QE is a side, none of PE, SP, and SR are sides, by Case 4f, Case 4b, and Case 4h, so triangles T_5, T_6, and T_7 must be placed on those three sides, respectively. Let X be the east vertex of T_5, and let Y be the northwest vertex of T_6. Then these must be the vertices of the final triangle, so Y is also the west vertex of T_7, and T_7 is triangle SRY, T_6 is triangle PSY, and T_5 is triangle PEX. Since only two triangles meet at E, by Lemma 1 $\gamma = \pi/2$. Then because $c = 2a$ we have $\alpha = \pi/6$ and $\beta = \pi/3$. But at S, there are four angles, two of which are α and β, so the angles of T_6 and T_7 at S must add to $3\pi/2$, which is impossible. That disposes of Case 4i.

Case 4j: RQ and SR are sides of the final triangle. By Case 4c, Case 4g, and Case 4i, neither SP, PE, nor QE can be sides along with RQ. Therefore we must place triangles T_5, T_6, and T_7 along those sides, respectively. Let X be the southeast vertex of T_7, and N the north vertex of T_5. Then X, N, and R must be the vertices of the final triangle. Hence X lies on RQ extended and N lies on RS extended. The vertex Y of T_6 that does not lie on PE extended must lie on NX. Then Y must lie on SP extended, so that SPY is the south side of T_5; and Y must lie on QE extended, so that QEY is the north side of T_7. That is, lines SP and QE

meet at Y. Since only two triangles lie to the north of E and QEY is a straight line, by Lemma 1 we have $\gamma = \pi/2$. Then because $c = 2a$ we have $\alpha = \pi/6$ and $\beta = \pi/3$. Because the angle sum at P of the angles SPV, QPE, and EPY is π, we must have angle $EPY = \beta$. Because the sum of angles QEP and YEP must be π, we have angle $YEP = \gamma$. Hence angle $EYP = \alpha$. But side $PE = b$ since it is opposite angle PQE. Hence angle $EYP = \alpha = \beta$, which is a contradiction since $\alpha = \pi/6$ and $\beta = \pi/3$. That disposes of Case 4j.

That completes all ten sub-cases of Case 4, and with them, the proof that case (iii) of the previous lemma's conclusion is impossible.

We now take up showing that case (ii) of the previous lemma's conclusion is impossible. In that case we start with a five-triangle configuration M, and we must show it is not possible to make it into a triangle by adding two more copies of T. Since SP is parallel to RQ, those two sides cannot both be sides of the final triangle. Suppose, for proof by contradiction, that triangle T_6 is placed south of RQ. Let X be the south vertex of T_6. Then we have vertices at S, E, and X. Suppose, for proof by contradiction, that RQ is not a side of T_6. Then we will have to place T_7 with a vertex on RQ. Then we still have vertices at X, S, and E, so the final triangle must be SXE. Then T_6 and T_7 must 2-tile triangle RQX. Then angle QRX is less than a right angle, so angle WRX is not a straight angle, since angle WRQ is a right angle. This contradiction shows that RQ is a side of T_6. Since $RQ = b$, T_6 has a right angle at R and α at Q, or the other way around. Unless the right angle of T_6 is at R and SPE is straight, the resulting six-triangle configuration will have five or more convex vertices, and cannot become a triangle by placing one more copy of T. Therefore T_6 must be placed with its right angle at R. Let X be its south vertex. Then SX is longer than c since $SR = 2a = c$. Hence we cannot create a triangle by placing T_7 west of SR. Also SPE, even if it is a straight line, has length $b + a > c$, so we cannot create a triangle by placing T_7 north of SPE. The only remaining possibilities are south of T_6 or east of QE. In either case the triangle T_7 will share vertex Q. The angle already at Q is $\pi/2 + 2\alpha$, so to make a triangle we must add $\pi/2 - 2\alpha$. If SPE is not straight, we cannot possibly create a triangle; hence SPE is straight. Then $\alpha = \arctan \frac{1}{2}$, not $\pi/6$, so $\pi/2 - 2\alpha$ is neither α nor β, and it is not possible to eliminate the vertex at Q by placing T_7. This contradiction shows that T_6 cannot be placed south of RQ and completed to a triangle.

Therefore RQ will be one of the sides of the final triangle. Then SP cannot be a side of the final triangle, since it is parallel to RQ. Hence we must place T_6 north of SP. We must not create a concave vertex anywhere on SP by placing T_6, so the vertices of T_6 must include S and P, unless SPE is straight and SE is one side of T_6; but $SE = b + a > c$, so that is not possible. Hence SP is a side of T_6. If SPE is straight, i.e. $\alpha = \arctan \frac{1}{2}$, then we have created a concave vertex at P, so we must have $\alpha = \pi/6$. In that case the total angle at P after placing the α angle of T_6 there is $\pi/2 + \alpha + \beta = \pi$, so if N is the north vertex of T_6 we have NPE straight. We cannot make a triangle by placing T_7 west of NR, or north of NPE, since these segments are longer than c, so that leaves east of QE as the only possibility, since we know RQ must be a side of the final triangle. Since $QE = b$, we must place the b side of T_7 along QE, with the right angle at E, because the sides of the final triangle must be NSR, NPE (extended), and RQ (extended). The angle of T_7 at Q must then be α. The total angle at Q is then $\pi/2 + 2\alpha = 5\pi/6$, not enough to eliminate the vertex at Q. That completes the proof that case (ii) of the previous lemma's conclusion is impossible, and that completes the proof of the lemma.

Lemma 8 *A 7-tiling cannot contain more than one non-strict vertex.*

Proof. The previous lemmas have shown that each non-strict vertex is of type $2:1$ and occurs in a certain 3-triangle configuration (shown in Figure 8.) Suppose a 7-tiling contains two (or more) non-strict vertices. We have to consider the following cases: Case 1, the two 3-triangle configurations overlap (share a triangle), thus requiring five or fewer triangles; Case 2, the two 3-triangle configurations do not share a triangle, but share a side; Case 3, the two 3-triangle configurations do not share a triangle or a side.

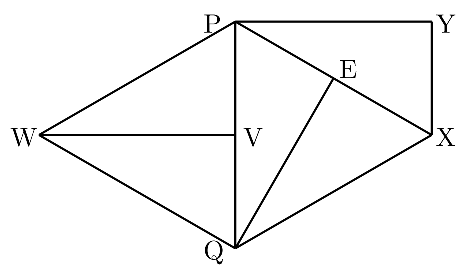


Figure 10: Two non-strict vertices in five triangles, first configuration

We take up Case 1. Each of the two 3-triangle configurations exposes only one side a on its boundary. If the (or a) shared triangle is one of the two triangles whose a sides are on the maximal segment in one configuration, then that triangle must be the T_3 of the other configuration. There are just two such configurations possible, when $\alpha = \pi/6$. When $\alpha = \arctan \frac{1}{2}$, we will show no such configuration is possible: let PQ be the maximal segment, with triangles T_1 and T_2 west of PQ and T_3 east of PQ with its b side equal to PQ and its right angle at P. Then the north side of T_3 is the only exposed side of length a, and it cannot occur as part of a pair in another configuration, since its b side is already used.

Therefore $\alpha = \pi/6$. We will show that the two configurations with $\alpha = \pi/6$ cannot be completed to a 7-tiling. Let PQ be a (north-south) maximal segment of length c, and T_1 and T_2 west of PQ with their a sides together matching PQ and shared west vertex W; then T_3 is east of PQ, with angle α at Q and east vertex E, and T_4 shares side QE and has angle α at Q and east vertex X, and T_5 shares side PX, and can be placed in either of two orientations. Let Y be the north vertex of T_5. Suppose, for proof by contradiction, that the β angle of T_5 is placed at X, and the α angle at P. (See Figure 11) Then $WPXQ$ is a parallelogram: since angle $XPQ =$ angle PQW, PX is parallel to WQ, and since angle $XQP =$ angle QPW, QX is parallel to WP. Since only two more triangles can be placed, the three sides of the final triangle are among the five sides of the pentagon $WPYXQ$. The remaining two triangles must therefore each share a side with one of the existing triangles. Since WP is parallel to XQ, we must place a triangle on one of those sides. It is not possible to place T_6 on any existing side in such a way as to create a concave exterior angle of $\pi/3$ or less. Hence, T_6 must be placed so as to create a quadrilateral by creating straight angles where two vertices were before. There is only one position in which that is possible: triangle T_6 must be placed along PY with its right angle at Y. (Technically, we should compute the angle sums for all the other possibly positions of T_6, rather than rely on inspection of Figure 11 for a "proof".) Then straight angles are created at Y and P. Then since WP and QX are parallel, we must place T_7 along QX. To make a straight angle at X we need to place the β angle of T_7 at X; but to make a straight angle at Q, we need to place the β angle there. Indeed this six-triangle configuration can be completed to an 8-tiling, but not to a 7-tiling. This contradiction shows that T_5 cannot be successfully placed with its β angle at X.

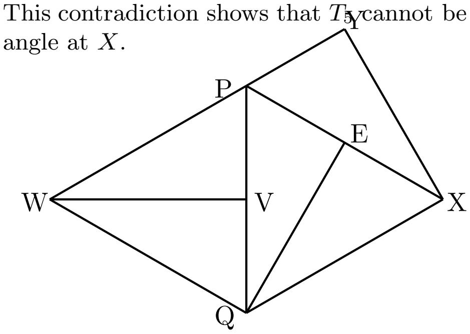


Figure 11: Two non-strict vertices in five triangles, second configuration

Therefore the β angle of T_5 is placed at P. (See Figure 12). Then the total angle there is $3\beta = \pi$, and WPY is straight. The quadrilateral $WYXQ$ is convex. If we place a triangle to share any of its sides, and we do not match the side exactly, then another triangle will be required on that side, and a large triangle cannot be created. Hence when we place a triangle, it must match one of the sides of the quadrilateral exactly. That means that two of the sides of the quadrilateral must be sides of the final triangle. If any triangle is placed north of WPY then we must place two triangles there, since the length of WPY is $c + a$. That will leave at least four vertices; hence WPY is one of the sides of the final triangle. Since QX is parallel to WPY, it cannot be a side of the final triangle. Hence T_6 must be placed south of QX, sharing side QX. There are two possible orientations. If the β angle of T_6 is at X, then the resulting figure has vertices at W, Y, X, Q, and the south vertex Z of T_6. It is convex, so it is not possible to complete it to a triangle by placing T_7. Therefore, the α angle of T_6 is at X. That creates a straight angle is created at Q, so the resulting figure is a quadrilateral. Now we ask if it is possible to place T_7 along any side of this quadrilateral so as to create a triangle. Sides WPY and WQZ are longer than c, so it is not possible to place T_7 there. If T_7 is placed along YX, then the β angle would go opposite YX, the right angle would have to go at Y to avoid creating a fourth vertex there, leaving the α angle to go at X, where the angle sum would then be $3\alpha + \beta = 5\alpha < \pi$, so a triangle is not created. Hence T_7 must be placed south of T_6. That means the sides of the final triangle are WY, YX extended, and WQ extended. Let Z be the intersection point of lines WQ and YX. This point lies to the southeast, because of the angles made by the transversal WY. Triangle ZXQ is similar to the tile, because it has angle α at Z (because the angle sum of triangle WYZ must be π), and angle β at Q (because WQZ is a straight angle). But it has side $QX = c = 2a$ opposite angle α. Hence its area is four times that of the tile, and thus it cannot be tiled by T_6 and T_7. That contradiction completes the proof of Case 1.

Now we take up Case 2, in which the two configurations share a side but not a triangle. When we join two convex quadrilaterals along a side, we get a figure with at least six convex vertices, and possibly with one or two more vertices, which might be concave. Placing one triangle can reduce the number of convex vertices by at most 2 (that can happen if a concave vertex occurs in just the right position). But even if that happens, there will still be four vertices left after placing T_7. That completes the proof in Case 2.

Finally we consider Case 3, in which the two configurations do not even share a side. Since the quadrilaterals are convex, they share at most one point. At that shared point there may be a vertex with a concave exterior angle, so placing one triangle could possibly eliminate two of the six remaining vertices, but that would still leave four, too many for a triangle. That completes the proof of the lemma.

Theorem 5 (Main Theorem) *There is no 7-tiling.*

Proof. Suppose triangle ABC is 7-tiled by seven copies of triangle T. Then according to our previous theorems, it is not a strict tiling, and there is exactly one non-strict vertex V, and triangle T has a right angle, and its small angle α is either $\pi/6$ or $\arctan(1/2)$, and the non-strict vertex occurs in one of two specific configurations of three triangles (one for each α). (Those configurations are illustrated in Figure 8.) To finish the proof, we have to show that it is impossible, starting from either of those configurations, to add four more copies of T to create a triangle. We need only consider placements of new copies of T that share sides with existing copies, since no additional non-strict vertices can occur in a 7-tiling.

There are 7π angles total in the seven copies of T. Of these, π are used at the vertices A, B, and C, and π at the non-strict vertex V, leaving 5π to be used at boundary and interior vertices (other than V). An interior vertex uses 2π, and a boundary vertex uses π. The possibilities are thus: one boundary and two interior vertices, or three boundary and one interior vertex, or five boundary vertices and no interior vertex. In particular there are at most two interior vertices.

First we take up the case $\alpha = \pi/6$. The starting configuration is the first one shown in Figure 8. The non-strict vertex V is at the midpoint of north-south line PQ. Triangles T_1 and T_2 are west of PQ, with a shared west vertex W, a right angle at their shared vertex V, and angle α at W. Triangle T_3 is east of PQ, with angle $PQE = \alpha$, and angle $QPE = \beta$.

Consider adding T_4 north of PE with its third vertex N on QE extended. If we then add T_5 north of PN, two additional triangles T_6 and T_7

will be required to fill the angle 2π at P. Vertices Q and N remain, so if this were to create a triangle, the west vertex Y would have to lie on QW extended. That would require at least two more triangles to share vertex W, one of which might be T_6, but there is no second one available. Hence the indicated placement of T_5 fails. Similarly, if we add T_5 southeast of EN, with southeast vertex X, then the exterior angle at vertex E will be concave, so we will have to add T_6 sharing vertex E. That can fill vertex E to 2π only if both T_5 and T_6 have a right angle at E, which will make a six-triangle convex pentagon; such a configuration cannot be completed to a triangle. If instead T_5, T_6, and T_7 are all placed with a vertex at E, the result cannot be a triangle since there are vertices at W, N, Q, and at least one more southeast of QN. Hence it fails to place T_5 southeast of EN. Since T_5 cannot be placed east of EN or north of PN, with this placement of T_4, two sides of ABC must be WPN and NEQ. We must then add the remaining three triangles to the southwest of WQ. Since P and E will now be boundary vertices, we are allowed only one interior vertex in the process. That vertex S will be created when we add triangle T_5 immediately south of WQ, with southwest vertex S. There are two ways to place T_5; first consider placing its α angle at Q. Then $WS = a$ and we must place T_6 west of WS with its right angle at S (since otherwise WS must be the third side of the final triangle, and there is not enough area south of SQ and north of NQ extended to hold two more copies of the tile). Let X be the west vertex of T_6 (placed west of WS). We now have a non-strict 6-tiling of triangle XNQ (Figure 13). This cannot be made into a triangle by adding one more triangle south of XSQ. Hence the indicated placement of T_5 fails.

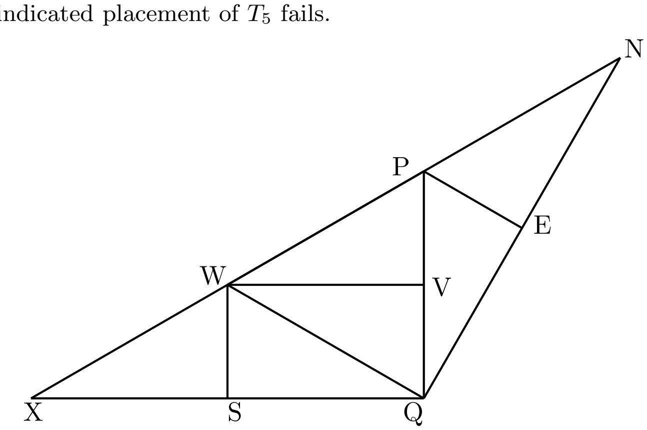


Figure 12: A six-tiling that cannot be completed to a 7-tiling

Therefore T_5 has to be placed with its β angle at Q, and its α angle at W. Let S be the third vertex of T_5. Then we cannot place T_6 with its α angle at Q to have its side extend NEQ, since that creates a non-strict vertex at S. Hence QS must be the third side of the final triangle. But that is not possible, either, since QS is parallel to WPN (since transversal PQ makes equal alternate interior angles SQP and QPN, both equal to 2β). this placement of T_5 also fails. Hence the indicated placement of T_4 (north of PE with its third vertex N on QE extended) fails.

Now consider adding T_4 north of PE with is third vertex N not on EQ extended, i.e. its right angle is at P instead of E. Then the exterior angle at vertex P is concave, with total interior angle $7\pi/6$. At least two more triangles T_5 and T_6 must share vertex P. If we use T_7 also at P, then we will still have vertices four vertices W, Q, E, and N, so we must use only T_5 and T_6 at P. We must therefore place T_5 along $NP = b$ with its right angle at P. Let Y be its third vertex. Then T_6 must be placed with its c side along WP and its β angle at P, so $YP = a$ is matched. The angle of T_6 at Y is $\pi/2$, so the total angle at Y is $5\pi/6$, not π, and our 6-triangle configuration has vertices at W, Y, N, E, and Q. (Figure 14.) Since this configuration is convex, the best we could hope to do by placing T_7 is to reduce the number of vertices by one to four. Hence this placement of T_4 also fails.

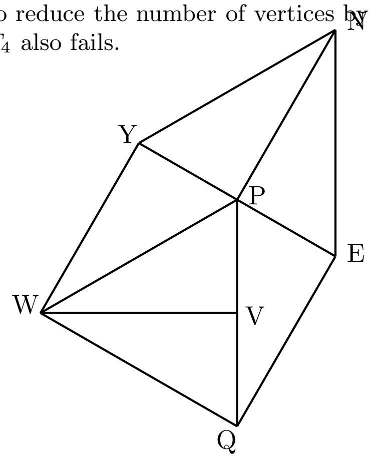


Figure 13: Another configuration that cannot be completed to a 7-tiling

Thus T_4 cannot be placed north of PE in either orientation. Hence PE lies on the boundary of triangle ABC. Since WQ is parallel to PE, it follows that WQ is *not* a side of triangle ABC. We must therefore place T_4 south of WQ; call its third vertex S. There are two orientations to consider: either the angle of T_4 at Q is β or it is α. First suppose T_4 has angle α at Q. Then SW is a north-south line. Consider placing T_5 along WP with α at P. Let X be the third vertex of T_5. Then XP is parallel to SQ, so T_6 must be placed either north of XP or south of SQ. In either case we are committed to making WS extended a side of ABC, since placing another triangle west of WS extended will create concave vertices. We must therefore definitely add T_6 north of XP to reach the intersection point Y of PE and WS extended. Now we have vertices Y, E, Q, and S, and we must remove the vertex at Q by placing T_7. The angle at Q is presently $2\alpha + \beta = 2\pi/3$. To remove it we would have to put the β angle of T_7 at Q, but the two sides SQ and QE are both b, so the β angle of T_7 cannot be placed at Q. Hence the placement of T_5 along WP with α at P fails. Now consider placing T_5 along WP with β at P. Then the third vertex X of T_5 lies on PE extended, and XW is perpendicular to PE. If we place T_6 along XW, we again reach the intersection point Y of PE and WS, and we have the same convex quadrilateral 6-tiled as with the previous placement of T_5, and again T_7

cannot be placed to make a triangle. Hence we cannot place T_6 along XW. But then XW must be a side of the final triangle ABC. Since PE is a side, the vertex at W will have to be eliminated, but this is not possible, since we would have to place angle α at W, but the side WS is a, which cannot be adjacent to angle α. Hence the placement of T_5 along WP with β at P fails. Now both possible placements of T_5 along WP have failed. Hence WP is contained in one of the sides of triangle ABC. We therefore must add T_5 west of WS, matching its a side to WS. Let X be the third (westernmost) vertex of T_5. If XSQ is not a side of ABC, we would have to add two more triangles south of XSQ, but that would not make a triangle. Hence XSQ is the third side of ABC. Let Y be the intersection point of EP and SQ. Then the remaining two triangles would have to tile triangle EQY. Triangle EQY is similar to T, since it has angle β at Q and a right angle at E, but the side opposite angle α is $EQ = b = \sqrt{3}/2$. So the area of triangle EQY is 3 times the area of T, not twice the area of T. Hence the placement of T_4 with angle α at Q fails.

Now consider the other possible placement of T_4, namely south of WQ with angle β at Q. Let S be the third vertex of T_4. If WS is a side of ABC, then we will need to use three more triangles north of WP to reach the intersection point Y of WS and EP, and that will leave four vertices Y, S, E, and Q. Hence WS is not a side of ABC, so we must place T_5 along $WS = b$. There are two possible orientations, with the angle of T_5 at W either α or a right angle. Consider first placing T_5 on WS with a right angle at W. Let X be the third vertex of T_5. Then X lies on WP extended and XS is parallel to WQ and PE. Then XS cannot be a side of ABC (since PE is a side), so we must place T_6 south of XS. Let Y be the third vertex of T_6. Then we have vertices Y, X, P, E, Q, and possibly S. Since this figure is convex, we cannot possibly reduce the number of vertices to three by placing T_7. So the placement of T_5 on WS with a right angle at W fails. Now consider the other possible placement of T_5, on WS with angle α at W. Let X be the third vertex of T_5. Now WX cannot be a side of ABC, since in that case three more triangles would be needed to reach the intersection point Y of PE and WX. Therefore we must add T_6 west of WX. Let Z be the westernmost vertex of T_6. Then we have vertices Z, X, Q, E, P at least, and the figure is convex, so it cannot be made into a triangle by placing T_7. Hence both orientations of T_5 on WS fail. Hence the second possible orientation of T_4 (south of WQ with angle β at Q) fails. That exhausts the possibilities, and completes the proof in case $\alpha = \pi/6$.

Now we consider the case $\alpha = \arctan \frac{1}{2}$, which is about 26.565 degrees. Then $\beta = \arctan 2$ is about 63.435 degrees. The starting configuration is shown in the second part of Figure 8. The non-strict vertex V is at the midpoint of north-south line PQ. Triangles T_1 and T_2 are west of PQ, with a shared west vertex W, a right angle at their shared vertex V, and angle α at W. Triangle T_3 is east of PQ, with angle $PQE = \alpha$, and angle QPE is a right angle.

We first consider placing T_4 north of PE with a right angle at P. Let N be its northern vertex. That creates a concave vertex at P with exterior angle $\pi - \beta$. That will require T_5 and T_6 to be used north of WP and west of NP,respectively. Even if we managed to solve the problem of the concave vertex at P, we would then have only one more triangle T_7 to place, and we cannot reduce the number of vertices by placing it on NE, QE, or WQ, but it must be placed on one of those sides since not all three can be sides of the final triangle ABC. Hence the placement of T_4 north of PE with a right angle at P fails. Next consider placing T_4 north of PE with a right angle at E, and let N be its northern vertex. This also creates a concave vertex at P, with exterior angle $3\pi/2 - 2\beta$. We cannot use all three remaining triangles at P, as that will leave vertices W, Q, E, and N. Since we can place only two new triangles with vertices at P, they must have their c sides along WP and PN, so they cannot have right angles at P. Therefore their maximum contribution to the angle sum at P is 2β, which is not enough to fill the angle at P, since $4\beta + \pi/2$ is about 343.74 degrees, not 360. Hence the placement of T_4 north of PE with a right angle at E fails. Hence T_4 cannot be placed north of PE at all. Hence PE is (contained in) a side of the final triangle ABC. Suppose, for proof by contradiction, that WQ is a side of the final triangle. Then we must place T_4 north of WP, and T_5 west of T_4. Let X be the west vertex of T_5. Then XQE is 5-tiled. We must place T_6 along QE, since XWQ and XPE are sides of the final triangle. There are two possible orientations of T_6, with angle α at E or angle β at E. First consider placing T_6 on QE with angle α at E. Let Y be the third vertex of T_6. Since $QE = c$, the angle of T_6 at Y is a right angle and QY is parallel to side PE of the final triangle. Hence T_7 must be placed south of $QY = a$, and its right angle must go at Y or a vertex will be created there. Hence the angle of T_7 at Q will be β, making the total angle at Q equal to $3\beta + \alpha = \pi/2 + 2\beta > \pi$, contradicting our assumption that WQ is a side of the final triangle. Hence the placement of T_6 on QE with angle α at E fails. Next consider the other possible placement of T_6, on QE with angle β at E. Let Y be the eastern vertex of T_6. We cannot place T_7 north of EY, since $3\beta > \pi$ and T_7 would then extend north of PE. Hence the third side of ABC must be EY, and T_7 must be placed south of QY with its right angle at Y. Then the angle of T_7 at Q is α and the total angle at Q is $3\alpha + \beta = \pi/2 + 2\alpha < \pi$, so a triangle has not been created. Hence the placement of T_6 on QE with angle β at E fails. Now both possible placements of T_6 have failed. This contradicts our assumption that WQ is a side of the final triangle. Hence WQ is not a side of the final triangle.

Therefore we must place T_4 south of WQ. First consider placing T_4 along WQ with angle α at Q, as shown in the first part of Figure 15. Let R be the third vertex of T_4. Suppose, for proof by contradiction, that RW is a side of the final triangle. Then we must place T_5 north of WP. Call its north vertex N. Since RQ is parallel to PE, RQ is not a side of the final triangle, and we must place T_6 south of RQ. Call its south vertex X. Since PE is (contained in) a side of the final triangle, the east vertex of the final triangle must lie on PE (extended); but since the exterior angle between EQ and PE extended is more than $\pi/2$, triangle T_7 cannot extend to the east of E on PE extended. Hence E is a vertex of the final triangle. It is not possible to create a final triangle by placing triangle T_7 along the east side of EQ, since then we will have four distinct vertices

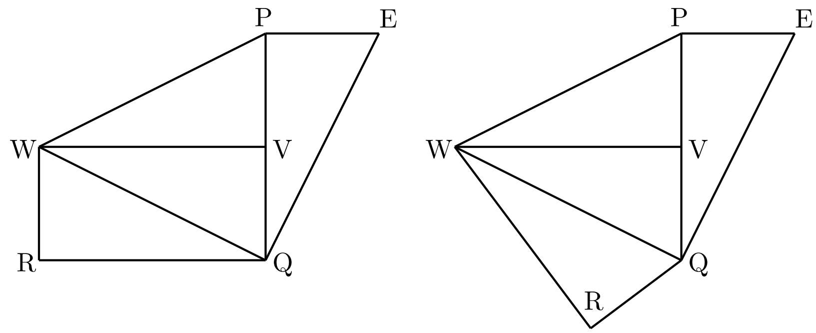


Figure 14: Two possible placements of T_4

N, E, X, and the east vertex of T_7. The latter two cannot coincide since X is south of RQ and west of PQ extended, while the east vertex of T_7 would be (in either possible orientation of T_7) east of PQ extended. Since T_7 cannot be placed east of EQ, the third side of the triangle must be EQ extended. Hence triangle T_6 does not have its right angle at Q (or it would extend east of EQ extended). Since $RQ = b$, triangle T_6 has angle β at X. Hence it has either angle α at Q. Then the angle sum at Q is $3\alpha + \beta = 2\alpha + \pi/2 < \pi$. Now we have a six-triangle configuration with four vertices N, E, Q, and X, and NR, NE, and QE are sides of the final triangle. But T_7 cannot be placed south of XQ so as to create a triangle, since to do so it would need an obtuse angle at X. This contradicts our assumption that RW is a side of the final triangle. Therefore RW is not a side of the final triangle, and we are back to where only T_4 has been placed (along WQ with angle α at Q), as shown in the first part of Figure 15.

Since RQ is parallel to PE, RQ is not a side of the final triangle, and we must place T_5 south of RQ. If its right angle is placed at R, then since RW is not a side of ABC, we will require both T_6 and T_7 west of SW extended, since we are not allowed to create another non-strict vertex. That will leave vertices at P, E, Q, and points west. Hence T_5 cannot be placed with its right angle at R. But then the right angle of T_5 is at Q. This creates a concave vertex at Q with exterior angle greater than $\pi/2$, so two more triangles are required at Q, leaving none to place west of RW, where we need one since RW is not a side of ABC. This contradiction shows that placing T_4 along WQ with angle α at Q fails.

Hence we must place T_4 south of WQ with angle β at Q, as shown in the second part of Figure 15. Let R be the third vertex of T_4. Suppose, for proof by contradiction, that WP is not a side of triangle ABC. Then we must add T_5 with vertex α at P, side c along WP, third vertex N on PE extended, with angle $WNP = \pi/2$. The total angle at W is now $3\alpha + \beta = \pi/2 + 2\alpha$. To eliminate the vertex at W (making a straight angle at W) we would need an angle of $\pi - (\pi/2 + 2\alpha) = \pi/2 - 2\alpha$, which is about $90 - 2 \cdot 26.5 = 37$ degrees, more than α, but less than both 2α and β, and hence impossible to supply. Possibly W might be eliminated as a vertex of ABC by becoming an internal vertex. Then the total angle at W would have to be made equal to 2π. After placing T_5 it is $\pi/2 + 2\alpha$, so we would need $2\pi - (\pi/2 + 2\alpha) = 3\pi/4 - 2\alpha$ more, which is about 217 degrees. Even if we used the right angles of T_6 and T_7 we could not make it. Hence W is definitely a vertex of triangle ABC (under the assumption that WP is not a side.) This is, however, impossible, since two of the vertices must be on line PE extended, and W cannot be the southernmost vertex, since for example point Q lies farther to the south than W. This contradiction shows that WP is, in fact, one of the sides of the final triangle ABC, along with PE.

Thus P is one of the vertices of ABC, and the other two lie on PE (extended) and PW (extended). W cannot be a vertex of ABC, since then W would be the southernmost vertex of ABC, but R lies farther south than W. Hence the vertex at W must be eliminated by placing more triangles with a vertex at W. The angle sum at W (from the three triangles already there) is 3α, about 79.5 degrees. To reach an angle sum of π at W, we could place three triangles with angle β at W, but that would use all seven triangles and still leave vertices at P, E, and Q, as well as somewhere southwest of W on PW extended, so no triangle would be formed. At least two must be used since $3\alpha + \pi/2 < \pi$. The possible angle sums resulting from placing two more angles at W are among

$$\begin{array}{rcl} 5\alpha & < & \pi \\ 4\alpha + \beta & = & 3\alpha + \pi/2 < \pi \\ 4\alpha + \pi/2 & > & \pi \\ 3\alpha + 2\beta & = & \pi + \alpha > \pi \\ 3\alpha + \beta + \pi/2 & = & 2\alpha + \pi > \pi \end{array}$$

None of these possibilities would succeed to eliminate vertex W. This final contradiction completes the proof of the theorem.

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