NO TRIANGLE CAN BE CUT INTO SEVEN CONGRUENT TRIANGLES

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Abstract. We prove the theorem in the title, and prove the theorem for 11 as well as 7. By previous work of others, the problem reduces to a number of cases. The cases not solved already are solved here.

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1. Introduction

An N-tiling of triangle ABC by triangle T is a way of writing ABC as a union of N triangles congruent to T, overlapping only at their boundaries. The triangle T is the "tile". We consider the problem of cutting a triangle into N congruent triangles. Shortly we shall give a number of examples of N-tilings, for various small values of N. These examples will be tilings that have, for the most part, been known a very long time. But it will be obvious that $N = 7$ is not in this list of examples, nor is $N = 11$.

These two values of N are the focus of this paper. We will prove here that there are no 7-tilings or 11-tilings. Originally it was the question of 7-tilings that attracted us to this subject. This question could easily have been understood by the Greek geometers working in Alexandria with Euclid three centuries BCE, and possibly could have been solved by them too. We were able to give a purely Euclidean proof, but it was very long and complicated. Once sufficient machinery is developed, non-existence of tilings for many N, including all primes congruent to 3 mod 4, is a consequence, but also that development is long and complicated. Therefore we were happy, in October 2018, to discover a relatively simple proof of the non-existence of any 7-tiling, which we present here. It was also possible to treat $N = 11$ with very little extra work–something we could not do with a purely Euclidean proof. One might say that here Descartes is victorious over Euclid, as algebra and computation is shorter and more efficient than geometry. Following Euclid we could do $N = 7$, but not 11.

We checked most of the algebra both by hand and by computer, using SageMath [14], and we provide the short snippets of code we used. In only one place is it too laborious to do by hand.[1]

These results fit into a larger research program, begun by Lazkovich [6]. He studied the possible shapes of tiles and triangles that can possibly be used in tilings, and obtained results that will be described below. It is our contribution to focus

[1]SageMath code, being written in Python, needs to contain tabs for indentation. When you cut and paste from a pdf file, you will get spaces, not tabs. Therefore you must paste into a file and supply tabs using the Unix utility `unexpand`. Also single quote marks are a different character in pdf. For very short snippets you may find it easier just to retype.

attention on N as well. One may say that Laczkovich studied the pair (ABC, T), and we want to refine his work to study the triple (ABC, T, N).

2. Some examples of tilings

Figures 1 through 4 show the simplest examples of N-tilings.

Figure 1. Two 3-tilings

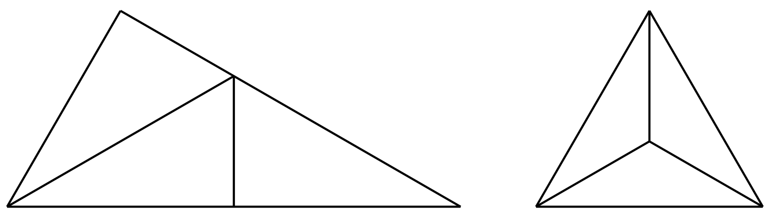
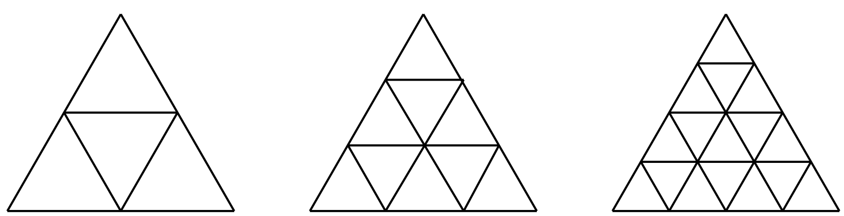
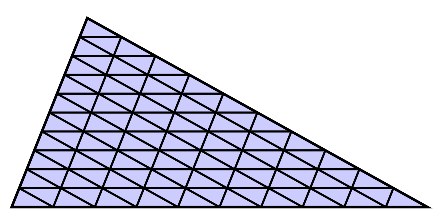


Figure 2. A 4-tiling, a 9-tiling, and a 16-tiling



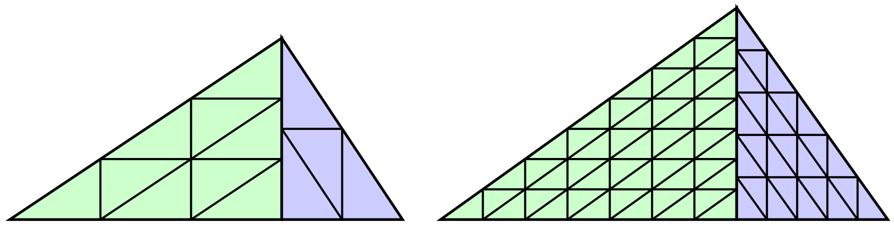
The method illustrated for $N = 4$,9, and 16 generalizes to any perfect square N. While the two exhibited 3-tilings clearly depend on the exact angles of the triangle, *any* triangle can be decomposed into n^2 congruent triangles by drawing $n-1$ equally spaced lines parallel to each of the three sides of the triangle, as illustrated in Fig. 3. Moreover, the large (tiled) triangle is similar to the small triangle (the "tile"). We call such a tiling a *quadratic tiling.*

Figure 3. A quadratic tiling of an arbitrary triangle



It follows that if we have a tiling of a triangle ABC into N congruent triangles, and m is any integer, we can tile ABC into Nm^2 triangles by subdividing the first tiling, replacing each of the N triangles by m^2 smaller ones. Hence the set of N for which an N-tiling of some triangle exists is closed under multiplication by squares.

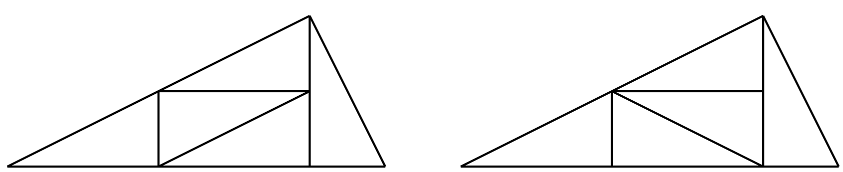
FIGURE 4. Biquadratic tilings with $N = 13 = 3^2 + 2^2$ and $N = 74 = 5^2 + 7^2$



Sometimes it is possible to combine two quadratic tilings (using the same tile) into a single tiling, as shown in Fig. 4. We will explain how these tilings are constructed. We start with a big right triangle resting on its hypotenuse, and divide it into two right triangles by an altitude. Then we quadratically tile each of those triangles. The trick is to choose the dimensions in such a way that the same tile can be used throughout. If that can be done then evidently N, the total number of tiles, will be the sum of two squares, $N = n^2 + m^2$, one square for each of the two quadratic tilings. On the other hand, if we start with an N of that form, and we choose the tile to be an n by m right triangle, then we can construct such a tiling. We call these tilings "biquadratic." More generally, a *biquadratic tiling* of triangle ABC is one in which ABC has a right angle at C, and can be divided by an altitude from C to AB into two triangles, each similar to ABC, which can be tiled respectively by n^2 and m^2 copies of a triangle similar to ABC. A larger biquadratic tiling, with $n = 5$ and $m = 7$ and hence $N = 74$, is shown in at the right of Fig. 4.

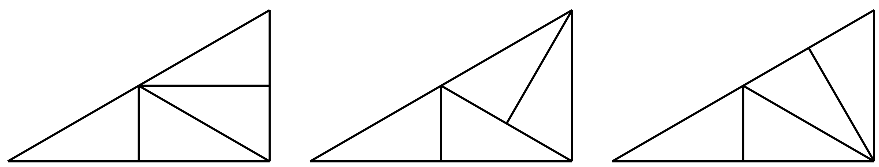
Since $5 = 2^2 + 1^2$, the simplest case of a biquadratic tiling is $N = 5$. The second 5-tiling in Fig. 5 shows that a biquadratic tiling can sometimes be more complicated than a combination of two quadratic tilings. Symmetry can permit rearranging some of the tiles. The symmetrical tile used in Fig. 6 also allows for variety.

FIGURE 5. Two 5-tilings



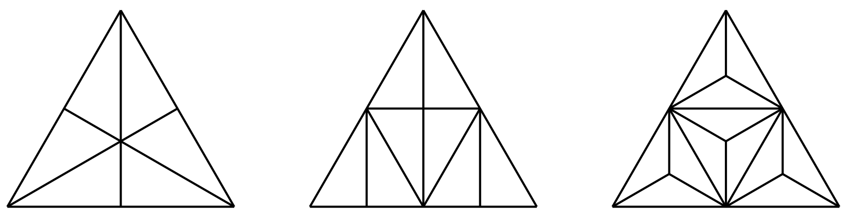
If the original triangle ABC is chosen to be isosceles, and is then quadratically tiled, then each of the n^2 triangles can be divided in half by an altitude; hence any isosceles triangle can be decomposed into $2n^2$ congruent triangles. If the original triangle is equilateral, then it can be first decomposed into n^2 equilateral triangles, and then these triangles can be decomposed into 3 or 6 triangles each, showing that any equilateral triangle can be decomposed into $3n^2$ or $6n^2$ congruent triangles. For example we can 12-tile an equilateral triangle in two different ways, starting with

FIGURE 6. Three 4-tilings



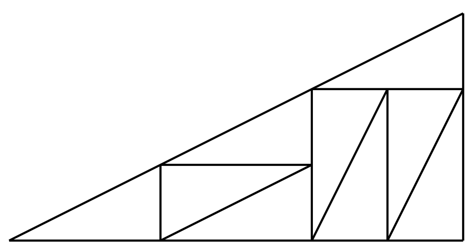
a 3-tiling and then subdividing each triangle into 4 triangles ("subdividing by 4"), or starting with a 4-tiling and then subdividing by 3.

FIGURE 7. A 6-tiling, an 8-tiling, and a 12-tiling



The examples above do not exhaust all possible tilings, even when N is a square. For example, Fig. 8 shows a 9-tiling that is not produced by those methods; again this seems attributable to symmetry permitting a rearrangement of tiles in a quadratic tiling.

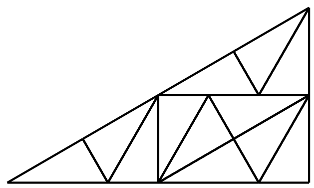
FIGURE 8. Another 9-tiling



There is another family of N-tilings, in which N is of the form $3m^2$, and both the tile and the tiled triangle are 30-60-90 triangles. We call these the "triple-square" tilings. The case $m = 1$ is given in Fig. 1; the case $m = 2$ makes $N = 12$. There are two ways to 12-tile a 30-60-90 triangle with 30-60-90 triangle. One is to first quadratically 4-tile it, and then subtile the four triangles with the 3-tiling of Figure 1. This produces the first 12-tiling in Fig. 9. Somewhat surprisingly, there is another way to tile the same triangle with the same 12 tiles, also shown in Fig. 9. The next member of this family is $m = 3$, which makes $N = 27$. Two 27-tilings are shown in Fig. 10. Similarly, there are two 48-tilings (not shown).

Until October 12, 2008, we did not know any more complicated tilings than those illustrated above (and there also none in [13]). Then we found the beautiful

FIGURE 9. Two 12-tilings



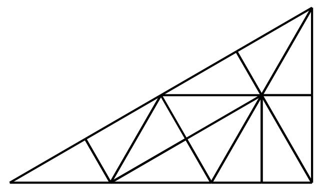
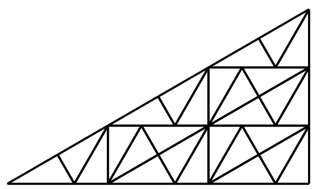
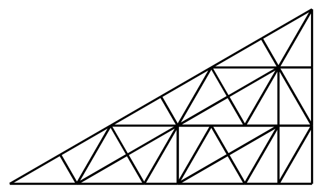


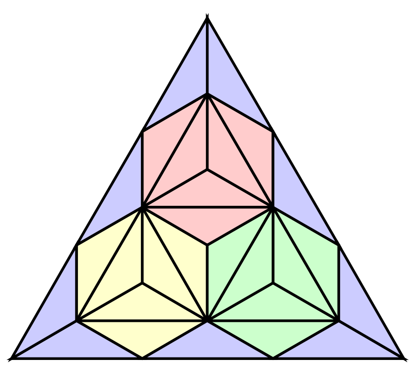
FIGURE 10. Two 27-tilings





27-tiling shown in Fig. 11. This tiling is one of a family of $3k^2$ tilings (the case $k = 3$). The next case is a 48-tiling, made from six hexagons (each containing 6 tiles) bordered by 4 tiles on each of 3 sides. In general one can arrange $1+2+\ldots+k$ hexagons in bowling-pin fashion, and add $k+1$ tiles on each of three sides, for a total number of tiles of $6(1+2+\ldots+k)+3(k+1) = 3k(k+1)+3(k+1) = 3(k+1)^2$. Fig. 12 shows more members of this family, which we call the "hexagonal tilings."[2]

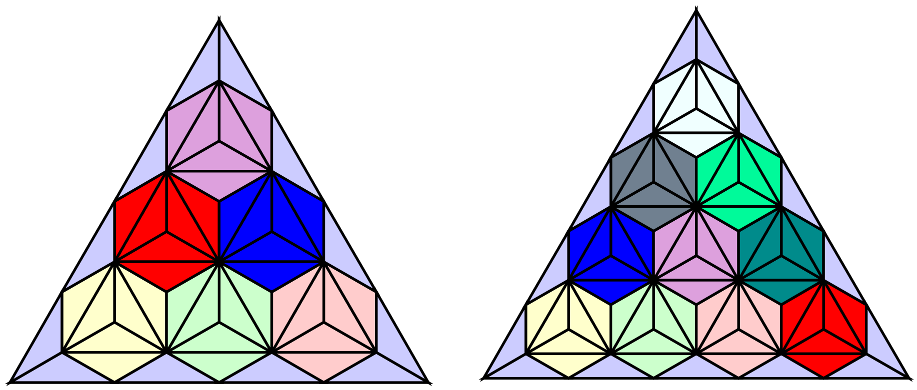
FIGURE 11. A 27-tiling due to Major MacMahon 1921, rediscovered 2011



Whenever there is an N-tiling of the right triangle ABM, there is a $2N$-tiling of the isosoceles triangle ABC. Using the biquadratic tilings (see Fig. 5 and Fig. 4)

[2] In January, 2012, I bought a puzzle at the exhibition at the AMS meeting, which contained the tiling in Fig. 11 as part of a tiling of a larger hexagon. The tiling is attributed to Major Percy Alexander MacMahon (1854-1929) [10].

FIGURE 12. $3m^2$ (hexagonal) tilings for $m = 4$ and $m = 5$



and triple-square tilings (see Fig 9 and Fig. 10), we can produce $2N$-tilings when N is a sum of squares or three times a sum of squares. We call these tilings "double biquadratic" and "hexquadratic". For example, one has two 10-tilings and two 26-tilings, obtained by reflecting Figs. 4 and 5 about either of the sides of the triangles shown in those figures; and one has 24-tilings and 54-tilings obtained from Figs. 8 and 9. Note that in the latter two cases, ABC is equilateral.

In the case when the sides of the tile T form a Pythogorean triple $n^2+m^2+k^2 = N/2$, then we can tile one half of ABC with a quadratic tiling and the other half with a biquadratic tiling. The smallest example is when the tile has sides 3, 4, and 5, and $N = 50$. See Fig. 16. One half is 25-tiled quadratically, and the other half is divided into two smaller right triangles which are 9-tiled and 16-tiled quadratically. This shows that the tiling of ABC does not have to be symmetric about the altitude.

3. Definitions and notation

We give a mathematically precise definition of "tiling" and fix some terminology and notation. Given a triangle T and a larger triangle ABC, a "tiling" of triangle ABC by triangle T is a set of triangles $T_1, \ldots, T_n$ congruent to T, whose interiors are disjoint, and the closure of whose union is triangle ABC.

Let a, b, and c be the sides of the tile T, and angles α, β, and γ be the angles opposite sides a, b, and c. The letter "N" will always be used for the number of triangles used in the tiling. An N-tiling of ABC is a tiling that uses N copies of some triangle T. The meanings of N, α, β, γ, a, b,c, A, B, and C will be fixed throughout this paper, and we assume $\alpha \le \beta \le \gamma$, when there is no other assumption about α and β, such as $3\alpha + 2\beta = \pi$.

4. History

In our gallery of examples, we saw quadratic and biquadratic tilings in which the tile is similar to ABC, and also hexagonal tilings. These involve N being square, a sum of two squares, or three times a square. The biquadratic tilings were known in 1964, when the paper [3] was published. This is the earliest paper on the subject of which I am aware.[3] Snover *et. al.* [12] took up the challenge of showing that these are the only possible values of N. The following theorem completely answers the question, "for which N does there exist an N-tiling in which the tile is similar to the tiled triangle?"

Theorem 1 (Snover *et. al.* [12]). *Suppose ABC is N-tiled by tile T similar to ABC. If N is not a square, then T and ABC are right triangles. Then either*

(i) N is three times a square and T is a 30-60-90 triangle, or

(ii) N is a sum of squares $e^2 + f^2$, the right angle of ABC is split by the tiling, and the acute angles of ABC have rational tangents e/f and f/e,

and these two alternatives are mutually exclusive.

Soifer's book [13] appeared in 1990, with a second edition in 2009. He considered two "Grand Problems": for which N can *every* triangle be N-tiled, and for which N can *every* triangle be dissected into similar, but not necessarily congruent triangles. (The latter eventually became a Mathematics Olympiad problem.) The 2009 edition has an added chapter in which the biquadratic tilings and a theorem of Laczkovich occur.

Miklos Laczkovich published six papers [5, 6, 7, 2, 8, 9] on triangle and polygon tilings. According to Soifer, the 1995 paper was submitted in 1992. Laczkovich, like Soifer, studied dissecting a triangle into smaller *similar* triangles, not *congruent* triangles as we require here. If those similar triangles are rational (i.e., the ratios of their sides are rational) then if we divide each of them into small enough quadratic subtilings, we can achieve an N-tiling into *congruent* triangles, but of course N may be large. Laczkovich proved little about N, focusing instead on the shapes of ABC (or more generally, convex polygons) and of the tile. His theorems, for example, do not address the possibility of an N-tiling (of some ABC by some tile) for any particular N, but they do give us an exhaustive list of the possible shapes of ABC and the tile, which we will need in our proof that there is no 7-tiling. This list can be found in §5 (of this paper). However, his theorem published in the last chapter of [13] does mention N. It states that given an integer k, there exists an N-tiling for some N whose square-free part is k.

5. Laczkovich

A basic fact is that, apart from a small number of cases that can be explicitly enumerated, if there is an N-tiling of ABC by a tile with angles (α, β, γ), then the angles α and β are not rational multiples of π. This theorem is Theorem 5.1 of [6]. Laczkovich calls the angles of the tile commensurable if each of them is a rational multiple of π. He states his theorem conversely to the way we just described it: if there is a tiling of ABC by a tile T with commensurable angles, then the pair (ABC, T) belongs to a specific, fairly short list. It is important to note that Laczkovich's list in Theorem 5.1 is about dissections of ABC into *similar*, not necessarily congruent, triangles. His subsequent Theorem 5.3 shows that three possibilities for dissecting the right isosceles triangle ABC into similar triangles are impossible with congruent tiles.

Laczkovich's list of possibilities from the cited 1995 paper is given in Table 1. In the table, the triples giving the angles of the tile are (α, β, γ) after a suitable

[3]The simplest hexagonal tiling is attributed to Major MacMahon (1921) in the notes accompanying a plastic toy I purchased at an AMS meeting in 2012.

permutation, i.e., they are unordered triples. The reader who checks with [6] will need to remember that we have deleted the entries for the right isosceles ABC mentioned above.

TABLE 1. Laczkovich's 1995 list of tilings by tiles with commensurable angles

ABC	the tile
(α, β, γ)	similar to ABC
$(\alpha, \alpha, 2\beta)$	$\gamma = \pi/2$
equilateral	$(\frac{\pi}{6}, \frac{\pi}{6}, \frac{2\pi}{3})$
equilateral	$(\frac{\pi}{3}, \frac{\pi}{12}, \frac{7\pi}{12})$
equilateral	$(\frac{\pi}{3}, \frac{\pi}{30}, \frac{19\pi}{30})$
equilateral	$(\frac{\pi}{3}, \frac{7\pi}{30}, \frac{13\pi}{30})$

In subsequent work, specifically Theorem 3.3 of [9], Laczkovich proved that the table can be considerably shortened: the tilings of the equilateral triangle mentioned in the last three rows cannot occur (when the tiles are required, as in this paper, to be congruent rather than just similar). Thus the final version is as shown in

TABLE 2. Laczkovich's 2012 list of tilings by tiles with commensurable angles

ABC	the tile
(α, β, γ)	similar to ABC
$(\alpha, \alpha, 2\beta)$	$\gamma = \pi/2$
equilateral	$(\frac{\pi}{6}, \frac{\pi}{6}, \frac{2\pi}{3})$

It is possible to prove by direct computation that the last two rows of Table 1 do not correspond to actual tilings. Namely, the area equation for the equilateral triangle with side X tells us $X^2 = Nbc$, if angle $\alpha = \pi/3$. Then writing $X = pa + qb + rc$ and calculating $(a, b, c) = (\sin\alpha, \sin\beta, \sin\gamma)$ for the specific angles involved, we get equations in certain algebraic number fields, that one then has to show impossible. For example,

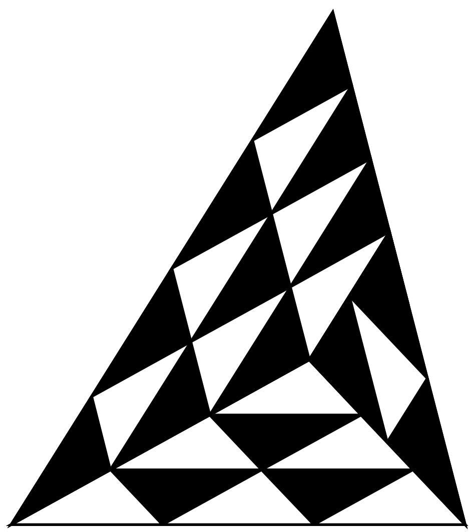
$$\left(p\left(\xi - \frac{1+\sqrt{5}}{8}\right) + q\frac{\sqrt{3}}{2} + r\left(\xi + \frac{1+\sqrt{5}}{8}\right)\right)^2 \quad = \quad N\left(\frac{3}{8} - \frac{\sqrt{5}}{8}\right) \tag{1}$$

One interesting thing about this approach is that SageMath is fully capable of performing all the required calculations, including determining whether certain expressions lie in certain algebraic number fields or not. We did not succeed, however, in entirely eliminating the row mentioning $\pi/12$ by computation; in that case, using the area equation as described only tells us that N is six times a square. That would be enough for this paper, where we only need that N cannot be 7 or 11; but Laczkovich entirely eliminated that possible tiling as well as the other two.

6. The coloring equation

In this section we introduce a tool that is useful for some, but not all, tiling problems. Suppose that triangle ABC is tiled by a tile with angles (α, β, γ) and sides (a, b, c), and suppose there is just one tile at vertex A. We color that tile black, and then we color each tile black or white, changing colors as we cross tile boundaries. Under certain conditions this coloring can be defined unambiguously, and then, we define the "coloring number" to be the number of black tiles minus the number of white tiles. An example of such a coloring is given in Fig. 13.

Figure 13. A tiling colored so that touching tiles have different colors.



The following theorem spells out the conditions under which this can be done. In the theorem, "boundary vertex" refers to a vertex that lies on the boundary of ABC or on an edge of another tile, so that the sum of the angles of tiles at that vertex is π. At an "interior vertex" the sum of the angles is 2π.

Theorem 2. *Suppose that triangle ABC is tiled by the tile (a, b, c) in such a way that*

(i) There is just one tile at A.

(ii) At every boundary vertex an odd number of tiles meet.

(iii) At every interior vertex an even number of tiles meet.

(iv) The numbers of tiles at B and C are both even, or both odd.

Then every tile can be assigned a color (black or white) in such a way that colors change across tile boundaries, and the tile at A is black. Let M be the number of black tiles minus the number of white tiles. Then the coloring equation

$$X \pm Y + Z = M(a + b + c)$$

holds, where Y is the side of ABC opposite A, and X and Z are the other two sides. The sign is $+$ or $-$ according as the number of tiles at B and C is odd or even.

Proof. Each tile is colored black or white according as the number of tile boundaries crossed in reaching it from A without passing through a vertex is even or odd. The hypotheses of the theorem guarantee that color so defined is independent of the path chosen to reach the tile from A. The total length of black edges, minus the total length of white edges, is $M(a+b+c)$, since $a+b+c$ is the perimeter of each tile. Each interior edge makes a contribution of zero to this sum, since it is black on one side and white on the other. Therefore only the edges on the boundary of ABC contribute. Now sides X and Y contain only edges of black tiles, by hypotheses (i) and (ii). Side Y is also black if the number of tiles at B and C is odd, and white if it is even. Hence the difference in the total length of black and white tiles is $X \pm Y + Z$, with the sign determined as described. That completes the proof.

7. Possible values of N in tilings with commensurable angles

We wish to add a third column to Laczkovich's Table 2, giving the possible forms of N if there is an N-tiling of ABC by the tile in that row. For example, when ABC is similar to the tile, then N must be a square, so we put n^2 in the third column. While we are at it, we add a fourth column with a citation to the result, and delete the rows corresponding to the tilings of the equilateral triangle that we have proved impossible. The revised and extended table is Table 3. All the entries in this table except the last one give necessary and sufficient conditions on N for the tilings to exist. The last one gives necessary conditions for certain tilings that probably do not actually exist.

Table 3. N-tilings by tiles with commensurable angles, with form of N

ABC	the tile	form of N	citation
(α, β, γ)	similar to ABC	n^2	[12]
(α, β, γ)	similar to ABC, $\gamma = \pi/2$	$e^2 + f^2$	[12]
$(\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2})$	similar to ABC	$3n^2$	[12]
$(\alpha, \alpha, 2\beta)$	$\gamma = \pi/2$	$2n^2$	[1]
$(\alpha, \alpha, 2\beta)$	$(\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2})$	n^2	[1]
$(\frac{\pi}{6}, \frac{\pi}{6}, \frac{2\pi}{3})$	$(\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2})$	$6n^2$	[1]
equilateral	$(\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2})$	$6n^2$	[1]
equilateral	$(\frac{\pi}{6}, \frac{\pi}{6}, \frac{2\pi}{3})$	$3n^2$	[1]

Theorem 3. *Suppose (α, β, γ) are all rational multiples of 2π, and triangle ABC is N-tiled by a tile with angles (α, β, γ). Then ABC, (α, β, γ), and N correspond to one of the lines in Table 3.*

Proof. As discussed above, Laczkovich characterized the pairs of tiled triangle and tile, as given in Table 2.[4] It remains to characterize the possible N for each line.

[4]Again, we remind readers who may check with [6] that there are three entries in Laczkovich's Theorem 5.1 that are shown in the subsequent Theorem 5.3 not to apply to tilings by congruent triangles, so they do not appear in our tables.

In several cases lines in Table 2 split into two or more lines in Table 3, which supplies the required possible forms of values of N. That table lists in its last column citations to the literature or theorems in this paper for each line. Finally, we have deleted the rows of Table 2 corresponding to the tilings that are impossible by Theorem 3.3 of [9]. That completes the proof.

8. Laczkovich's second table

Laczkovich also studied the case when not all the angles of the tile are rational multiples of π. Again a finite number of cases can arise. This is Theorem 4.1 of [6], and the list of cases is given in Table 4.

Table 4. Tilings when not all angles are rational multiples of π.

ABC	the tile
(α, β, γ)	similar to ABC
equilateral	$\alpha = \pi/3$
$(\alpha, \alpha, 2\beta)$	$\gamma = \pi/2$
$(\alpha, \alpha, \pi - 2\alpha)$	$\gamma = 2\alpha$
$(2\alpha, \beta, \alpha + \beta)$	$3\alpha + 2\beta = \pi$
$(2\alpha, \alpha, 2\beta)$	$3\alpha + 2\beta = \pi$
isosceles	$3\alpha + 2\beta = \pi$
$(\alpha, \alpha, \pi - 2\alpha)$	$\gamma = 2\pi/3$
$(\alpha, 2\alpha, \pi - 3\alpha)$	$\gamma = 2\pi/3$
$(\alpha, 2\beta, 2\alpha + \beta)$	$\gamma = 2\pi/3$
$(\alpha, \alpha + \beta, \alpha + 2\beta)$	$\gamma = 2\pi/3$
$(2\alpha, 2\beta, \alpha + \beta)$	$\gamma = 2\pi/3$
equilateral	$\gamma = 2\pi/3$

Table 2 and 4 together constitute an exhaustive list of tilings. If we have some conditions on the tile, such as for example $3\alpha + 2\beta = \pi$, then we look to see what entries in Table 2 satisfy those conditions. That gives some tilings with commensurable angles. Then we look in the other table for tilings in which not all the angles are rational multiples of 2π. To fix the ideas we spell out the details for the case $3\alpha + 2\beta = \pi$.

Lemma 1. *Let $3\alpha + 2\beta = \pi$. Suppose there is an N-tiling of triangle ABC by tile T with angles (α, β, γ). Suppose also that ABC is not similar to T. Then α and β are not rational multiples of π, and every linear relation between π, α, and β is a multiple of $3\alpha + 2\beta = \pi$.*

Proof. Suppose there is an N-tiling as in the statement of the lemma. Then if angles of the tile are all rational multiples of π, the pair ABC and the tile must occur in Table 2. So we have to check if any of the triples in that table satisfy $3\alpha + 2\beta = \pi$. And they do not, so that completes the proof.

Remark. The reader of Laczkovich's paper [6] should beware: Theorem 5.1 includes the triple $(\pi/4, \pi/8, 5\pi/8)$, which does satisfy $3\alpha+2\beta=\pi$. But as discussed above, it is included since the theorem is about dissections into *similar* triangles, and Theorem 5.3 of [6] rules it out for tilings into *congruent* triangles. Hence we have deleted it from Table 2 and do not need to consider it here.

9. Adding a column for N to Laczkovich's second table

The research program that we have been pursuing in this subject is to study triples (ABC, T, N) instead of just pairs (ABC, T), where there is an N-tiling of ABC by tile T. Another way to say that is that we wish to add a third column to Laczkovich's second table, entering the possible forms of N in that column, as we did to the first table. This has proved to be a longer business than I had originally imagined, although also more interesting, since several new tilings have been discovered in the process, and this research program is not complete. The point of the present paper is that we have pursued it far enough to reach the goal of showing that 7-tilings are impossible. Presently, we can supply entries in the third column down to the cases with $\gamma = 2\pi/3$, but some of them are only necessary conditions, not necessary and sufficient, leaving open many questions about particular values of N that are not ruled out by those necessary conditions.

10. Some number-theoretic facts

The facts in this section may not be well-known to all our readers, so we collect them here with short proofs or citations.

Lemma 2. *An integer N can be written as a sum of two integer squares if and only if the squarefree part of N is not divisible by any prime of the form $4n+3$.*

Proof. See for example [4], Theorem 366, p. 299.

Lemma 3. *A quotient of sums of two rational squares is a sum of two rational squares.*

Proof. A sum of two rational squares is the square of the absolute value of some complex number. The quotient of the absolute values is the absolute value of the quotient. Explicitly:

$$\begin{aligned}
\frac{a^2+b^2}{c^2+d^2} &= \frac{|a+bi|^2}{|c+di|^2} \\
&= \left|\frac{a+bi}{c+di}\right|^2 \\
&= \left|\frac{(a+bi)(c-di)}{c^2+d^2}\right|^2 \\
&= \left(\frac{ac+bd}{c^2+d^2}\right)^2 + \left(\frac{bc-ad}{c^2+d^2}\right)^2
\end{aligned}$$

That completes the proof of the lemma.

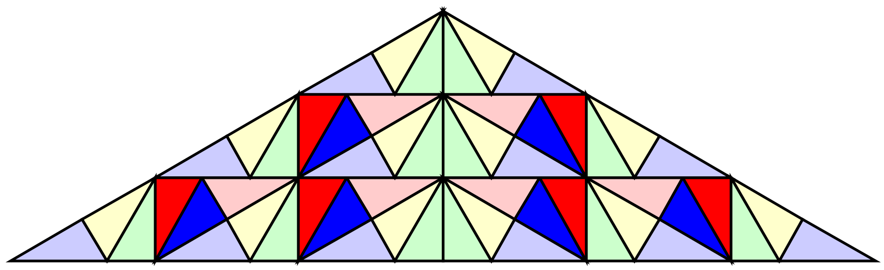
The following lemma identifies those relatively few rational multiples of π that have rational tangents or whose sine and cosine satisfy a polynomial of low degree over $\mathbb{Q}$.

Lemma 4. *Let $\theta = 2m\pi/n$, where m and n have no common factor. Suppose $\cos\theta$ is algebraic of degree 1 or 2 over $\mathbb{Q}$. Then n is one of $5, 6, 8, 10, 12$. If both $\cos\theta$ and $\sin\theta$ have degree 1 or 2 over $\mathbb{Q}$, then n is $6, 8,$ or 12.*

Proof. Let φ be the Euler totient function. Assume $\cos\theta$ has degree 1 or 2. By [11], Theorem 3.9, p. 37, $\varphi(n) = 2$ or 4. The stated conclusion follows from the well-known formula for $\varphi(n)$. The second part of Theorem 3.9 of [11] rules out $n = 5$ or 10 when $\sin\theta$ is also of degree 1 or 2.

11. Isosceles ABC tiled by a right triangle

Figure 14. A 54-tiling; $N/2$ is three times a square. Tile is 30-60-90.



The possible ways of tiling an isosceles triangle have been studied in [1]. Let the angles of the tile be (α, β, γ). Laczkovich's results imply that there are only four cases to consider: $\gamma = \pi/2$, $\gamma = 2\alpha$, $2\alpha + 3\beta = \pi$, or $\gamma = 2\pi/3$, with α not a rational multiple of $\pi/2$ in the last two cases. The first three cases have been studied in [1]. Here is the result for the case of a right-angled tile:

Theorem 4. *Suppose isosceles triangle ABC with base angles β is N-tiled by a right-angled tile. Then N is a square, or a sum of two squares, or six times a square.*

Corollary 1. *N is not a prime congruent to 3 mod 4; in particular it is not 7, 11, or 19; nor can N be twice such a prime.*

Proof of Corollary. Primes congruent to 3 mod 4 cannot be sums of squares, by Lemma 2.

Remark. For N even, all the listed possibilities can occur. See Figs. 15 and 16, and additional examples in [1]. We conjecture that there are no such tilings with N odd.

Remark. Our original plan was to avoid citing [1] by exhibiting simple SageMath code that verifies directly that there are no such tilings for $N = 7, 11, 14, 19$, by showing that the area equation $(pa + qb + r)^2 = N/2$ together with the right-angle equation $a^2 + b^2 = 1$ has no solutions in non-negative integers (p, q, r). Although this code is straightforward, to justify it we had to prove two lemmas: no tile except the one at the vertex B can touch both equal sides, and we can assume $p + q + r < N/3$. In total then the computational approach is not much shorter than the proof in [1].

FIGURE 15. N is a twice a square or a twice a sum of squares. 50 is both.

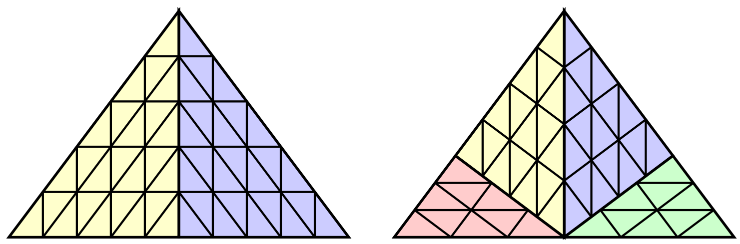
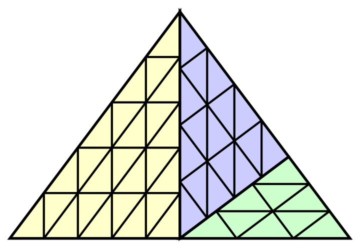


FIGURE 16. 50 is both twice a square and twice a sum of squares.



12. Useful lemmas

In this section, we collect some facts that will be applied when we start eliminating the possibilities for 7-tilings and 11-tilings case by case, according to the cases of Laczkovich's second table.

12.1. **Angles.**

Lemma 5. *Suppose triangle ABC is N-tiled by a tile in which $3\alpha + 2\beta = \pi$. Then $\gamma > \pi/2$.*

Proof.

$$\begin{aligned} \pi &= 3\alpha + 2\beta \\ &= \alpha + 2(\alpha + \beta) \\ &= \alpha + 2(\pi - \gamma) \\ \gamma &= \frac{\pi}{2} + \frac{\alpha}{2} \; > \; \frac{\pi}{2} \end{aligned}$$

That completes the proof.

Lemma 6. *Let triangle ABC be N-tiled by a tile with angles (α, β, γ). Suppose that either $3\alpha + 2\beta = \pi$ and ABC is not isosceles with base angles α, or $\gamma = 2\pi/3$. Then no tile has its γ angle at a vertex of ABC.*

Proof. By Lemma 1, α and β are not rational multiples of π. Hence the angles of ABC are linear integral combinations of α, β, and γ. First assume $3\alpha + 2\beta = \pi$. Then the angles of ABC are each equal to α, 2α, $\alpha + \beta$, β, or 2β. Of these angles, all but 2β are less than γ, as we now show. Then $\gamma = \beta + 2\alpha$, and

$$\begin{array}{rcrcl} \alpha & < & \beta + 2\alpha & = & \gamma \\ \beta & < & \beta + 2\alpha & < & \gamma \\ \alpha + \beta & < & \beta + 2\alpha & = & \gamma \\ 2\alpha & < & \beta + 2\alpha & = & \gamma. \end{array}$$

Since ABC is not similar to the tile, there cannot be a γ angle alone at any vertex, since that would leave $\alpha + \beta$ for the other two vertices, making ABC similar to the tile, since α is not a rational multiple of β.

Since all the possible angles but 2β are less than γ, it only remains to deal with the case where angle C is equal to 2β and $\gamma < 2\beta$, and there is a tile with its γ angle at C. We do not have $2\beta = \gamma$, by Lemma 1. Then there must be another tile at C as well. If the angle of that tile at C is α, then the total angle at C is at least $\gamma + \alpha = 2\alpha + \beta + \alpha = 3\alpha + \beta$, leaving only β for the other two angles of ABC. But that is impossible, since α is not a rational multiple of β. If the second angle at C is β, then the total angle at C is at least $\gamma + \beta = 2\alpha + 2\beta$, leaving just α for the other two angles, which is again impossible. Hence the second angle at C cannot be β. That completes the proof under the assumption $3\alpha + 2\beta = \pi$.

We now take up the case $\gamma = 2\pi/3$. Then the possible angles of ABC are α, β, $\alpha + \beta$, $\alpha + 2\beta$, $2\alpha + \beta$, 3α, and 3β. All but 3α and 3β are less than $2\alpha + 2\beta = \gamma$, so a γ tile can occur, if at all, only at a vertex angle of 3α or 3β. Suppose vertex C has angle 3α and there is a γ angle of a tile at C. Then $\gamma < 3\alpha$ and angles A and B together are $\pi - 3\alpha < \pi - \gamma$, which is impossible since the three angles of ABC add up to π. Similarly if vertex C has angle 3β and $\gamma < 3\beta$. That completes the proof of the lemma.

12.2. **Two c edges on each side of ABC.**

Lemma 7. *Suppose triangle ABC is N-tiled by a tile with angles (α, β, γ) and $\gamma > \pi/2$. Suppose all the tiles along one side of ABC do not have their c sides along that side of ABC. Then there is a tile with a γ angle at one of the endpoints of that side of ABC.*

Proof. Let PQ be the side of ABC with no c sides of tiles along it. Then the γ angle of each of those tiles occurs at a vertex on PQ, since the angle opposite the side of the tile on PQ must be α or β. Let n be the number of tiles along PQ; then there are $n-1$ vertices of these tiles on the interior of PQ. Since $\gamma > \pi/2$, no vertex on the boundary has more than one γ angle. By the pigeonhole principle, there is at least one tile whose γ angle is not at one of those $n-1$ interior vertices; that angle must be at P or Q. That completes the proof of the lemma.

Lemma 8. *Suppose triangle ABC is N-tiled by a tile T with angles (α, β, γ). Suppose*

(i) $\gamma > \pi/2$, and

(ii) α is not a rational multiple of π, and

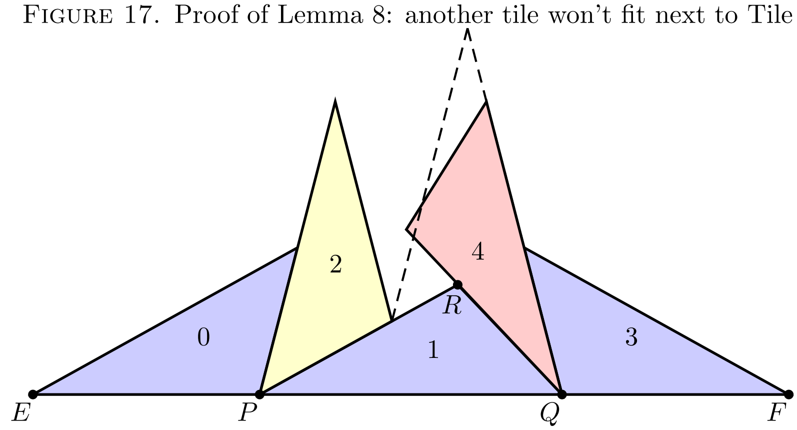
(iii) every angle of triangle ABC is less than γ.

Then there are at least two c edges of tiles on side AC.

Remarks. One can prove by the same method that the c edges must occur in adjacent blocks of at least two edges, but we found no use for that result.

Proof. By hypothesis (ii), every boundary vertex P (except A, B, and C) that has a γ angle (i.e., some tile with a vertex at P has its γ angle at P) touches exactly three tiles, which contribute angles of α, β, and γ. By Lemma 7, each side of ABC has at least one c edge. The present lemma, however, claims more: there must be at least two c edges. Suppose, to the contrary, that there is just one c tile, Tile 1, with an edge on one side EF of triangle ABC. Then all the other tiles with an edge on EF have a γ angle on EF. We visualize EF as horizontal with triangle ABC above, and use the word "north" and "northwest" accordingly. See Fig. 17.

FIGURE 17. Proof of Lemma 8: another tile won't fit next to Tile 2



Since there cannot be a γ angle at the vertices of ABC, it follows that both the tiles on AC adjacent to Tile 1 (if there are two, or otherwise, only the one) have their γ angles adjacent to Tile 1. Let PQ be the c edge of Tile 1 lying on AC. Let R be the northern vertex of Tile 1. Suppose (without loss of generality) that Tile 1 has its β angle at Q. Then the side PR of Tile 1, opposite Q, has length b. Let Tile 2 be the tile adjacent to PR.

Since the hypotheses of the theorem remain true if (the names of) α and β are interchanged, we may assume without loss of generality that $\alpha < \beta$. Then by the law of sines, $a < b$. Since $\gamma > \pi/2$ we also have $a < c$ (by the law of cosines).

Assume, for proof by contradiction, that neither P nor Q is a vertex of ABC. Then there exist Tile 0 and Tile 3 on AC sharing vertices P and Q with Tile 1. Tile 2, between Tile 0 and Tile 1, must have its β angle at P, since Tile 1 has its α angle there and Tile 0 has its γ angle at P. There is an open α angle between Tile 1 and Tile 3; let Tile 4 be the tile that fills that notch. Then Tile 4 has its b or c edge along QR. Since Tile 1 has its a edge along QR and $a < b$ and $a < c$, the edge of Tile 4 on QR extends past R. Then the segment PR is of length b and its northwest side is composed of a number of tile edges, starting with Tile 2 at P. These must all be a edges, since a is the only edge less than b. Since the tiles northwest of PR all have their a edges on PR, they all have a γ angle on PR. But Tile 2 does not have its γ angle at P, since Tile 0 has its γ angle at P. And the last tile cannot have its γ angle at R, since Tile 4 extends along QR past R, and Tile 1 has its γ angle at R. So if there are n tiles northwest of PR, there are only $n-1$ possible places for their γ angles, contradicting the pigeon-hole principle. This contradiction proves that one of P or Q is a vertex of ABC.

Now we argue by cases.

Case 1: Q is a vertex of ABC, i.e., $Q = C$. If the angle of ABC at Q is strictly between β and 2β, then Tile 4 must have its α angle at Q, and we argue exactly as before. If the angle of ABC at Q is exactly β, then we argue as above, except that RQ is now extended past R by one side of ABC rather than an edge of Tile 4. The argument about the γ angles of the tiles northwest of PR is unchanged, if P is not a vertex of ABC. If P is a vertex of ABC, then we still can argue that Tile 2 must have its a side on PR, because it cannot fit next to Tile 1 with its b or c side on PR.

Therefore we may assume that the angle of ABC at Q is at least 2β, and that Tile 4 has its β angle at Q and its a edge against Tile 1. Hence there is a double angle at Q. Then by hypothesis (iv), b is not a multiple of a. Tile 4 cannot have its γ angle at Q, by hypothesis (iii). Therefore Tile 4 has its γ angle at R, and since $\gamma > \pi/2$ by hypothesis (i), PR does not extend past R as part of the tiling. The tiles northwest of PR must all have their a edges on PR, since a is the only edge less than b. Similarly, the tiles supported by the west edge of Tile 4 must all have their a edges against that west edge, which has length b. All those tiles northwest of PR have their γ angles on PR (since they have their a edges on PR), and by the pigeonhole principle those γ angles are all at the northwest. Therefore the tile supported by PR at R, call it Tile 5, has its γ angle there. Since Tiles 1 and 4 already have their γ angles at R, Tile 5 shares an edge with Tile 4, and as just shown that edge has length a. But now Tile 5 has two a edges, contradiction. That completes Case 1.

Case 2: P is a vertex of ABC, and Q is not. Then Tile 4 is placed as shown in the figure. Therefore the angle of ABC at vertex P must be greater than α, since if it were equal to α, Tile 4 would not lie inside ABC. Then Tile 2 exists, and Tile 2 must have its a side on PR, because it cannot fit next to Tile 1 with its b or c side on PR. From there the argument proceeds as before. That completes Case 2.

That completes the proof of the lemma.

If there are enough tiles on the boundary of ABC then N must be at least 12. How many is "enough"? As it turns out we do not need a precise answer; the following lemma is helpful enough and easy to prove. No doubt the number 10 can be improved, but this is good enough.

Lemma 9. *Let ABC be N-tiled, and suppose the total number of tiles with an edge on the boundary of ABC is at least k, with at least two tiles on each side of ABC, and only one tile at B, and a total of five tiles at the vertices of ABC. Suppose $\gamma \neq \pi/2$. Then $N \geq k + 2$.*

Proof. We must produce at least two non-boundary tiles. Case 1, two vertices, say A and B, of ABC have only one tile each. Since $\gamma \neq \pi/2$, at three tiles (at least) meet at each boundary vertex. Therefore, the tile that shares an edge with the tile at A is not a boundary tile, and the same for the tile next to the tile at B. That makes at least $k + 2$ tiles.

Case 2, only B has a single tile, while vertices A and C have two tiles each. Then the tile adjacent to the tile at B is a non-boundary tile. Consider the two tiles at vertex A, say Tile 1 and Tile 2. If they do not share a common edge then one of them, say Tile 1, has a shorter edge along their common boundary. Then the tile adjacent to that edge is not a boundary tile, and hence it is a second non-boundary tile. If they do share a common edge, then let Tile 3 and Tile 4 be the tiles adjacent to Tile 1 and Tile 2, respectively. At most one of Tile 3 and Tile 4 can have a boundary extending past the common interior vertex E of Tile 1 and Tile 2, and the one that does not cannot be a boundary tile. Hence it is a second non-boundary tile. That completes the proof of the lemma.

13. The case $3\alpha + 2\beta = \pi$

Three of the rows of Table 4 fall under the case $3\alpha + 2\beta = \pi$, with α not a rational multiple of π. For some of those cases we have proved necessary and sufficient conditions for the existence of an N-tiling; and for all of them we have strong necessary conditions. In other words, we have added a fourth column to Table 4, at least for the rows corresponding to $3\alpha + 2\beta = \pi$. From those entries we can simply read off that $N = 7$ and $N = 11$ are impossible. In fact $N = 28$ is the smallest possible N. But the proofs, which are unpublished, occupy approximately a hundred pages. (except for its dependence on [6] and [12]). Therefore we give a short, self-contained, algebraic and computational proof that N-tilings do not exist when $N < 12$ and $3\alpha + 2\beta = \pi$ and α is not a rational multiple of π.

An important tool in the analysis of these tilings is the "coloring equation" given in Theorem 2. That theorem applies here, as we now show. If $3\alpha + 2\beta = \pi$ and α is not a rational multiple of π, then every boundary vertex is composed of three tiles $(\alpha+\beta+\gamma)$ or five tiles $(3\alpha+2\beta)$, and every interior vertex is either a "center" with four tiles $(3\gamma + \beta)$ or has six tiles $(2\alpha + 2\beta + 2\gamma)$ or eight tiles $(4\alpha + 3\beta + \gamma)$ or ten tiles $(6\alpha + 4\beta)$.

Since there are five tiles at the angles of ABC, by renaming the vertices we may assume that only one tile is at B. Let (X, Y, Z) be the lengths of sides AB, BC, and AC. Then we have the "coloring equation"

$$M(a + b + c) \quad = \quad X + Z \pm Y \tag{2}$$

where the $+$ sign is taken if the angles at A and C have an odd number of tiles, and the $-$ sign is taken if they have an even number.

Besides the coloring equation, we have the "area equation", which says that the area of ABC is equal to N times the area of the tile. We use the formula for the area of a triangle that says twice the area is the product of two adjacent sides and the sine of the included angle. By the law of sines, $a/c = \sin\alpha/\sin\gamma$. Then the area equation can be written

$$XZ\sin\alpha \quad = \quad Nbc\sin\alpha \tag{3}$$

$$XZ \quad = \quad Nbc \qquad \text{if angle } B = \alpha \tag{4}$$

$$XZ \quad = \quad Nac \qquad \text{if angle } B = \beta \tag{5}$$

Definition 1. *Let a triangle have angles (α, β, γ). We define*

$$s = 2\sin(\alpha/2).$$

This definition is useful because the ratios a/c and b/c can be expressed simply in terms of s, as shown in the following lemma.

Lemma 10. *Suppose* $3\alpha + 2\beta = \pi$. *Let* $s = 2\sin\alpha/2$. *Then we have*

$$\begin{aligned} \sin\gamma &= \cos\frac{\alpha}{2} \\ \frac{a}{c} &= s \\ \frac{b}{c} &= 1 - s^2 \end{aligned}$$

Proof. Since $\gamma = \pi - (\alpha + \beta)$, we have

$$\begin{aligned} \sin\gamma &= \sin(\pi - (\alpha + \beta)) \\ &= \sin(\alpha + \beta) \\ &= \cos(\pi/2 - (\alpha + \beta)) \\ &= \cos\frac{\alpha}{2} \qquad \text{since } \pi/2 - \beta = 3\alpha/2 \end{aligned}$$

Then $c = \sin\gamma = \cos\alpha/2$, and $a = \sin\alpha = 2\sin(\alpha/2)\cos(\alpha/2)$. Hence

$$\frac{a}{c} = 2\sin\alpha/2.$$

Since $3\alpha + 2\beta = \pi$, we have

$$\begin{aligned} \sin\beta &= \sin(\pi/2 - 3\alpha/2) \\ &= \cos(3\alpha/2) \\ &= 4\cos^3\frac{\alpha}{2} - 3\cos\frac{\alpha}{2} \end{aligned}$$

Hence

$$\begin{aligned} b/c &= 4\cos^2(\alpha/2) - 3 \\ &= 4(1 - \sin^2\alpha/2) - 3 \\ &= 1 - 4\sin^2\alpha/2 \end{aligned}$$

Then we have

$$\begin{aligned} \frac{a}{c} &= s \\ \frac{b}{c} &= 1 - s^2 \end{aligned}$$

establishing the second equation of the lemma. That completes the proof of the lemma.

Theorem 5. *Suppose* $3\alpha + 2\beta = \pi$, *and triangle* ABC *is* N*-tiled by a tile with angles* (α, β, γ) *not similar to* ABC, *and* α *is not a rational multiple of* π. *Then* $N \geq 12$.

Proof. We first discuss the possibility of applying of Lemma 8. Do the hypotheses hold? By Lemma 6, no tile has a γ angle at a vertex of ABC; and by Lemma 5, $\gamma > \pi/2$. Since the tile is not similar to ABC, and α is not a rational multiple of π, each angle of ABC is less than γ. Therefore Lemma 8 is applicable.

We now explain the idea of the proof. The tiling provides an expression for each side of ABC as a linear combination of abc. Thus

$$\begin{aligned} X &= pa + qb + rc \\ Z &= ua + vb + wc \\ Y &= ka + \ell b + mc \end{aligned}$$

Substitute these expressions for (X, Y, Z) in the coloring equation. With $P = p + u \pm k$, $Q = q + v \pm \ell$, $R = r + w \pm m$ we have $M(a + b + c) = Pa + Qb + Rc$. Dividing by c and use $a/c = s$ and $b/c = 1 - s^2$ we have

$$M(2 + s - s^2) \quad = \quad Ps + Q(1 - s^2) + R$$

For given (M, P, Q, R) that quadratic can be solved for s (provided its discriminant is nonnegative). The area equation too can be expressed in terms of s, and we can check if it is satisfied for the s from the coloring equation. For a given N, we need to consider only values of the integer parameters between 0 and N, so this search will terminate. Moreover, as discussed in the first paragraph of this proof, Lemma 8 tells us that we can restrict the search by only examining values of r, w, and m that are at least 2, provided ABC is isosceles or s is rational. Finally, Lemma 9 allows us to not consider cases in which there would be ten or more boundary tiles, i.e., when $p + q + r + u + v + w + k + \ell + m \geq 10$. SageMath code to carry out this plan for isosceles ABC with base angles α or β is exhibited in Fig. 18. Run that code passing 7 as the function parameter, and then again passing 11. It runs in about 12 seconds, and produces no output except the reassuring progress reports as M changes. That shows that there is no 7 or 11 tiling in the case of isosceles ABC with angles α at A and B, or β at A and B, i.e., when the coloring equation is $M(a + b + c) = X + Y + Z$.

The other possible shapes of ABC satisfy the coloring equation $M(a + b + c) = X - Y + Z$. That code differs from the code in Fig. 18 in two respects. First, because of the minus sign in the coloring equation, negative values of (P, Q, R) are allowed, and the upper limits of (P, Q, R) go up to N, $N - |P|$, and $N - |P| - |R|$, respectively, and the values of (k, ℓ, m) are preceded by a minus sign, with a continue statement inserted to reject negative values. Second, we are only allowed to assume each side contains at least two c edges in case s is rational, so the variable `looplimit` has to be recalculated each time s is recalculated, and set to 2 if s is rational, and otherwise to 1. Although it adds a page to the length of the paper, we also include enough of this code so that any reader can reproduce our results. See Fig. 19.

To prove the theorem as stated, we ran both programs for all N between 3 and 11, inclusive. The second program is slower, requiring 27 seconds for $N = 7$, over three minutes for $N = 11$, and about 8 minutes for all values 3 to 11. But it gets the answer: no solutions are found. That completes the proof.

Remark. This method cannot be used for $N = 14$ or 19, as some solutions are found. As discussed in §18 below, we do have a proof that there is no 14-tiling or 19-tiling with $3\alpha + 2\beta = \pi$; but the short direct computational proof given here will not work.

FIGURE 18. SageMath code used in the proof of Theorem 5, *ABC* isosceles

```
def oct22(N):  # ABC isosceles
  var('P,Q,R,M,s,p,q,r,u,v,w,k,ell,m')
  epsilon = 0.0000001
  lowerlimit = 2  # each side has at least 2 c edges
  for M in range(1,N):
    print("M=%d" %M)
    for P in range(0,N):
      for Q in range(0,N-P):
        for R in range(6,N-P-Q):
          eq1 = M*(2+s-s^2) - P*s - Q*(1-s^2) - R
          discriminant = (M-P)^2 - 4*(Q-M)*(2*M-Q-R)
          if discriminant < 0:
            continue
          answers = solve(eq1,s)
          for x in answers:
            if x.rhs() <= 0 or x.rhs() >= 1:
              continue
            for r in range(lowerlimit,R+1):
              for w in range(lowerlimit,R-r):
                m = R -r-w
                if m < lowerlimit:
                  continue
                for p in range(0,P+1):
                  for u in range(0,P-p):
                    k = P-p-u;
                    for q in range(0,Q+1):
                      for v in range(0,Q-q):
                        ell = Q-q-v
                        boundarytiles = p+q+r+u+v+w+k+ell+m
                        if boundarytiles >= N-2:
                          continue
                        X = r + p*S + q*(1-S^2)
                        Y = w + u*S + v*(1-S^2)
                        area1 = abs( X*Y - N*(1-S^2))
                        area2 = abs(X*Y - N*S)
                        if n(area1) < epsilon:   #  B = alpha
                          print("alpha",N,M,p,q,r,u,v,w,k,ell,m)
                          print(area1)
                        if n(area2) < epsilon:  #  B = beta
                          print("beta",N,M,p,q,r,u,v,w,k,ell,m)
```

FIGURE 19. SageMath code used in the proof of Theorem 5, ABC not isosceles

```
def oct22b(N):  # case when ABC is not isosceles
  var('P,Q,R,M,s,p,q,r,u,v,w,k,ell,m')
  epsilon = 0.0000001
  for M in range(1,N):
    print("M=%d" %M)
    for P in range(-N,N+1):
      for Q in range(-(N-abs(P)),N-abs(P)+1):
        for R in range(-(N-abs(P)-abs(Q)),N-abs(P)-abs(Q)+1):
          eq1 = M*(2+s-s^2) - P*s - Q*(1-s^2) - R
          discriminant = (M-P)^2 - 4*(Q-M)*(2*M-Q-R)
          if discriminant < 0:
            continue
          answers = solve(eq1,s)
          for x in answers:
            S = x.rhs()
            if S <= 0 or S >= 1:
              continue
            lowerlimit=1; #will be set to 2 when S is rational
            if S < 1-S^2 and (S in QQ or not (1-S^2)/S in ZZ):
              lowerlimit = 2
            else:
              if 1-S^2 < S and (not (S/(1-S^2) in ZZ)):
                lowerlimit = 2
              else:
                lowerlimit = 1
            for r in range(lowerlimit,R+1):
              for w in range(lowerlimit,R-r):
                m = -(R -r-w)
                if m < lower3limit:
                  continue
                for p in range(0,P+1):
                  for u in range(0,P-p):
                    k = -(P-p-u);
                    if k < 0:
                      continue
                    for q in range(0,Q+1):
                      for v in range(0,Q-q):
                        ell = -(Q-q-v)
                        if ell < 0:
                          continue
                        # ... the rest as in the previous figure
```

14. The case $\gamma = 2\pi/3$ and α not a rational multiple of π

In this case, $\alpha + \beta = \pi/3$, so a boundary vertex can be composed of angles contributed by 3 or 6 tiles. Hence it is not in general possible to color the tiles black and white in a way that leads to a "coloring equation."

There are several shapes possible for ABC, listed in Table 4, but for our purposes there are just two cases to consider: either one of the vertices of ABC has just one tile (in which case we rename α and β so that the standalone angle is α, and we rename the vertices so it occurs at A), or there are two tiles at each of the three vertices, in which case we may assume that the angle at A is $\alpha+\beta$. We do not need to consider the case ABC similar to the tile, so no γ angles occur at the vertices of ABC. The shape of the tile can be expressed using the law of cosines, since $\cos(2\pi/3) = -\frac{1}{2}$, by the equation

$$c^2 = a^2 + b^2 + ab. \tag{6}$$

For example, $(3, 5, 7)$ and $(8, 7, 13)$ are rational tiles satisfying this equation.

Although there is no coloring equation, we do still have the "area equation" that says the area of ABC is N times the area of the tile. That equation takes different forms depending on the shape of ABC. In case the angle at A is α, and the sides AB and AC have length X and Y, the area equation is $XY \sin\alpha = Nab \sin\alpha$. After canceling $\sin\alpha$ we have

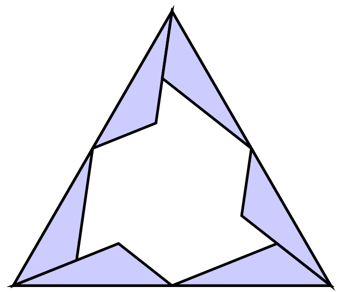
$$XY = Nab \tag{7}$$

We do have some general results about this kind of tiling, but the theory is incomplete and we do not go into it here. For our present purposes it suffices to show that any such tiling requires at least 12 tiles; that is Theorem 6 below.

Theorem 6. *Let triangle ABC be N-tiled by a tile with angles $(\alpha, \beta, 2\pi/3)$, not similar to ABC, and suppose α is not a rational multiple of π. Then $N \geq 12$. In particular, N is not equal to 7 or 11.*

Remark. The idea of the proof of this case is that, because of Lemma 8, each side of ABC is at least $2c$ in length, and that makes the area more than the area of 12 tiles. See Fig. 20, which illustrates an equilateral ABC with six tiles placed, and more area remaining than six tiles can cover. (This figure is only illustrative.) We first proved this theorem by a geometrical argument about placing tiles, but algebra is shorter and simpler. Both ideas can be seen in Fig. 20: It is geometrically impossible to complete the tiling, and also the untiled area is more than the area of six tiles.

FIGURE 20. The case of Theorem 6 when there are exactly six boundary tiles



Proof. Since the tile is not similar to ABC, and α is not a rational multiple of π, there can be no γ angle at a vertex of ABC. Then there must occur a total of six tiles at the vertices of ABC, contributing three α angles and three β angles to make up the angles of ABC. Lemma 8 is applicable, since no vertex of ABC can have a $\gamma = 2\pi/3$ angle, so each side of ABC has at least two c edges. That is the key idea of this proof.

We divide the proof into two cases. Case 1: One vertex of ABC has an angle δ with $\pi/3 \le \delta \le 2\pi/3$. The point of that inequality is that it implies $\sin \pi/3 < \sin \delta$. Let X and Y be the lengths of the sides adjacent to that angle. Then we have the area equation

$$XY \sin \delta \quad = \quad Nab \sin \frac{2\pi}{3}$$

Since $\sin \frac{2\pi}{3} = \sin \frac{\pi}{3} \le \sin \delta$, we have

$$XY \quad \le \quad Nab$$

According to Lemma 8, there are at least two c edges on each side of ABC. Hence $X \ge 2c$ and $Y \ge 2c$. Therefore $Nab \ge XY \ge 4c^2$. Therefore

$$3ab \quad \ge \quad \frac{12c^2}{N} \tag{8}$$

Recall (6):

$$\begin{aligned} c^2 \quad &= \quad a^2 + b^2 + ab \\ c^2 - 3ab \quad &= \quad a^2 + b^2 - 2ab \;=\; (a-b)^2 \;>\; 0 \qquad (9) \end{aligned}$$

Substituting on the left from (8) we have

$$c^2 \left(1 - \frac{12}{N}\right) \quad > \quad 0$$

We have strict inequality since the hypothesis that α is not a rational multiple of π implies $a \ne b$. Since $c^2 > 0$ we have

$$\begin{aligned} 1 - \frac{12}{N} \quad &> \quad 0 \\ N \quad &> \quad 12 \end{aligned}$$

That completes the proof in Case 1.

Case 2: Every vertex angle of ABC is either more than $2\pi/3$ or less than $\pi/3$. They cannot all be less than $\pi/3$ since they add up to π. Therefore one angle is more than $2\pi/3$. Renaming the vertices if necessary, we can assume the angle at B is more than $2\pi/3$. Renaming α and β if necessary, we can assume $\alpha < \beta$. Then the angles at A and C are either (α, α) or $(\alpha, 2\alpha)$ or $(2\alpha, \alpha)$, since otherwise the angle at B is $\le \pi - (\alpha + \beta) = 2\pi/3$.

I say that no tile has one vertex on AB and another vertex on BC. Suppose, to the contrary, that Tile 1 has vertex P on AB and vertex Q on BC. Then Tile 1 does not have a vertex at B, since the tiles at B have at least one vertex interior to ABC. Consider triangle PBQ. Angle B is either 3β or $3\beta + \alpha$, either of which is more than $2\pi/3$, so $|\cos B| > 1/2$. Hence angles P and Q of triangle PBQ are acute. Consider the rays emanating from B along the dividing lines between tiles with vertices at B. All three or four of the tiles at B have one c edge emanating along such a ray, and the whole tile is contained in PBQ. If one of the tiles at B supported by PB or BQ has its c edge there, then $PB \ge c$ or $BQ \ge c$. Otherwise let BR be the ray containing the other edge of the tile supported by PB at B, with R on PQ. In triangle BRP, BR is opposite an acute angle P, and PB is opposite an angle greater than $\pi/2$, since that angle is π minus angle P minus angle PBR, and both the subtracted angles are acute. Since the greater side is opposite the greater angle, $BP > BR \ge c$. Thus either $BP \ge c$ or $BQ \ge c$. Relabeling P and Q if necessary, we can assume $BP \ge c$. Since BQ is also composed of tile edges, at least $BQ \ge a$. Then by the law of cosines we have

$$\begin{aligned} PQ^2 &= BP^2 + BQ^2 - 2PB \cdot QB \cos B \\ &= BP^2 + BQ^2 + 2PB \cdot QB|\cos B| \\ &\ge c^2 + a^2 + 2ac|\cos B| \\ &= c^2 + a^2 \\ &> c^2 \end{aligned}$$

Hence the length of PQ is greater than c, and hence cannot be just one tile edge. Hence, as I said, no tile has one vertex on AB and another vertex on BC.

I say that c is not a linear integral combination of (a, b) unless it is a multiple of a. For suppose $c = ua + vb$. Then

$$\begin{aligned} c^2 &= a^2 + b^2 + ab \\ (ua + vb)^2 &= a^2 + b^2 + ab \\ (u^2 - 1)a^2 + (v^2 - 1)b^2 + (2uv - 1)ab &= 0 \end{aligned}$$

which is a contradiction if both u and v are positive. Therefore $u = 0$ or $v = 0$. If $v = 0$, then c is a multiple of a as claimed. Therefore we may assume $c = vb$ with $v > 1$. Then by the law of cosines,

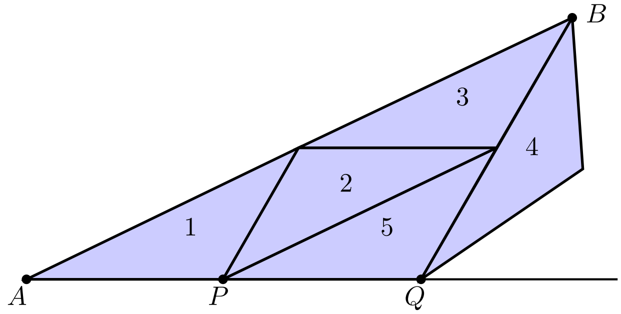
$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos\alpha \\ &> b^2 + c^2 - 2bc \\ &= b^2(v^2 + 1 - 2v) \qquad \text{since } c = vb \\ &= b^2(v - 1)^2 \\ &> b^2 \qquad \text{since } v > 1 \end{aligned}$$

Hence $a > b$, contradiction. Hence, as I said, c is not a linear integral combination of (a, b) unless it is a multiple of a.

We now divide into further cases. Case 2A: each of the two edges AB and BC supports at least three tiles. Then, there are two "notches" between the three tiles, so there are five tiles touching AB in at least one point, and five tiles touching BC. None of these have been double-counted, since no tile has a vertex on AB and a vertex on BC, so that is ten tiles. If ABC is isosceles then there are four tiles with vertices at B, two of which we have not yet counted, so that makes 12. If ABC is not isosceles then there are three tiles with vertices at B, and one more with a vertex at A or C, making again two uncounted tiles for 12 total. That completes the proof in Case 2A.

Case 2B: AB supports exactly two tiles, and ABC has angle α at A. As already noted, by Lemma 8, there are at least two c edges on each side of ABC, so both the tiles supported by AB have their c edges on AB. See Fig. 21 for an illustration of the following argument. Let Tile 1 be the tile at A; then Tile 1 has its b edge on

Figure 21. Case 2B of Theorem 7



AC. Let Tile 2 be the tile east of Tile 1. Tile 2 shares the a edge of Tile 1, since that edge terminates at both ends on the boundary of ABC. Tile 2 cannot have its γ angle to the south, since that would make two γ angles at a vertex on the boundary. Therefore it has its γ angle on AB and its β angle on AC. Let Tile 3 be the tile north of Tile 2. Then Tile 3 is supported by AB and hence has its c edge on AB, and shares the b edge of Tile 2 on its southern border. Let Tile 4 be the tile east of Tile 3. Then Tile 4 has a vertex at B. Either Tile 4 has its b or c edge along Tile 3 (either of which extends beyond Tile 3, since $a < b < c$), or it has its a edge shared with Tile 3 and its γ angle to the southwest. In all three of those cases, it terminates the line forming the southeast border of Tile 2. Let Tile 5 be the tile south of Tile 2, with its southwest vertex P on AC shared with Tile 1.

Suppose, for proof by contradiction, that Tile 5 does not share the c edge of Tile 2. Since its α angle is towards the west, it has its b edge along Tile 2. Since the tiles south of Tile 2 must terminate at the eastern vertex of Tile 2, the remaining $c-b$ of the southeast border of Tile 2 must be filled by some tile edges. But then c would be an integral linear combination of a and b, including at least one b, which is impossible, as proved above. That contradiction completes the proof that Tile 5 shares the c edge of Tile 2. Hence Tile 5 has its b edge on AC, as shown in Fig. 21. Then Tiles 1,2,3,5 are definitely as shown in Fig. 21.

Now consider BQ, the eastern border of Tiles 3 and 5. On the west side of BQ are two a edges. These cannot be matched on the east by two a edges, since then the γ angles of those two tiles would occur on BQ, either both to the north, or both to the south, but both are impossible. If Tile 4 has its b edge on BQ then $2a-b$ is an integer linear combination of (a,b,c), which is impossible. Hence Tile 4 has its c edge on BQ. Hence $c = 2a$ and Tile 4 has a vertex on AC as shown in

Fig. 21. Let Tile 6 be south of Tile 4. Then Tile 6 has its β angle at Q and hence does not have its b edge along Tile 4. Since $2a = c > b$, the tile or tiles south of Tile 4 do not terminate at the eastern boundary of Tile 4, but continue to the east. Let Tile 7 be the tile east of Tile 4. Then Tile 7 must share its a edge with Tile 4, since it cannot extend to the south. But that is impossible, as then it would have a γ angle on the shared a edge, but that cannot occur at B on the north or at the southeast vertex of Tile 4 either. We have reached a contradiction. That completes the proof in Case 2B.

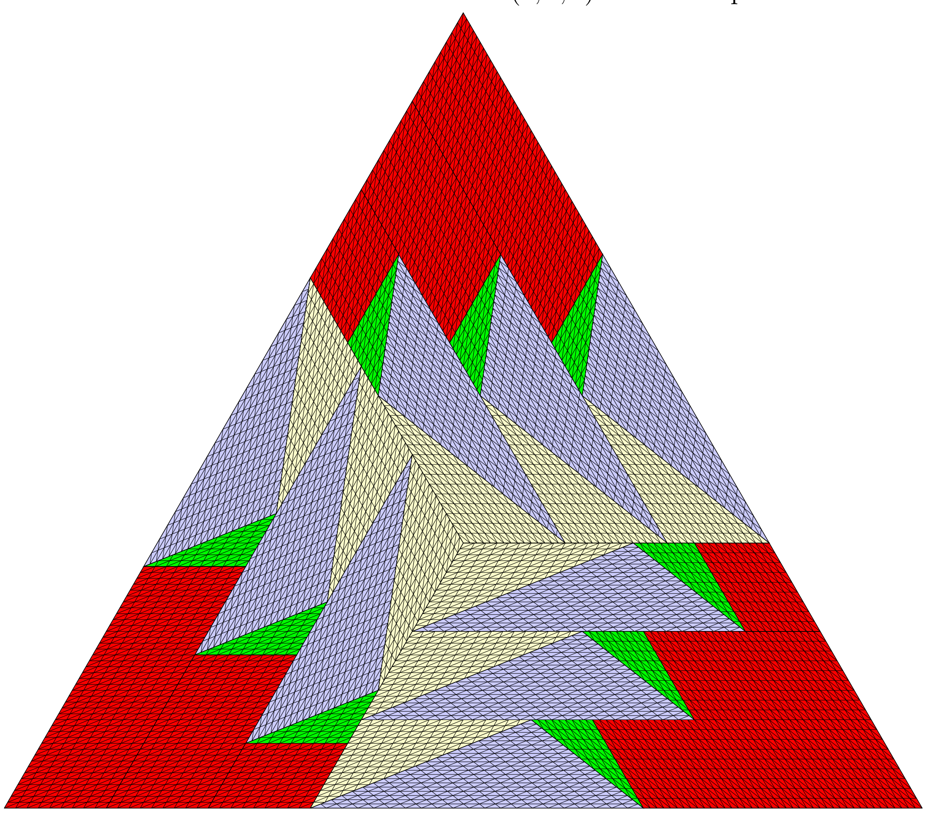
Case 2C: The angle of ABC at A is 2α and AB supports exactly two tiles. Then again the two tiles on AB have their c edges on AB. As before let Tile 1 have a vertex at A. Let Tile 8 be the other tile with a vertex at A, south of Tile 1. Then Tile 8 either extends east of Tile 1, or has its γ angle at the shared eastern vertex P. Tile 2 cannot be placed with its γ angle also at P, making P a "center" with three γ angles, since in that case the b side of Tile 2 would extend past Tile 8, which is impossible as that would go outside ABC. Therefore, whatever the position of Tile 8, Tile 2 must be placed as before, with its a edge shared with Tile 1. Then Tile 3 must be placed as before also, and as before Tile 4 must terminate the southern boundary of Tile 2 from extending eastwards. Let Tile 5 be the tile south of Tile 2. Since the southern boundary of Tile 2 is terminated at both ends. Assume, for proof by contradiction, that Tile 5 does not share its c edge with Tile 2. Then c is a combination of tile edges a and b, which implies that c is a multiple of a. Then Tile 5 has its a edge against Tile 2. Then its β angle or γ angle is at P, which implies that Tile 8 shares its b edge with Tile 1 and has its γ angle at P. But then, there is not room for Tile 5 to also have its γ angle at P. Hence Tile 5 has its β angle at P. Then there exists Tile 9 between Tile 8 and Tile 5, with the α angle of Tile 9 at P. Tile 9 lies next to the a edge of Tile 8, but since Tile 9 has its α angle at P, it must have its b or c side next to Tile 9, which is impossible as that side would extend outside ABC. That contradiction completes the proof that Tile 5 does share its c edge with Tile 2. Hence Tile 5 must occur in the position shown in Fig. 21. That is, Fig. 21 correctly shows Tiles 1,2,3,5 also in Case 2C, regardless of the position of Tile 8 (which is not shown in the figure).

Now, in Case 2B, the southern vertex of Tile 4 had to be the southeastern vertex of Tile 5. If that is so, then as before $c = 2a$, and the proof is completed as in Case 2B. Let P and Q be the southeastern vertices of Tiles 1 and 5, respectively. If some tile south of APQ blocks line BQ from continuing south of PQ, then the southern vertex of Tile 4 is Q, and as in Case 2B, $c = 2a$ and the proof can be completed. Hence, we can assume BQ does extend through APQ. Then the tiles south of Tiles 1 and 5 must share the b edges of those tiles. But that is impossible, as those two tiles would have their γ angles to the east, crossing BQ. That contradiction completes the proof in Case 2C.

If Case 2 holds then either AB or BC supports exactly two tiles. Renaming A and C if necessary, we can assume it is AB that supports exactly two tiles. Then either Case 2B or Case 2C applies. That completes the proof of the theorem.

Laczkovich proved that N-tilings of the kind discussed in this section exist, but did not actually exhibit any. Although in [6], he did not explicitly consider N at all, he did consider N in a theorem that he allowed Soifer to publish in [13]. In that theorem, he proved that for tilings of an equilateral triangle by a tile $(\alpha, \beta, 2\pi/3)$, the square-free part of N could be anything desired. Following Laczkovich's ideas, we found the tiling of Fig. 22, with $N = 10935$. Other tilings that we found require more than $32,000$ tiles (and so are too big to draw nicely on a normal page). What the smallest possible N is, we have no idea. For all we know, the construction method used for this tiling might yield a smaller N for some tile with very large sides; or there might be a much more efficient tiling construction yet to be discovered. In 2012 it was not known if there is an N-tiling of the equilateral triangle for every sufficiently large N, or if instead there are arbitrarily large N for which, like 7 and 11, there is no N-tiling at all. In unpublished work, we have proved N cannot be prime. Therefore there are arbitrarily large N for which there is no N-tiling.

Figure 22. $N = 10935$. The tile is $(3, 5, 7)$. ABC is equilateral.

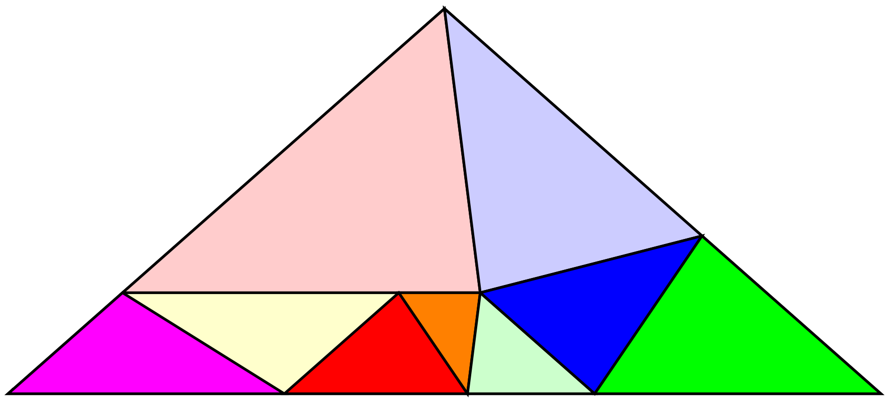


15. Tilings of an isosceles triangle with $\gamma = 2\alpha$

In this section we take up the row of Laczkovich's second table in which ABC is isosceles with base angles α and is tiled by a tile with $\gamma = 2\alpha$, and α is not a rational multiple of π. The condition $\gamma = 2\alpha$ can also be written as $3\alpha + \beta = \pi$. Unlike the similar-looking condition $3\alpha + 2\beta = \pi$, this condition does not imply $\gamma > \pi/2$. The vertex angle of ABC is then $\pi - 2\alpha = \alpha + \beta$.

Laczkovich [6] proves that, given any tile with $\gamma = 2\alpha$ and α not a rational multiple of π, an isosceles triangle can be dissected into triangles *similar* to the tile. Following the steps of his proof with the tile $(4, 5, 6)$, one finds the dissection shown in Fig. 23. To make an N-tiling, we have to tile each of these triangles

FIGURE 23. Laczkovich's dissection of isosceles ABC into triangles similar to $(4, 5, 6)$ can be used to produce a 5861172-tiling.



and the parallelogram with many copies of the same tile. Along each edge in the figure there is an arithmetical condition to satisfy. Working out those conditions, we find that more than five million tiles are required: 5861172 to be precise. It is not possible to print such a large tiling (unless one could use the side of a large building), and we do not know a smaller one. But at least, some such tilings do exist. Indeed, *many* such tilings exist.

The theory of such tilings has progressed far enough in [1] to prove that that tile is necessary rational, and from that plus the characterization of the tile given above, one can show that N cannot be prime. That is proved in [1]; in fact, it is proved that N cannot be twice a prime either. In particular, N cannot be 7, 11, 14, or 19; so we need not consider the case $\gamma = 2\alpha$ further in this paper. We merely remark that there is still a big gap between the smallest N for which a tiling of this kind is known to exist (over five million) and the lowest N for which we have not ruled out the existence of a tiling, currently $N = 20$. We are far from a characterization of the N for which there are N-tilings; but we did prove N cannot be prime.

16. Tilings of an equilateral triangle with α/π irrational

Laczkovich's Theorem 3.1 [6] says that, given a rational tile with an angle $\pi/3$, and the other angles not rational multiples of π, that tile will tile *some* an equilateral ABC. There are infinitely many such rational tiles, as Laczkovich proves. The two simplest ones are $(7, 5, 8)$ and $(7, 3, 8)$. Similarly when the tile has a $2\pi/3$ angle. Hence there are plenty of tilings of the kind considered here.

Laczkovich's second table has an entries for the cases when ABC is equilateral and the tile has either a $\pi/3$ or a $2\pi/3$ angle. The second table assumes not all the angles are rational multiples of π, so this entry also assumes that β is not a rational multiple of $\pi/3$. In Theorem 6, we proved that if the tile has a $2\pi/3$ angle then $N > 12$. Thus, for the purpose of proving N cannot be 7 or 11, we are already done with that case. Nevertheless, for very little extra work, we can push the lower limit on N higher for both cases of tilings of an equilateral triangle.

The main difference between the two cases is that when the tile has a $\pi/3$ angle, we can color the tiles black and white in such a way that the coloring theorem applies. That is not possible when the tile has a $2\pi/3$ angle, for example because it is possible (and necessary) for three tiles to meet at some vertex where all three have a $2\pi/3$ angle, and three tiles at one vertex cannot be colored.

In unpublished work, we have interesting results about the existence or non-existence of such tilings, including a proof that the tile ratios b/a and c/a can be computed from N and M, and that N cannot be prime, which certainly covers the cases 7 and 11. These proofs will not be presented here; instead we treat the problem computationally. Of course that covers only sufficiently small N. But it at least deals with $N = 7$ and 11.

There are two computational approaches to the problem of tiling an equilateral triangle; one uses the coloring equation and hence applies only to the case of a tile with a $\pi/3$ angle, but the other uses only the area equation, and applies just as well to both cases. The latter method, however, needs to use the fact that the tile is rational, which Laczkovich proved in 2012 [9], Theorem 3.3. The approach via the coloring equation does not use that result, which we may count in its favor, but the method based on the area equation works better, so we present the code for that method.

We will label the angles of the tile so that γ is the $\pi/3$ angle or the $2\pi/3$ angle. Then we have the law of cosines:

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos\gamma \\ a^2 &= b^2 + c^2 \pm bc \qquad (10) \end{aligned}$$

since $\cos\gamma = \pm\pi/3$. The plus sign corresponds to the case $\gamma = 2\pi/3$ and the minus sign corresponds to $\gamma = \pi/3$. The area equation is the same in both cases, since $\sin\gamma = \sin\pi/3$ in either case. If X is the length of each side of ABC,

$$\begin{aligned} X^2 \sin\pi/3 &= Nab\sin\gamma \\ X^2 &= Nab \qquad (11) \end{aligned}$$

In case $\gamma = \pi/3$, we also have the coloring equation

$$M(a + b + c) = 3X \qquad (12)$$

where M is the coloring number of the tiling.

Both computational approaches depend on writing

$$X = pa + qb + rc$$

for non-negative integers (p, q, r); this expression describes how the sides of ABC are composed of tile edges.

The following lemma gives some useful restrictions on the possibilities for (p, q, r).

Lemma 11. *Let the equilateral triangle ABC be tiled by a tile with (α, β, γ) and sides (a, b, c), where γ is either $(\pi/3)$ or $(2\pi/3)$. Possibly after a relabeling of the vertices of ABC, let $X = pa + qb + rc$ where X is the length of AB. Then*

(i) If $\gamma = \pi/3$, then $p \geq 1$ and $q \geq 1$

(ii) $r \geq 2$

Proof. Ad (i). If $\gamma = \pi/3$, then at the vertices of ABC there are tiles with altogether three a edges and three b edges. Therefore at least one side of ABC has one a edge and one b edge at its endpoints. Choosing that side for the decomposition of X, we have $p \geq 1$ and $q \geq 1$. We relabel the vertices so that side is AB.

If $\gamma = 2\pi/3$, we do not assert $p \geq 1$ and $q \geq 1$. But we do have, by Lemma 8, that there are at least two c edges on each side of ABC, so $r \geq 2$.

Now suppose $\gamma = \pi/3$. We have $r \geq 1$, since if there are no c edges on AB, then every tile supported by AB has a γ angle on AB, which contradicts the pigeonhole principle since the edges at the endpoints are a or b edges. I say that $r \geq 2$ holds also in case $\gamma = \pi/3$. Indeed if there is only one c edge on AB, and a total of n tiles supported on AB, then there are $n-1$ tiles with a γ angle on AB, and $n-1$ possible vertices for them, so the one tile with a c edge is bordered by two tiles with their γ angles adjacent. Let PQ be the c edge of Tile 3 supported by AB, and let Tiles 1,2,3,4,5 be adjacent tiles in order, so that Tiles 1,3,5 are supported by AB. Tiles 1 and 5 have their γ angles at P and Q respectively. Since $\gamma = \pi/3$ we have $\alpha + \beta = 2\pi/3$, so $\alpha < \gamma < \beta$. Hence $a < c < b$. Renaming P and Q if necessary, we may assume that Tile 3 has its α angle at P and its β angle at Q. Then Tile 2 has its β angle at P and Tile 4 has its α angle at Q. Let R be the third vertex of Tile 3. Then RQ has length a since it is opposite the α angle of Tile 3. The adjacent edge of Tile 4 is not the a edge, since Tile 4 has its α angle at Q. Hence Tile 2 has its b edge matching the b edge of Tile 3 along PR, since the c edge is too long to fit, and if the a edge were there, the remaining part $b-a$ cannot be tiled unless b is an integer multiple of a. But b cannot be an integer multiple of a, for then by the law of cosines,

$$\begin{aligned} c^2 &= a^2 + b^2 - ab \\ &= a^2 + (ma)^2 - a(ma) \\ &= a^2 \end{aligned}$$

Hence $c = a$. Hence by the law of sines $\alpha = \gamma$, contradicting the assumption that α is not a rational multiple of π. Hence Tile 2 has its b edge along PR. But that is impossible, since it has its β angle at P. That completes the proof of the lemma.

Our algorithm is going to check all possible values of (p, q, r) and try to solve the area equation. The time that takes will clearly depend on how large (p, q, r) can be in terms of N. Of course each of (p, q, r) is at most N since there are at most N tiles altogether. But for efficiency of computation, we want a better bound. We improve the bound in the next two lemmas by a factor of 6; still crude, but enough for our purposes.

Lemma 12. *Let the equilateral triangle ABC be N-tiled by a tile with sides (a, b, c) and α/π irrational. If $\gamma = 2\pi/3$ then no tile touches two different sides of ABC. If $\gamma = \pi/3$ then exactly three tiles touch two different sides of ABC.*

Proof. First assume $\gamma = 2\pi/3$. Since we can relabel the vertices of ABC, it suffices to show that no tile touches both AB and BC. We have $a < c$ and $b < c$ since $\alpha + \beta = \pi/3 < \gamma$. Renaming α and β if necessary, we may suppose $\alpha < \beta$. Suppose PQ is a tile edge with P on AB and Q on AC. Then angle APQ plus angle AQP is $2\pi/3$, since the sum of the angles in triangle APQ is π. Then one of those two angles is $\leq \pi/3$. Without loss of generality we can assume it is angle APQ. Then angle APQ is either equal to $\pi/3$ or to α. If AQ does not support a tile with its c edge on AQ, then each tile supported by AQ has its γ angle on AQ, and by the pigeonhole principle, the tile at Q has its γ angle at Q, contradiction. Therefore

$\overline{AQ} \geq c$. In triangle APQ, the angle opposite PQ (namely $\pi/3$) is greater than or equal to the angle opposite AQ (namely angle APQ). Therefore the length x of PQ is greater than or equal to the length of AQ. Hence $x \geq c$. But x is the length of a tile edge, and c is the longest tile edge, so $x = c$ and we have equality throughout, i.e., $AQ = c$ and triangle APQ is equilateral. Now we have an equilateral triangle APQ tiled by some number n of tiles, with the side of APQ equal to c. The area equation tells us $c^2 = nab$. Let $g = gcd(a, b)$. Then g^2 divides c; but (a, b, c) have no common factor, so $g = 1$. By the law of cosines,

$$\begin{aligned} c^2 &= a^2 + b^2 + ab \\ (n-1)ab &= a^2 + b^2 \qquad \text{since } c^2 = nab \end{aligned}$$

Taking this equation mod a we find $b^2 \equiv 0 \bmod a$. Since b and a are relatively prime, that is a contradiction. That completes the proof in case $\gamma = 2\pi/3$.

Now we assume $\gamma = \pi/3$. Then $\alpha < \gamma < \beta$, so $a < c < b$. At each vertex of ABC, there is a single tile with its γ angle at the vertex. Its c side therefore does touch two sides of ABC. We have to prove that no other tile touches both AB and AC. Suppose to the contrary that P lies on AB and Q on AC and PQ is an edge of a tile in the tiling. Let x be the length of PQ, so x is one of (a, b, c). Assume, for proof by contradiction, that triangle APQ is equilateral. What is the length x of the sides of APQ? Since the tiles at A have their γ angles at A, they have their a or b edges on AP and AQ. If either has its b edge there, then $x \geq b$, so $x = b$. If neither has its b edge there, then they both have their b edges along a line from A to an interior point on PQ. Such a line is shorter than the side of an equilateral triangle, so again $x \geq b$. In this case $x > b$, which is impossible since x is one of (a, b, c) and b is the largest of these. Therefore $x = b$. Now triangle APQ is n-tiled for some n, and the area equation tells us

$$\begin{aligned} b^2 &= nab \\ b &= na \end{aligned}$$

By the law of cosines

$$\begin{aligned} c^2 &= a^2 + b^2 - ab \\ &= a^2 + (na)^2 - a(na) \\ &= a^2 \end{aligned}$$

Hence $a = c$, contradiction. Therefore triangle APQ is not equilateral.

Now consider angles APQ and AQP; one of these angle is less than or equal to $\pi/3$, since the sum of the angles of triangle APQ is π and angle A is $\pi/3$. Since triangle APQ is not equilateral, one of those two angles is strictly less than $\pi/3$. Without loss of generality we may assume it is angle APQ. Since PQ is part of the tiling, angle APQ must be α. Then angle AQP is β, and triangle APQ is similar to the tile. Then for some $\lambda > 0$ we have $x = \lambda c$ and $AQ = \lambda b$ and $AP = \lambda a$. Since x is one of (a, b, c) we consider the possibilities one by one. If $x = c$ then $\lambda = 1$ and $AQ = b$ and $AP = a$. Then the two tiles at A form a parallelogram with sides a and b, and PQ is the diagonal, hence not a part of the tiling. That rules out the case $x = c$. If $x = b$ then $\lambda = b/c$ and $AQ = b^2/c$ and $AP = ab/c$. Since $a < b < c$ that would make $AP < a$, which is impossible since AP must support at least one tile. That rules out $x = b$. Therefore $x = a$. Then $\lambda = a/c$ and $AQ = ab/c < a$, which is also impossible. That completes the proof of the lemma.

Lemma 13. *Let the equilateral triangle ABC be N-tiled by a tile with sides (a, b, c), and let X be the length of AB. Suppose $X = pa+qb+rc$. Then $p+q+r \le N/6+1$.*

Proof. There are three tiles at each boundary vertex of ABC. Suppose there are k tiles supported by AB. Then there are $2k-1$ tiles with an edge or vertex on AB.

First we assume $\gamma = 2\pi/3$. Then there are two tiles at each vertex, and by Lemma 12, no tile has a vertex on two different sides of ABC. Then there will be no double-counting of tiles when we triple that number: there are at least $3(2k-1)$ different tiles with an edge or vertex on the boundary. Hence $6k-3 \le N$, so $k = p+q+rle(N+3)/6 \le N/6+1$.

Now assume $\gamma = \pi/3$. Then when we triple the number $3(2k-1)$, we have double-counted the single tiles at the vertices of ABC, and also there are three tiles with a vertex on two sides of ABC that will be double-counted. But by Lemma 12, only those three tiles touch two different sides of ABC. Hence

$$\begin{aligned} 3(2k-1)-6 &\le N \\ 6k &\le N+6 \\ p+q+r &\le N/6+1 \end{aligned}$$

That completes the proof of the lemma.

Lemma 14. *Let the equilateral triangle ABC be N-tiled by a tile with $\gamma = \pi/3$ or $2\pi/3$, and α not a rational multiple of π. Then $N \ge 40$. If $N \le 75$ then the possible values of N and the associated tiles are given in Table 5.*

TABLE 5. Possible N and (a, b, c) for equilateral tilings

N	γ	the tile
40	$\frac{\pi}{3}$	$(5, 8, 7)$
54	$\frac{\pi}{3}$	$(3, 8, 7)$
56	$\frac{2\pi}{3}$	$(7, 8, 13)$
60	$\frac{2\pi}{3}$	$(3, 5, 7)$
65	$\frac{\pi}{3}$	$(9, 65, 61)$
66	$\frac{2\pi}{3}$	$(11, 24, 31)$
70	$\frac{\pi}{3}$	$(7, 40, 37)$
80	$\frac{2\pi}{3}$	$(5, 16, 19)$
84	$\frac{\pi}{3}$	$(16, 20, 19)$
85	$\frac{\pi}{3}$	$(17, 80, 73)$

Proof. After a suitable relabeling of the vertices of ABC so that Lemma 11 will apply, let X be the length of AB and let the tiling determine the integers (p, q, r) such that $X = pa+qb+rc$. According to the area equation (11) we have

$$\begin{aligned} X^2 &= Nab \\ (pa+qb+rc)^2 &= Nab \end{aligned}$$

By the law of cosines (10), we have

$$c = \sqrt{a^2+b^2 \pm ab}$$

Putting that into the area equation, we have

$$(pa + qb + r\sqrt{a^2 + b^2 \pm ab})^2 \quad = \quad Nab$$

Define $s := a/b$ and divide the equation by b^2:

$$(ps + q + r\sqrt{s^2 + 1 \pm s})^2 \quad = \quad Ns$$

Expanding the left side we have

$$\begin{aligned}(ps + q)^2 + r^2(s^2 + 1 \pm s) + 2(ps + q)r\sqrt{s^2 + 1 \pm s} \quad &= \quad Ns \\ 2(ps + q)r\sqrt{s^2 + 1 \pm s} \quad &= \quad Ns - (ps + q)^2 - r^2(s^2 + 1 \pm s)\end{aligned}$$

Squaring both sides we have

$$4(ps + q)^2 r^2(s^2 + 1 \pm s) \quad = \quad (Ns - (ps + q)^2 - r^2(s^2 + 1 \pm s))^2 \tag{13}$$

That is a fourth-degree polynomial equation in s. Since the tile is known to be rational, and since we can assume $a < b$, we are looking for rational solutions s with $0 < s < 1$. If there is a tiling, there will be such a solution. Since SageMath can solve quartic equations, and test whether the solutions are rational, we are almost finished: we just need to bound the possible values of (p, q, r).

We make use of Lemma 13 to use the (still crude) bound $N/6 + 1$. Now the algorithm is simple: given N, check all the possibilities (p, q, r) that satisfy the conditions of Lemma 11 and satisfy $p + q + r \leq N/6 + 1$. Solve (13). Reject solutions that are not real and solutions that are not rational and solutions that are not between 0 and 1. If there are no solutions remaining, then there is no N-tiling. If there is a solution, the method is inconclusive. The code is shown in Fig. 24. We put the function defined there in an outer loop and ran it over N from 3 to 47. It finds solutions only for the values of N mentioned in the statement of the lemma. That completes the proof, at least, if you believe the code is correctly written and correctly executed.

We took the following steps to check this code for correctness. We wrote this code independently in SageMath and in C, and, commenting out the line to reject rational solutions, printed out the solutions that are found. The C code only works with floating-point numbers, so to compare the results, we had SageMath numerically evaluate its exact solutions. The C and SageMath code printed out the same decimal values, which we found reassuring. Since the C code works with floating-point numbers, it is not so easy to have it reject irrational solutions, so we could not replace the SageMath code with more efficient C code.

FIGURE 24. SageMath code for ruling out equilateral N-tilings

```
def feb24(N,an):  # an = -1 for pi/3, 1 for 2pi/3
# search for equilateral gamma = pi/3 or 2pi/3 solutions
# of area equation and boundary conditions.
   var('s')
   if an == -1:    # gamma at A,B,C
      Abound = 1   # at least one a edge
      Bbound = 1    # at least one b edge
   else:           # alpha + beta at A,B,C
      Abound = 0
      Bbound = 0
   upperbound = N/6 + 2
   for p in range(Abound,upperbound):
      for q in range(Bbound,upperbound):
         for r in range(2,upperbound):  # at least 2 c edges in either case
            if p+q+r >= floor(upperbound):
               continue
            eq = 4*(p*s + q)^2*r^2*(s^2+1+an*s)
                - (N*s-(p*s+q)^2 - r^2*(s^2+1+ an*s))^2
            answers = solve(eq,s)
            for t in answers:
               S = t.rhs();
               if not S in RR:
                  continue
               if not S in QQ:
                  continue      # tile is known to be rational
               if S <= 0 or 1 <= S:  # we can assume a < b so s<1
                  continue
               A = S.numerator()
               B = S.denominator()
               C = B * sqrt(S^2+1+ an*S)
               if not C in QQ:
                  continue;
               if not C in ZZ:
                  print("oops")   # it better be in ZZ!
                  print(A,B,C);
               g = gcd(A,gcd(B,C));
               a = A/g
               b = B/g
               C = C/g
               print("found (%d, %d, %d)" %(a,b,c))
   return true
```

17. No 7-tilings

We break the proof that there are no 7-tilings or 11-tilings into two cases, according as the angles are commensurable or not. All the required cases have already been dealt with: it only remains to put the pieces together.

Theorem 7. *Suppose (α, β, γ) are all rational multiples of 2π. Then there is no 7-tiling of any triangle ABC by a tile with angles (α, β, γ). Moreover, there is no N-tiling by such a tile for $N = 11, 14, 19, 31$ or any number which is neither a square, sum of squares, or 2, 3, or 6 times a square.*

Remark. Any odd N which is not divisible by 3 but whose squarefree part is divisible by some prime congruent to 3 mod 4 meets the conditions of the theorem.

Proof. Assume, for proof by contradiction, that there is such a tiling. By Theorem 3, the pair ABC and (α, β, γ) (after a suitable renaming of the angles) occurs in Table 3. But 7 does not match any of the forms of N listed in that table, which are the forms listed in the final sentence of the theorem. That completes the proof.

Finally we have arrived at the main theorem.

Theorem 8. *There are no 7-tilings or 11-tilings.*

Proof. Suppose, for proof by contradiction, that triangle ABC is N-tiled by a tile with angles (α, β, γ), with $N = 7$ or 11. We also note the cases where the proof works for $N = 14$ and 19. Since N is not a square or a sum of two squares, then by [12], ABC is not similar to the tile. Then according to Theorem 7, not all the angles (α, β, γ) are rational multiples of π. Then according to [6], the tiling must correspond to one of the rows in Table 4 in this paper.

For our purpose these rows will be combined into five cases: Either $3\alpha + 2\beta = \pi$, or $\gamma = 2\pi/3$, or ABC is isosceles with base angles α and $\gamma = \pi/2$, or ABC is isosceles with base angles α and $\gamma = 2\alpha$, or ABC is equilateral and $\alpha = \pi/3$.

In case ABC is equilateral and $\alpha = \pi/3$, Lemma 14 tells us there is no 7-tiling or 11-tiling (but the proof does not work for $N = 14$ or $N = 19$).

In case ABC is isosceles with base angles α and $\gamma = \pi/2$, then by [1], N is a square, or a sum of squares, or six times a square. Hence N cannot be $7, 11, 14$, or 19.

In case ABC is isosceles with base angles α and $\gamma = 2\alpha$, by [1], N is not prime or twice a prime. In particular N is not $7, 11, 14$, or 19.

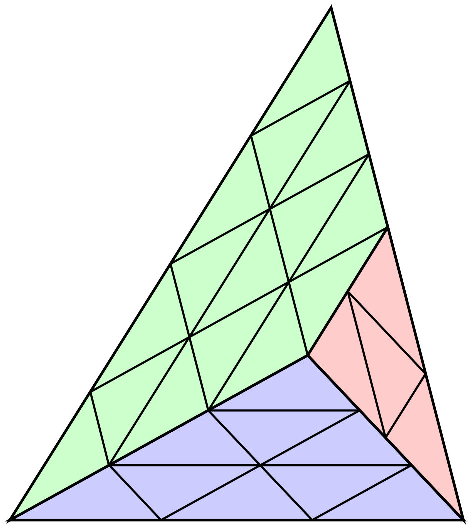
In case $3\alpha + 2\beta = \pi$, Theorem 5 tells us there is no 7-tiling or 11-tiling. In case $\gamma = 2\pi/3$, by Theorem 6 there is no 7-tiling or 11-tiling. That completes the proof of the theorem.

18. Concluding Remarks

This paper has successfully avoided the need to appeal to the hundred pages of theoretical work on the case $3\alpha + 2\beta = \pi$, as well as more than thirty pages on the equilateral case, although we did appeal to our work on the isosceles case, since the computational approach was not much shorter. Instead we have used algebraic and computation shortcuts that work only for small values of N.

In this section we nevertheless mention some results of those lengthier investigations. First, in each of the three cases ($3\alpha + 2\beta = \pi$, isosceles, equilateral), we used techniques pioneered by Laczkovich to prove that the tile has to be rational. Then we used the area equation and (for the $3\alpha + 2\beta = \pi$ case and one equilateral case) the "coloring equation" to derive necessary conditions. We used these equations to prove that N cannot be prime in some cases. The exceptions are as follow: There are the biquadratic tilings of a right isosceles triangle, in which case if N is prime

FIGURE 25. A tiling with $N = 28$ and $3\alpha + \beta = \pi$, and tile $(2, 3, 4)$



it must be congruent to 1 mod 4. Also, for an isosceles triangle tiled by a right triangle, we could only prove N cannot be a prime congruent to 3 mod 4, although we conjecture N has to be even (and hence not prime at all). And in case ABC is isosceles with base angles α, and $\gamma = 2\pi/3$, and α/π is not rational, we were unable to settle the question, although if N is prime in that case, $N = 2b + a$.

Generally we want to we expand the lines with in Table 4 by adding a third column with restrictions on the possible form of N. In some cases this is a necessary and sufficient condition; in others it is only a necessary condition. Where the necessary and sufficient conditions do not match, there are open questions. Whether there are yet-undiscovered tilings, or our necessary conditions are too weak, we do not know. Table 6 gives a summary of what we know about $N \le 100$.

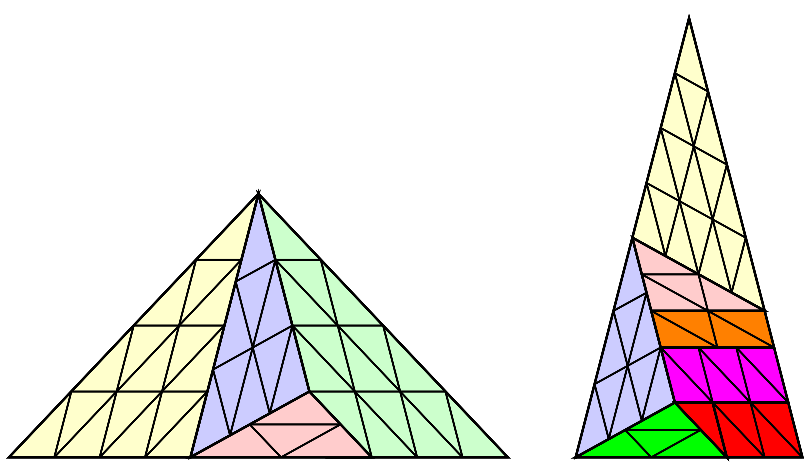
Figures 25 and 26 show some examples of tilings unknown before 2011 and 2108, respectively. We also have many examples of larger tilings.

Very little is known about the possible values of N for tilings of isosceles and equilateral triangles; proving that N cannot be prime is an advance, since for example in 2012 it was not known whether there are arbitrarily large N such that no equilateral triangle can be N-tiled by a tile whose angles are not all rational multiples of π. Similar for the case of tiling an isosceles ABC by a tile with $\gamma = 2\alpha$. Now we know N can't be prime, but we still don't know if N can be even or not, and we don't know if N can be less than five million, although Laczkovich proved that any such tile must tile *some* isosceles ABC, so there do exist a lot of such tilings–we just don't know how big (or small) N can be.

TABLE 6. Knowledge about tilings with $N \le 100$ as of March, 2019

ABC shape	the tile	known N	values $<= 100$	least unknown N
equilateral	$(\frac{\pi}{6}, \frac{\pi}{6}, \frac{2\pi}{3})$	$6n^2$	$24, 72, 96$	
	$(\frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3})$	$3n^2$	$12, 27, 48, 75$	
	$(\alpha, \beta, \frac{\pi}{3})$	5861172	?	40?
	$(\alpha, \beta, \frac{2\pi}{3})$	10395	?	40?
right$(\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2})$	$(\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2})$	$n^2, 3n^2$	$4, 9 \ldots 81, 100$	
right$(\alpha, \beta, \frac{\pi}{2})$	$(\alpha, \beta, \frac{\pi}{2})$	$N = e^2 + f^2$		
isosceles-α	$(\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2})$	$6n^2$	$24, 72, 96$	
	$(\alpha, \beta, \frac{\pi}{2})$	$2n^2$	$2, 8, 18 \ldots$	
	$(\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2})$	$6n^2$	$24, 72, 96$	
	$(\alpha, \beta, 2\alpha)$		not p or $2p$	20?
	$(\alpha, \beta, \frac{2\pi}{3})$	1878500		33?
	$3\alpha + 2\beta = \pi$		84	70?
isosceles-β	$3\alpha + 2\beta = \pi$		44	59? 66? 71? 74?
				83? 92? 99?
isosceles-$\alpha + \beta$	$3\alpha + 2\beta = \pi$		48	45? 72? 75? 99?
$(\alpha, 2\alpha, 2\beta)$	$3\alpha + 2\beta = \pi$		77	
$(2\alpha, \beta, \alpha + \beta)$	$3\alpha + 2\beta = \pi$		28	
$(\alpha, \alpha + \beta, \alpha + 2\beta)$	$(\alpha, \beta, \frac{2\pi}{3})$			13?
$(\alpha, 2\alpha, 3\beta)$	$(\alpha, \beta, \frac{2\pi}{3})$			13?
$(2\alpha, 2\beta, \alpha + \beta)$	$(\alpha, \beta, \frac{2\pi}{3})$			13?
any ABC	similar to ABC	n^2	4,9,16,... 100	

FIGURE 26. Tilings with $N = 44$ and 48, with $3\alpha + \beta = \pi$ and tile $(2, 3, 4)$



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