TILINGS OF AN ISOSCELES TRIANGLE

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ABSTRACT. An N-tiling of triangle ABC is a way to cut ABC into N congruent smaller triangles. The smaller triangle is the "tile." When ABC is isosceles with base angles α, and not equilateral, there are only four possible tiles (aside from a tile similar to ABC): a right-angled tile with one angle α, a tile with angles $(\alpha, \beta, 2\alpha)$, a tile with angles $(\alpha, \beta, 2\pi/3)$, or a tile with angles satisfying $3\alpha + 2\beta = \pi$ (and in all but the first case, with α not a rational multiple of π). We study the first three cases in this paper.

For tilings by a right triangle, N has to be a square, or an even sum of squares, or six times a square; in particular it cannot be a prime congruent to 3 mod 4; and all these possibilities actually occur. We prove that unless ABC is a right isosceles triangle, N has to be even.

For tilings by $(\alpha, \beta, 2\alpha)$, we show that the tile is necessarily rational (the ratios of its sides are rational), and we give a necessary condition for the existence of a tiling. This condition implies that when an isosceles and not equilateral ABC is N-tiled by such a tile, N cannot be a prime number, or even squarefree.

In the last case, when the tile has a 120 degree angle, we also prove that the tile must be rational, and find a necessary condition for the existence of a tiling. That condition rules out $N < 36$, but leaves open whether N can possibly be prime. The smallest known such tiling has $N = 2736$.

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1. INTRODUCTION

An N-tiling of triangle ABC by triangle T is a way of writing ABC as a union of N triangles congruent to T, overlapping only at their boundaries. The triangle T is the "tile". We consider here the case of an isosceles (but not equilateral) triangle ABC.

Our results fit into a larger research program, begun by Lazkovich [7]. Laczkovich studied the possible shapes of tiles and triangles that can possibly be used in tilings, and obtained results that will be described below. The reader who is new to the subject may want to see examples of N-tilings for various shapes of ABC; such pictures can be found in [1]. Here we give only examples relevant to the case of ABC isosceles.

First we point out that *any* triangle can be decomposed into n^2 congruent triangles by drawing $n - 1$ equally spaced lines parallel to each of the three sides of the triangle, as illustrated in Fig. 1. Moreover, the large (tiled) triangle is similar to the small triangle (the "tile"). We call such a tiling a *quadratic tiling*.

FIGURE 1. A quadratic tiling of an arbitrary triangle
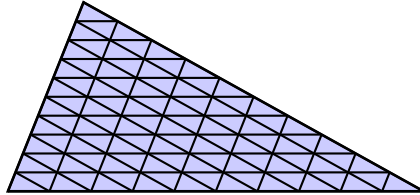
It follows that if we have a tiling of a triangle ABC into N congruent triangles, and m is any integer, we can tile ABC into Nm^2 triangles by subdividing the first tiling, replacing each of the N triangles by m^2 smaller ones. Hence the set of N for which an N-tiling of some triangle exists is closed under multiplication by squares.

Sometimes it is possible to combine two quadratic tilings (using the same tile) into a single tiling, as shown in Fig. 2. We will explain how these tilings are

FIGURE 2. Biquadratic tilings with $N = 13 = 3^2 + 2^2$ and $N = 74 = 5^2 + 7^2$
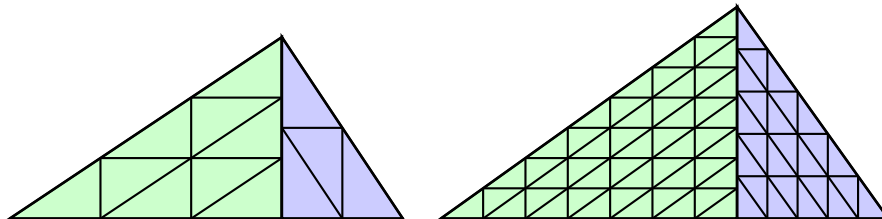
constructed. We start with a big right triangle resting on its hypotenuse, and divide it into two right triangles by an altitude. Then we quadratically tile each of those triangles. The trick is to choose the dimensions in such a way that the same tile can be used throughout. If that can be done then evidently N, the total number of tiles, will be the sum of two squares, $N = n^2 + m^2$, one square for each of the two quadratic tilings. On the other hand, if we start with an N of that form, and we choose the tile to be an n by m right triangle, then we can construct such a tiling. We call these tilings "biquadratic." More generally, a *biquadratic tiling* of triangle ABC is one in which ABC has a right angle at C, and can be divided by an altitude from C to AB into two triangles, each similar to ABC, which can be tiled respectively by n^2 and m^2 copies of a triangle similar to ABC. A larger biquadratic tiling, with $n = 5$ and $m = 7$ and hence $N = 74$, is shown in at the right of Fig. 2.

If the original triangle ABC is chosen to be isosceles, and is then quadratically tiled, then each of the n^2 triangles can be divided in half by an altitude; hence any isosceles triangle can be decomposed into $2n^2$ congruent triangles. If the original triangle is equilateral, then it can be first decomposed into n^2 equilateral triangles, and then these triangles can be decomposed into 3 or 6 triangles each, showing that any equilateral triangle can be decomposed into $3n^2$ or $6n^2$ congruent triangles. For example we can 12-tile an equilateral triangle in two different ways, starting with

a 3-tiling and then subdividing each triangle into 4 triangles ("subdividing by 4"), or starting with a 4-tiling and then subdividing by 3.

There is another family of N-tilings, in which N is of the form $3m^2$, and both the tile and the tiled triangle are 30-60-90 triangles. We call these the "triple-square" tilings. The case case $m = 2$ makes $N = 12$. There are two ways to 12-tile a 30-60-90 triangle with 30-60-90 triangle. One is to first quadratically 4-tile it, and then subtile the four triangles with the 3-tiling of Figure 1. This produces the first 12-tiling in Fig. 3. Somewhat surprisingly, there is another way to tile the same triangle with the same 12 tiles, also shown in Fig. 3. The next member of this family is $m = 3$, which makes $N = 27$. Two 27-tilings are shown in Fig. 4. Similarly, there are two 48-tilings (not shown).
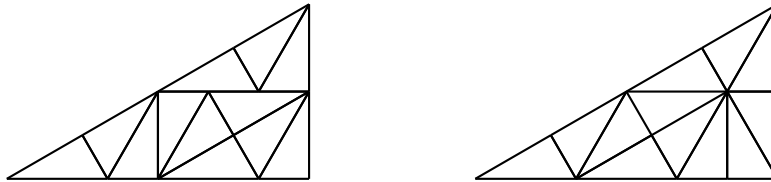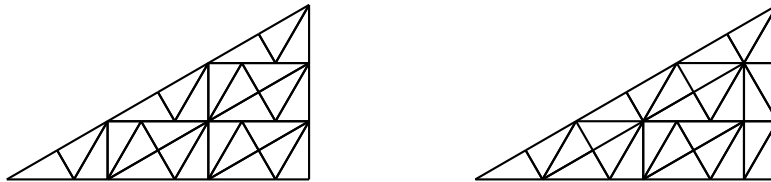
FIGURE 3. Two 12-tilings

FIGURE 4. Two 27-tilings

Whenever there is an N-tiling of the right triangle ABM, there is a $2N$-tiling of the isosceles triangle ABC. Using the biquadratic tilings (see Fig. 2) and triple-square tilings (see Fig 3 and Fig. 4), we can produce $2N$-tilings when N is a sum of squares or three times a sum of squares. We call these tilings "double biquadratic" and "hexquadratic". For example, one has two 10-tilings and two 26-tilings, obtained by reflecting Figs. 4 and 5 about either of the sides of the triangles shown in those figures; and one has 24-tilings and 54-tilings obtained from Figs. 8 and 9. Note that in the latter two cases, ABC is equilateral.

In the case when the sides of the tile T form a Pythogorean triple $n^2 + m^2 + k^2 = N/2$, then we can tile one half of ABC with a quadratic tiling and the other half with a biquadratic tiling. The smallest example is when the tile has sides 3, 4, and 5, and $N = 50$. See Fig. 7. One half is 25-tiled quadratically, and the other half is divided into two smaller right triangles which are 9-tiled and 16-tiled quadratically. This shows that the tiling of ABC does not have to be symmetric about the altitude.

As we shall see below, the work of Laczkovich implies that there are only four possible shapes of the tile: right-angled, $\gamma = 2\alpha$, $\gamma = 2\pi/3$, and $3\alpha + 2\beta = \pi$. The

last case is taken up in another paper, since the techniques apply also to tilings of non-isosceles triangles ABC with $3\alpha + 2\beta = \pi$. The first three are studied in this paper. We obtain, in the second and third case, necessary conditions on N, but not necessary and sufficient conditions. In the case of a right-angled tile, our conditions are necessary and sufficient.

All known examples of tilings of isosceles ABC with $\alpha \neq \frac{\pi}{2}$ have N even. We could prove that it must be so when the tile is right-angled, but we could not prove it in the other two cases, where indeed we know only a few tilings, all of which require N with five to seven digits.

1.1. **Acknowledgment.** I am grateful to Miklos Laczkovich for his valuable comments on my work and especially for his simplification of the proofs of Lemmas 7.1, 7.4, and 7.5, and of course for his many pioneering papers in this subject, on which this paper rests.

1.2. **Definitions and notation.** We first note that this paper is about triangles ABC that are isosceles and not equilateral. Let that be understood; then for the rest of this paper, "isosceles" means "exactly two sides are equal."[1]

We give a mathematically precise definition of "tiling" and fix some terminology and notation. Given a triangle T and a larger triangle ABC, a "tiling" of triangle ABC by triangle T is a set of triangles $T_1, \ldots, T_n$ congruent to T, whose interiors are disjoint, and the closure of whose union is triangle ABC.

Let a, b, and c be the sides of the tile T, and angles α, β, and γ be the angles opposite sides a, b, and c. The letter "N" will always be used for the number of triangles used in the tiling. An N-tiling of ABC is a tiling that uses N copies of some triangle T. The meanings of N, α, β, γ, a, b,c, A, B, and C will be fixed throughout this paper. We do not assume $\alpha \leq \beta$ in general; although that may sometimes be justified by symmetry, we often will consider some equation such as $3\alpha + 2\beta = \pi$, in which case we do not want to assume $\alpha \leq \beta$.

2. History

Above we exhibited quadratic and biquadratic tilings in which the tile is similar to ABC. There are hexagonal tilings, not exhibited in this paper, but see [1] for pictures. These involve N being square, a sum of two squares, or three times a square. The biquadratic tilings were known in 1964, when the paper [4] was published. This is the earliest paper on the subject of which I am aware.[2] Snover *et. al.* [13] took up the challenge of showing that these are the only possible values of N, when the tile is similar to ABC. The following theorem completely answers the question, "for which N does there exist an N-tiling in which the tile is similar to the tiled triangle?"

Theorem 2.1 (Snover *et. al.* [13])**.** *Suppose ABC is N-tiled by tile T similar to ABC. If N is not a square, then T and ABC are right triangles. Then either*

(i) N is three times a square and T is a 30-60-90 triangle, or

(ii) N is a sum of squares $e^2 + f^2$, the right angle of ABC is split by the tiling, and the acute angles of ABC have rational tangents e/f and f/e,

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[1] That was Euclid's definition of "isosceles."

[2] The simplest hexagonal tiling is attributed to Major MacMahon (1921) in the notes accompanying a plastic toy I purchased at an AMS meeting in 2012.

and these two alternatives are mutually exclusive.

Soifer's book [14] appeared in 1990, with a second edition in 2009. He considered two "Grand Problems": for which N can *every* triangle be N-tiled, and for which N can *every* triangle be dissected into similar, but not necessarily congruent triangles. (The latter eventually became a Mathematics Olympiad problem.) The 2009 edition has an added chapter in which the biquadratic tilings and a theorem of Laczkovich occur.

Miklos Laczkovich published six papers [6, 7, 8, 2, 9, 10] on triangle and polygon tilings. According to Soifer, the 1995 paper was submitted in 1992. Laczkovich, like Soifer, studied dissecting a triangle into smaller *similar* triangles, not *congruent* triangles as we require here. If those similar triangles are rational (i.e., the ratios of their sides are rational) then if we divide each of them into small enough quadratic subtilings, we can achieve an N-tiling into *congruent* triangles, but of course N may be large. Laczkovich focused primarily on the shapes of ABC (or more generally, convex polygons) and of the tile. His theorems give us an exhaustive list of the possible shapes of ABC and the tile, which we will need in our proof that there is no 7-tiling. This list can be found in §3 (of this paper). However, his theorem published in the last chapter of [14] does mention N. It states that given an integer k, there exists an N-tiling for some N whose square-free part is k.

3. Laczkovich

A basic fact is that, apart from a small number of cases that can be explicitly enumerated, if there is an N-tiling of ABC by a tile with angles (α, β, γ), then the angles α and β are not rational multiples of π. This theorem follows from Theorems 4.1, 5.1, and 5.3 of [7]. Laczkovich is dealing with a more general situation, tiling an arbitrary triangle by tiles that are only required to be similar, not congruent. We extract the following theorem from his results by specializing to isosceles ABC and congruent tiles.

TABLE 1. Possible tilings of isosceles triangles, according to Laczkovich.

ABC	the tile
$(\beta, \beta, 2\alpha)$	similar to ABC
$(\beta, \beta, 2\alpha)$	$\gamma = \pi/2$
$(\alpha, \alpha, \pi - 2\alpha)$	$\gamma = 2\alpha$
$(\alpha, \alpha, \pi - 2\alpha)$	$\gamma = 2\pi/3$
$(\alpha, \alpha, \alpha + 2\beta)$	$3\alpha + 2\beta = \pi$
$(\beta, \beta, 3\alpha)$	$3\alpha + 2\beta = \pi$
$(\alpha + \beta, \alpha + \beta, \alpha)$	$3\alpha + 2\beta = \pi$

Theorem 3.1 (Laczkovich [7]). *Let isosceles (and not equilateral) triangle ABC be N-tiled by a tile with angles (α, β, γ). Then the possible shapes of ABC and the tile are given by Table 1. In the table, the triples giving the angles of the tile are*

(α, β, γ) *after a suitable permutation, i.e., they are unordered triples. In all but the first two lines, α is not a rational multiple of π.*

Remark. For example, in the second line of the table, we do not list separately $(\alpha, \alpha, 2\beta)$, as that is already covered by the entry $(\beta, \beta, 2\alpha)$, and the fact that we do not assume $\alpha < \beta$.

Proof. This theorem is proved in [7], but it is not stated in quite this way; therefore we spell out in detail how this statement follows from theorems explicitly stated in [7]. Let isosceles (and not equilateral) triangle ABC be N-tiled by a tile with angles (α, β, γ). Then either all three angles are rational multiples of π, or not. Case 1, they are not all rational multiples of π. Then by Theorem 4.1 of [7], where T in that paper is our ABC, one of six cases holds. Cases (i), (ii), and (iv) are the first three lines of our table. Case (iii) says ABC is equilateral, which we have ruled out by hypothesis. Case(v) says $3\alpha + 2\beta = \pi$ and the base angles of isosceles ABC must be α or β or $\alpha + \beta$, by Theorem 2.4 of [7]; so that is lines 5, 6, and 7 of our table. Finally, case (vi) is has the tile $(\alpha, \alpha, \alpha + 3\beta)$ with $\gamma = 2\pi/3$, which is another way of writing line 4 of the table, since if $\gamma = 2\pi/3$, then $\alpha + 3\beta = \pi - 2\alpha$.

Case 2, all three of (α, β, γ) are rational multiples of π. Then Theorem 5.1 of [7] applies. That theorem is about dissections into similar (rather than congruent) triangles, and according to the subsequent Theorem 5.3, the last three cases (cases (v), (vi), and (vii)) in Theorem 5.1 cannot hold for dissections into congruent triangles. Cases (i) and (ii) are the first lines of our table. Case (iii) requires ABC equilateral, which we have ruled out by hypothesis. Case (iv) has ABC a right triangle with one angle $\pi/6$, which is not isosceles and hence irrelevant here. That completes the proof.

We note in passing the following immediate consequence of Laczkovich's theorem: If an isosceles triangle ABC is tiled by a right-angled tile $(\alpha, \beta, \frac{\pi}{2})$, then the base angles of ABC are either equal to β or to α. That follows, because in Table 1, there is only one entry corresponding to a right-angled tile, namely the second line. Readers are invited to try to prove that directly, without appeal to Laczkovich, in order to gain a deeper appreciation for Laczkovich's work.

4. Some number-theoretic facts

The facts in this section may not be well-known to all our readers, and their proofs are short.

Lemma 4.1. *An integer N can be written as a sum of two integer squares if and only if the squarefree part of N is not divisible by any prime of the form $4n + 3$.*

Proof. See for example [5], Theorem 366, p. 299.

Lemma 4.2. *N is a sum of two squares if and only if $2N$ is a sum of two squares.*

Proof. The lemma follows immediately from the identities

$$
\begin{aligned}
(p - q)^2 + (p + q)^2 &= 2(p^2 + q^2) \\
\left(\frac{p - q}{2}\right)^2 + \left(\frac{p + q}{2}\right)^2 &= \frac{1}{2}(p^2 + q^2).
\end{aligned}
$$

This lemma is also a corollary of Lemma 4.1, of course, but that is not needed.

The following lemma identifies those relatively few rational multiples of π that have rational tangents or whose sine and cosine satisfy a polynomial of low degree over $\mathbb{Q}$.

Lemma 4.3. *Let $\theta = 2m\pi/n$, where m and n have no common factor. Suppose $\cos\theta$ is algebraic of degree 1 or 2 over $\mathbb{Q}$. Then n is one of $5, 6, 8, 10, 12$. If both $\cos\theta$ and $\sin\theta$ have degree 1 or 2 over $\mathbb{Q}$, then n is $6, 8$, or 12.*

Proof. Let φ be the Euler totient function. Assume $\cos\theta$ has degree 1 or 2. By [12], Theorem 3.9, p. 37, $\varphi(n) = 2$ or 4. The stated conclusion follows from the well-known formula for $\varphi(n)$. The second part of Theorem 3.9 of [12] rules out $n = 5$ or 10 when $\sin\theta$ is also of degree 1 or 2.

Lemma 4.4 (Pythagorean triangles)**.** *The integer solutions of the equation $x^2 + y^2 = z^2$ have the form $(x, y, z) = (m^2 - k^2, 2mk, m^2 + k^2)$ for some integers (m, k)*

Proof. See any number theory textbook. But the proof is short, so we just give it here. By the Pythagorean theorem, (x, y, z) form a right triangle, with one angle α such that $x/z = \cos\alpha$ and $y/z = \sin\alpha$. We use the Weierstrass substitution, $t = \tan(\alpha/2)$. Then

$$\cos\alpha \;\; = \;\; \frac{1 - t^2}{1 + t^2} \qquad \sin\alpha \;\; = \;\; \frac{2t}{1 + t^2}$$

Setting $t = m/k$ in lowest terms, and replacing $\sin\alpha$ and $\cos\alpha$ by y/z and x/z, we find the formulas of the lemma for (x, y, z). That completes the proof.

Lemma 4.5. *If the integer n is a sum of two rational squares then it is a sum of two integer squares.*

Proof. Suppose $n = (p/q)^2 + (s/t)^2$. Then $(qt)^2 n = p^2 + s^2$. Then by Lemma 4.1, the square-free part of n is not divisible by any prime congruent to 3 mod 4. Then by a second application of Lemma 4.1, n is a sum of two integer squares. That completes the proof.

5. Tilings of an isosceles ABC by a right-angled tile: examples

FIGURE 5. A 54-tiling; $N/2$ is three times a square. Tile is 30-60-90.
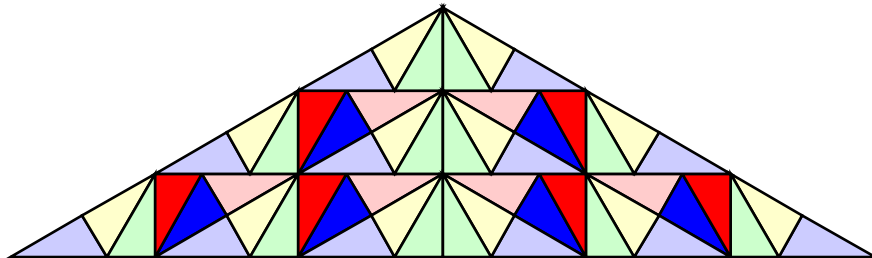
Is it possible to have more complicated tilings without essential segments? Yes, because when two tiles share their hypotenuses, they form a rectangle, and we can just draw the diagonal of that rectangle the other way. In this way we can produce (exponentially) many different tilings, but they differ only in this trivial way. And
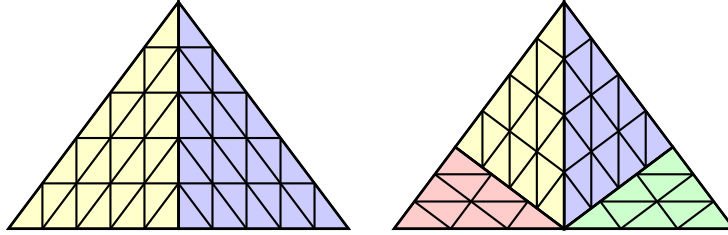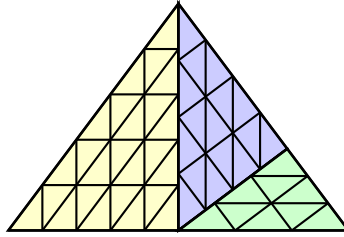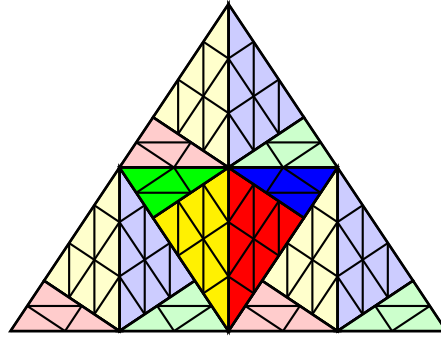
FIGURE 6. N is twice a square or twice a sum of squares.

FIGURE 7. 50 is both twice a square and twice a sum of squares.

FIGURE 8. $N = 104$, eight essential segments, base angles about $56°$

sometimes, as shown in Fig. 9, even those rectangles can be rotated. That figure also shows that a tiling need not necessarily include the altitude of ABC.

In the tilings based on two biquadratic tilings, there are no c edges on AB and BC, while in the tilings based on two quadratic tilings, there are only c edges. There are of course some hybrid tilings when a square is also a sum of squares, in which AB falls under one case and BC under the other. If $N/2$ is not a square (as is the case for the biquadratic tilings) then there are no c edges on AB and BC, as we see in the biquadratic tilings (and prove in the next section).

All these tilings, in which $N/2$ is a sum of squares, involve essential segments (where tiles of different lengths occur on the two sides of an internal line). One sees such linear relations in two of the tilings illustrated in Fig. 6.

FIGURE 9. The altitude need not be part of the tiling.
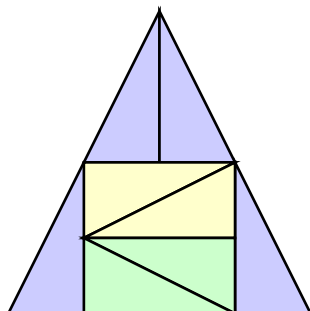
FIGURE 10. $N = 2312$, $N/2 = 34^2$, $(a, b, c) = (3, 4, 5)$
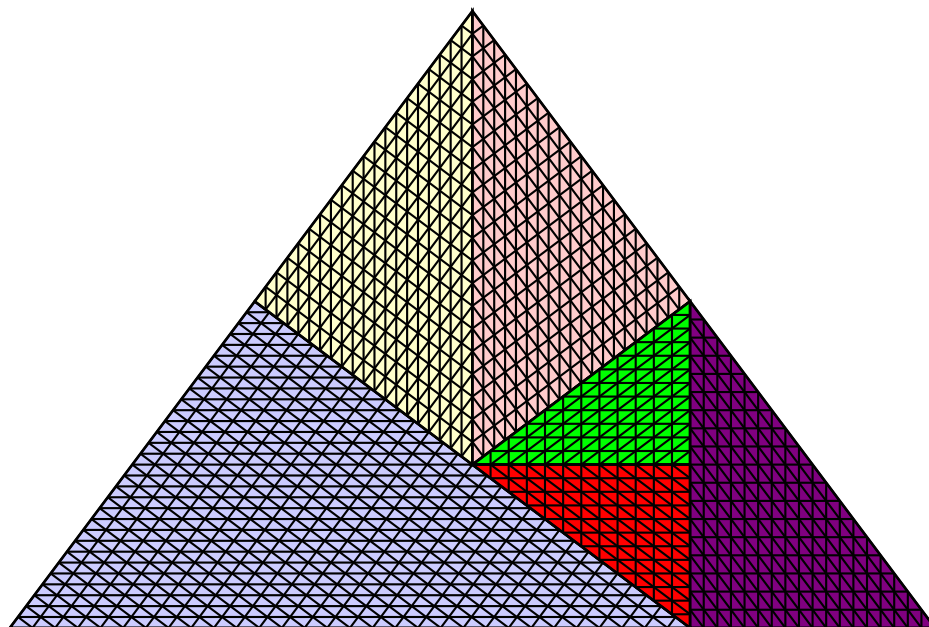
6. LACZKOVICH'S GRAPHS Γ_a

In trying to prove the impossibility of certain tilings directly, it is easy to become involved in complicated arguments with many cases, involving complicated diagrams. Laczkovich had the brilliant idea to abstract some of these arguments using graph theory. The definition will not be grasped immediately, but instead will require time and the study of examples to understand. But it leads to very elegant proofs of theorems that are much more complicated or impossible to prove more directly. To emphasize its importance, we devote a whole section just to the definition.

Given a tiling of a triangle ABC, an *internal segment* is a line segment connecting two vertices of the tiling that is contained in the union of the boundaries of the tiles, and lies in the interior of ABC except possibly for its endpoints. A *maximal*

segment is an internal segment that is not part of a longer internal segment. A segment is *terminated* at a vertex P if it has tiles on both sides with vertices at P. (In that case there may or may not be a continuation of that segment past P.) A *left-terminated segment* is an internal segment XY that is terminated at X. A *left-maximal segment* is an internal segment PQ that cannot be extended past P to a longer internal segment UPQ. (In these two concepts, we are using directed segments; so PQ is not the same as QP in this context. The "left" in "left maximal" refers to the fact that P is listed to the left of Q in PQ.) A tile is *supported by* XY if one edge of the tile lies on XY. The internal segment XY is said to have "all c's on the left" if the endpoints X and Y are vertices of tiles supported by XY and lying on the left side of XY, and all tiles supported by XY lying on the left of XY have their c edges on XY. Similarly for "all c's on the right." (Here again XY is a directed segment, so the concept "left side" of XY makes sense; but this is a different sense of the English word "left" than in "left-terminated.")

An internal segment XY is said to *witness the relation* $jc = \ell a + mb$ if one side has j more c edges than the other, and the other has ℓ more a edges and m more b edges than the first. The simplest example is when XY has all c's on one side, and exactly j of them (that is, the length of XY is jc), and on the other side XY supports ℓ tiles with their a edges on XY and m tiles with their b edges on XY (in any order) and no other tiles, and the endpoints X and Y are vertices of tiles on both sides of XY. Similarly we use the terminology "XY witnesses a relation $jb = \ell a + mc$."

An internal segment that witnesses a relation is called an *essential segment.* The definition allows that an essential segment might have different numbers of tiles of lengths a, b, c on its two sides, without necessarily having all the tiles on one side be the same length, but often it is the case that all the tiles on one side are the same length.

To be sure that you understand the concept of "essential segment", identify the eight essential segments in Fig. 8. Also identify in that figure some internal segments PQ that are not essential segments, because each side of PQ supports tiles with three b and two a sides on PQ. (Those PQ connect the midpoints of the sides of ABC in that figure.)

The following definition is equivalent to the one given in [10, p. 346], except that there condition (iv) is automatic because of an additional assumption about the tile.

Definition 6.1 (The directed graph Γ_a). *Given a tiling of some triangle, the nodes of the graph Γ_a are certain vertices of the tiling. A link of Γ_a connects vertices X and Y if*

(i) the segment XY is a left-maximal internal segment having all a edges on one side (say Side 1) of XY, and

(ii) On the other side of XY (say Side 2) the first tile (the one with a vertex at X) does not have its a edge on XY, and

(iii) At vertex Y, there is another tile supported by XY on Side 1 of XY with a vertex at Y, that does not have its a edge on Side 1, and

(iv) No tile supported by XY on Side 2 of XY has a vertex at Y.

The directed graphs Γ_b and Γ_c are defined similarly.

Remarks for clarification. We use "link" instead of "edge" with these graphs, to avoid confusion with tile "edges." If XY is a link in Γ_a, then XY does not terminate at Y, because there is another tile past Y whose a side is not on XY; and also because Y lies on the interior of an edge of a tile on the other side of XY.

The reader is recommended to identify all three graphs Γ_a, Γ_b, and Γ_c in Fig. 8. Hint: Γ_c is empty; Γ_b has four links, starting at the midpoints of the sides of ABC; Γ_a is, in this example, the same graph as Γ_b.

7. Tilings of an isosceles ABC by a right-angled tile: theory

Laczkovich studied the possible shapes of tiles that can tile an isosceles triangle, but did not characterize the possible N. We do so in this section for right-angled tiles and N even. We have two ways to tile an isosceles triangle by a right triangle: either tile each of its two halves by a quadratic tiling, in which case N is twice a square, or tile each of its halves with a biquadratic tiling, in which case N is twice a sum of squares. See Figs. 6 and 7. The main theorem in this section shows that these are the only possible values of N. But Fig. 10 shows that, when $N/2$ is a square, there are also more complicated tilings.

Lemma 7.1. *Suppose isosceles (or equilateral) triangle ABC with base angles β is N-tiled by tile $(\alpha, \beta, \pi/2)$ with sides (a, b, c), and α is not a rational multiple of π, or α is an odd multiple of β. Let PQ be a link in Γ_c. Then there are two adjacent tiles with vertices at Q whose common boundary contains an a or b edge of one tile, and a c edge of the other tile.*

Remarks. For short, there is an a/c edge at Q, or a b/c edge at Q. Consider Fig. 5, in which $\beta = \pi/6$ and $\alpha = \pi/3$, so α is not an odd multiple of β. Observe that the present lemma fails in the tiling of Fig. 5, showing that the hypothesis that α is an odd multiple of β cannot be removed.

Proof. Let $\Delta_1, \ldots, \Delta_k$ be tiles with vertices at Q, numbered so that Δ_1 is supported by PQ (and hence has side c on PQ), Δ_i and Δ_{i+1} are adjacent, and Δ_k has one edge extending PQ past Q. Since PQ is a link in Γ_c, there does exist such a tile Δ_k, and Δ_k has its a or b edge extending PQ. (There may or may not be tiles on the other side of PQ with a vertex at Q, but if so, we do not list them among the Δ_i.)

If there are an even number of tiles with vertices at Q and an α or β angle at Q, then each has a c edge ending at Q. Since Δ_1 has its c edge ending at Q and Δ_k does not, the remaining odd number of c edges cannot all be paired with other c edges supported by the same line. Therefore, there is an a/c or a/b edge, as claimed. Therefore, we may assume the number of such tiles is odd. At most one tile can have its right angle at Q, and it cannot be Δ_1.

Now, if α is not a rational multiple of π, then either $k = 2$ and both angles are right angles, or $k = 4$ and there are two α and two β angles, or $k = 3$ and there are one each of $(\alpha, \beta, \frac{\pi}{2})$. In all these cases, the above condition that there are an even number of tiles at Q with an α or β angle at Q is fulfilled.

If α is an odd multiple m of β, then the same argument works, as the number of tiles with vertices at Q that do not have their right angle at Q will still be even.

That completes the proof of the lemma.

Lemma 7.2. *Suppose isosceles (or equilateral) triangle ABC with base angles β is N-tiled by tile $(\alpha, \beta, \pi/2)$ with sides (a, b, c), and α is not a rational multiple of π,*

*or α is an odd multiple of β. Suppose there is no relation $jc = ua + vb$ with $j > 0$
and $u, v \geq 0$, and u, v, j integers. Then*

*(i) Let PQ be a link in Γ_c. Then there is a vertex R such that QR is a link in
Γ_c.*

(ii) The in-degree and out-degree of each node of Γ_c is exactly 1.

Remark. Note that in Fig. 8, it is not true that the in-degree of every vertex in Γ_b
is equal to the out-degree of that vertex. That is because, in that tiling, there is
a relation $2b = 3a$. Please take the time to verify that in Fig. 5, the conclusion of
this lemma fails, but so does the hypothesis that α is an odd multiple of β; this
shows the necessity of that hypothesis.

Proof. According to Lemma 7.1, there is an outgoing a/c edge from Q. Let R lie
on the internal segment containing that edge, as far as possible from Q such that
there are only c edges on one side of QR. Then R is not a vertex of a tile on the
other side of QR, since that would give rise to relation $jc = ua + vb$, where j is
the excess of the number of c edges on one side of QR over the other. Then, by
definition of Γ_c, QR is a link in Γ_c. That completes the proof of (i).

Next we observe that the in-degree of each node Q of Γ_c is at most 1. For, if PQ
is a link of Γ_c, then PQ is part of an internal segment of the tiling that extends
past Q and on one side, is not a vertex of any tile on that side. Hence no other
internal segment can pass through Q; hence there is no other link of Γ_c ending at
Q.

By part (i), the out-degree of each node of Γ_q that has positive in-degree is at
least 1. Hence, the out-degree always is greater than or equal to the in-degree.
But since every link has one head and one tail, the total in-degree is equal to the
total out-degree. Therefore, the in-degree and out-degree are equal at every node.
Therefore, if the in-degree is positive, both the in-degree and out-degree are 1. If
the in-degree is zero, so must the out-degree be zero, but then that node is not in
Γ_c at all. That completes the proof of the lemma.

Corollary 7.3. *Suppose isosceles (or equilateral) triangle ABC with base AC and
base angles β is N-tiled by tile $(\alpha, \beta, \pi/2)$ with sides (a, b, c), and α is not a rational
multiple of π, or α is an odd multiple of β. Suppose there is no relation $jc = ua + vb$
with $j > 0$ and $u, v \geq 0$, and u, v, j integers. Then AC is composed only of c edges
and there are no c edges on AB or BC, or AB and BC are composed only of c
edges and there is no c edge on AC.*

Proof. According to Lemma 7.2, each link PQ in Γ_c has a corresponding link QR.

Suppose there is a tile with a c edge on AB. Unless the entire segment AB
supports only tiles with their c edges on AB, there will be a segment PQ lying on
AB such that PQ is composed of c edges but beyond Q there is another tile with an
a or b edge on PQ. But that contradicts Lemma 7.2, as there will be an outgoing
link of Γ_c from Q, but at a boundary point, there can be no incoming link. Hence
if there is any c edge on AB, then AB is composed entirely of c edges. Similar for
BC and AC.

At vertex A, there is an angle β. There must be a single tile there, with its β
angle at A, since either $\beta < \alpha$ or α is not a rational multiple of π. The c edge of
that tile must lie on AB or AC. If it lies on AB, then AB is composed entirely of c
edges. If it lies on AC, then AC is composed entirely of c edges. If AC is composed
of c edges, then AB and BC do not contain any c edges, since if they contained

one, then there would also be a c edge at A or C, which is impossible since the tiles at A and C have only one c edge, and it is on AC. That completes the proof of the lemma.

Lemma 7.4. *Suppose isosceles (or equilateral) triangle ABC with base angles β at A and C is N-tiled by a tile with angles (α, β, γ) and sides $(a, b, 1)$. Suppose $\beta \neq \pi/6$ and $\sqrt{N/2}$ is irrational (i.e., N is not twice a square). Then a and b belong to $\mathbb{Q}(\sqrt{N/2})$.*

Remark. The tiling in Fig. 5 shows that the exception for $\beta = \pi/6$ is necessary.

Proof. Let $X = |AB|$. Twice the area of ABC is the cross product of the two equal sides, which is

$$
\begin{aligned}
X^2 \sin 2\alpha \;&=\; 2X^2 \sin\alpha\cos\alpha \\
&=\; 2X^2 \sin\alpha \sin\beta \;=\; 2X^2 ab
\end{aligned}
$$

Twice the area of the tile is ab. Since N tiles cover ABC we have the area equation

$$
2X^2 \;=\; N
$$

Define

$$
\lambda := \sqrt{N/2}
$$

Then $X = \lambda$. Let M be the midpoint of the base AC. Then triangle ABM has a right angle at M, angle α at B, and angle β at A, so it is similar to the tile. Therefore $AM = X \sin\alpha = \lambda a$. Therefore $|AC| = 2a\lambda$. Since there is a tiling of ABC, there are non-negative integers (p, q, r) and (s, t, u) such that

$$
\begin{aligned}
\lambda \;&=\; pa + qb + r \\
2a\lambda \;&=\; sa + tb + u
\end{aligned}
$$

We write this as a system of equations in unknowns a, b:

$$
\begin{aligned}
pa + qb \;&=\; \lambda - r \\
(s - 2\lambda)a + tb \;&=\; -u
\end{aligned}
$$

The determinant $D = pt - q(s - 2\lambda)$. If $D \neq 0$, then both a and b belong to $\mathbb{Q}(\lambda)$, since that field contains D and the right-hand sides of the system.

Now suppose, for proof by contradiction, that $D = 0$. Then since λ is irrational, we have $q = 0$ and $pt = qs = 0$. Since $pa + qb = \lambda - r \neq 0$, we have $p \neq 0$ and hence $t = 0$. Then the two equations become

$$
\begin{aligned}
pa \;&=\; \lambda - r \\
(s - 2\lambda)a \;&=\; -u
\end{aligned}
$$

Multiplying these two equations we have

$$
\begin{aligned}
(s - 2\lambda)(\lambda - r) \;&=\; -pu \\
-2\lambda^2 + \lambda(s + 2r) - sr \;&=\; -pu \\
-N + \lambda(s + 2r) - sr \;&=\; -pu \qquad \text{since } \lambda^2 = N
\end{aligned}
$$

Since λ is irrational, and s and r are non-negative, we have $s = r = 0$. Hence $AC = u$ is made up only of c edges (each of length 1) and $X = pa + qb$, so AB has no c edges. Since also $q = 0$ (as shown above), $X = pa$, so AB also has no b edges, and is therefore composed entirely of a edges. A similar argument applies to show that CA also is composed entirely of a edges.

Therefore $a = X/p = \lambda/p$ belongs to $\mathbb{Q}(\lambda)$. We cannot, however, immediately conclude that b is in $\mathbb{Q}(\lambda)$. But we can conclude that there is no relation $jc = ua+vb$ with integers j, u, v: Suppose there was such a relation. Since $c = 1$, that would mean a is a rational multiple of b, and since a belongs to $\mathbb{Q}(\lambda)$, so does b, and we are done. Therefore, as claimed, there is no relation $jc = ua + vb$.

Now consider the tiles with a vertex at B, where triangle ABC has an angle of 2α. Since sides AB and CB are composed entirely of a edges, the tiles at B supported by AB and CB have their a edges on AB or BC, and hence do not have their α angles at B. In particular, the case when there is only one tile at B is ruled out, since it cannot have an a edge on both AB and BC; and the case of just two tiles at B, both with α angles at B, is also ruled out, since neither would have an a edge on AB or BC. Therefore there must be some tiles with β angles at B. Either α, or 2α, or $2\alpha - \pi/2$ must be a multiple of β. In the latter case, $2\alpha - \pi/2 = \alpha - \beta$ is a multiple of β, so α is a multiple of β. In all of these cases, then, 2α is a multiple of β, say $2\alpha = m\beta$, with $m > 1$. Then

$$\begin{aligned}
\beta &= \frac{\pi}{2} - \alpha \\
2\beta &= \pi - 2\alpha = \pi - m\beta \\
(m+2)\beta &= \pi \\
\beta &= \frac{\pi}{m+2} = \frac{2\pi}{2(m+2)}
\end{aligned}$$

Now $\cos\beta = \sin\alpha = a$ belongs to $\mathbb{Q}(\lambda)$. By Lemma 4.3, $2(m+2)$ equal to one of $5, 6, 8, 10, 12$. Therefore $m+2$ is one of $3, 4, 5, 6$. Since $m > 1$, we have $m = 3$ or $m = 4$.

Case 1, $m = 3$. Then $2\alpha = 3\beta$. Then $\alpha = 3\pi/10$ and $\beta = \pi/5$. The 2α angle of ABC at B is filled with three tiles, each with their β angle at B. The two tiles on AB and BC have their a edges on AB and BC, and their c edges on the two interior segments. The a edge of the middle tile must lie on one of the c edges of the outer tiles. Hence, there is a c/a edge emanating from the vertex. Since there are no relations $jc = ua + vb$, there is a link of Γ_c emanating from B. Since α is an odd multiple of β, Lemma 7.2 is applicable; then there must be an incoming link at B, which is impossible, as links of Γ_c cannot terminate on the boundary of ABC. Hence Case 1 is impossible.

Case 2, $m = 4$. Then $2\alpha = 4\beta$, $\alpha = \pi/3$ and $\beta = \pi/6$, which is ruled out by hypothesis (and has to be, because of Fig. 5). Note that Lemma 7.2 does not apply, since 2α is an even multiple of β. That completes the proof of the lemma.

Lemma 7.5. *Suppose isosceles triangle ABC with base angles β at A and C is N-tiled by a tile with angles (α, β, γ) and sides (a, b, c). Suppose $\beta \neq \frac{\pi}{6}$ and $\sqrt{N/2}$ is irrational. Then*

(i) a and b are rational multiples of $\lambda = \sqrt{N/2}$, and

(ii) AC is composed only of c edges, and there are no c edges on AB or BC.

Proof. Without loss of generality, we may assume $c = 1$. Since by hypothesis, $\sqrt{N/2}$ is irrational and $\beta \neq \frac{\pi}{6}$, we can apply Lemma 7.4 to conclude that a and b belong to $\mathbb{Q}(\lambda)$.

I say that

$$\lambda = X = |AB|.$$

Twice the area of ABC is equal on the one hand to Nab, and on the other to

$$X^2 \cos 2\alpha = X^2 2 \sin\alpha \cos\alpha = 2X^2 ab.$$

Hence $N = 2X^2$. But $N = 2\lambda^2$ by definition of λ. Hence $X = \lambda$, as claimed.

Let $a = x\lambda + y$ and $b = z\lambda + w$, where x, y, z, w are rational. Then for some nonnegative integers p, q, r,

$$\lambda = pa + qb + r = (px + qz)\lambda + (py + qw + r).$$

Since λ is irrational,

$$(1) \qquad\qquad py + qw + r = 0.$$

We also have

$$1 = a^2 + b^2 = (x\lambda + z)^2 + (y\lambda + w)^2 = (x^2 + z^2)\frac{N}{2} + 2(xy + zw)\lambda.$$

Since λ is irrational we have

$$(2) \qquad\qquad xy + zw = 0.$$

Case 1: $xy \neq 0$. Then xy and zw have different signs. I say that $x > 0$ and $z > 0$. We will prove this by cases, according to the sign of xy. First suppose $xy > 0$. Then x and y are of the same sign. Since $a = \lambda x + y > 0$, it follows that $x, y > 0$. Since the tile at B has either its b or c edge on AB, not both q and r are zero; hence (1) implies that $qw \leq 0$; hence $w \leq 0$. Since $zw < 0$, we have $z > 0$ and $w < 0$. Thus the claim $x > 0$ and $z > 0$ holds if $xy > 0$.

Now suppose $xy < 0$. Then $zw > 0$, and we find $z > 0$ and $w > 0$, since $b = z\lambda + w > 0$. Since not both q and r are zero, and $w > 0$, we have $qw + r > 0$. Then (1) implies $py < 0$; hence $y < 0$. Since $xy < 0$, we conclude $x > 0$, establishing the claim $x > 0$ and $z > 0$ also in case $xy < 0$. Thus we have proved that $xy \neq 0$ implies $x > 0$ and $z > 0$.

I say there is no relation $jc = j = ua + vb$ with non-negative integers j, u, v and $j > 0$. Suppose such a relation exists; then

$$\begin{aligned}
j &= ua + vb \\
&= u(x\lambda + y) + v(z\lambda + w) \\
&= (ux + vz)\lambda + (uy + vw)
\end{aligned}$$

Since λ is irrational, we have $ux + vz = 0$. Since x and z are positive and $u, v \geq 0$, this implies $u = v = 0$, which is a contradiction, since $j > 0$.

Then by Corollary 7.3, AC is composed of all c edges, and there are no c edges on AB. Since the length of AC is $2a\lambda$, we conclude that $a = s/(2\lambda) = (s/N)\lambda$, since $2\lambda^2 = N$. Hence a is a rational multiple of λ. Since AB has no c edges, $r = 0$ and $X = \lambda = pa + qb$. Hence $b = (\lambda - pa)/q$ is also a rational multiple of λ. That completes the proof in Case 1.

Case 2: $xy = 0$. Then also $zw = 0$. I say this case implies $y = w = 0$. We argue by cases on whether $x = 0$ or not. If $x = 0$, then $a = y$ is rational and $y > 0$. If $z = 0$, then $b = w > 0$, contradicting (1). If $w = 0$, then $b = z\lambda$,

$$\begin{aligned}
2a\lambda = sa + tb + u &= - & tz\lambda + (sy + u) \\
sy + u &= & 0 \\
s &= & u = 0
\end{aligned}$$

and AB is composed only of b edges, which is impossible, since the tile at A cannot have its b edge on AB.

On the other hand, if $x \neq 0$ then since $xy = 0$ we have $y = 0$. Then $qw + r = 0$ by (1). If $z = 0$, then $b = w > 0$ and therefore $q = r = 0$, contradiction. The only remaining possibility is $y = w = 0$, as claimed. Then $a = x\lambda$ and $b = z\lambda$. Then $X = \lambda = pa + qb + r = (x + z)\lambda + r$, so $r = 0$ and

$$
\begin{aligned}
Y &= 2a\lambda \\
&= 2x\lambda^2 \\
&= xN \\
&= sa + tb + u \\
&= (sx + tz)\lambda + u,
\end{aligned}
$$

which implies $s = t = 0$. That completes the proof in Case 2, and also the proof of the lemma.

Theorem 7.6. *Suppose ABC is isosceles with base angles β, or ABC is equilateral, and ABC is tiled by triangle T similar to half of ABC. If α is a rational multiple of π, then either*

(i) N is even and $N/2$ is a square, or

(ii) N is a square and $\beta = \pi/4$ and ABC has base angles $\pi/4$, or

(iii) $N/2$ is three times a square and $\beta = \pi/6$ and $\alpha = 2\beta = \pi/3$.

Remark. One possible tiling under case (iii) of the theorem is illustrated in Fig. 5. N can be odd in case (ii), since half the triangle ABC is similar to ABC, so quadratic tilings are allowed.

Proof. We begin by remarking that $N/2$ is a rational square if and only if it is an integer square, since if it is a rational square then $2N$ is an integer that is a rational square, hence it is an integer square. Then it is the square of an even integer $2m$, so $N/2 = m^2$. Hence if $\sqrt{N/2}$ is rational, then N is even and $N/2$ is a square. Then condition (i) holds.

Therefore we may assume that $\sqrt{N/2}$ is irrational. We now divide into cases according as $\beta = \frac{\pi}{6}$ or not.

First assume $\beta \neq \frac{\pi}{6}$. Then by Lemma 7.4, $a = \cos\alpha$ and $b = \sin\alpha$ belong to $\mathbb{Q}(\sqrt{N/2})$. By hypothesis, α is a rational multiple of π. These two facts make Lemma 4.3 applicable, so we can drastically limit the possible values of α. Namely, by Lemma 4.3, α and β are odd multiples of $2\pi/n$, where n is one of $6, 8, 12$; that is, they are odd multiples of $\frac{\pi}{3}, \frac{\pi}{4}, or \frac{\pi}{6}$. Since they are both less than $\frac{\pi}{2}$, α and β must be exactly $\frac{\pi}{3}, \frac{\pi}{2}$, or $\frac{\pi}{6}$. Those are the values of α and β allowed in the statement of the lemma. We arrived at that conclusion under the assumption that $\beta \neq \frac{\pi}{6}$, but since that conclusion includes $\beta = \frac{\pi}{6}$, it holds without that assumption. That is, we have proved outright that α and β are equal to $\frac{\pi}{3}, \frac{\pi}{2}$, or $\frac{\pi}{6}$.

It remains to show that N has one of the stated values. Let $X = pa + rb + q$ be the length of AB. Then the area equation is

$$
Nab = X^2 \cos 2\alpha = 2X^2 ab
$$

so $N = 2X^2$.

Case 1. $\alpha = \beta = \frac{\pi}{4}$. Then with $c = 1$ we have $a = b = 1/\sqrt{2}$, so the area of each tile is $\frac{1}{2}$. By Lemma 7.5, $q = 0$, so $X = pa + rb$. Therefore

$$
\begin{aligned}
X &= (p + r)(1/\sqrt{2}) \\
X^2 &= (p + r)^2/2 \\
N &= 2X^2 \qquad \text{as shown above} \\
N &= (p + r)^2/2 \qquad \text{by the previous two lines}
\end{aligned}
$$

Hence $2N$ is a rational square. As remarked at the beginning of the proof, it is therefore an integer square, and hence N is even. Therefore conclusion (i) of the theorem holds. That completes Case 1.

Case 2, $\alpha = \frac{\pi}{6}$. (This is the case when ABC is equilateral.) Then $a = \cos \alpha = \sqrt{3}/2$ and $b = \sin \alpha = 1/2$; hence $X = |AB| = pa + qb + r$ belongs to $\mathbb{Q}(\sqrt{3})$. Then $\sqrt{N/2}$ has the form $u + v\sqrt{3}$ with u and v rational. Squaring both sides we have $N/2 = u^2 + 3v^2 + 2uv\sqrt{3}$. Hence $uv = 0$. Hence either $u = 0$ or $v = 0$.

In case $u = 0$ then $N/2$ is three times a rational square (which is possible, for example see Fig. 5). Then let $v = s/t$ with s and t relatively prime integers, so $N/2 = 3(s/t)^2$. Then $6N = (6s/t)^2$ so $Nt^2 = 6s^2$. Since s and t are relatively prime, 6 divides N. Hence $N/6 = (s/t)^2 = v^2$ is a integer that is a rational square; hence $N/6$ is an integer square. Hence $N/2$ is three times a square, so conclusion (iii) holds.

In case $v = 0$ then $N/2 = u^2$ is a rational square, so it is an integer square by the first paragraph of this proof. Hence conclusion (i) holds.

Case 3, $\alpha = \frac{\pi}{3}$. Then $\beta = \frac{\pi}{6}$, which is no Then $a = \cos \alpha = \sqrt{3}/2$ and $b = \sin \alpha = 1/2$; hence $X = |AB| = pa + qb + r$ belongs to $\mathbb{Q}(\sqrt{3})$. Then the proof is completed verbatim as in Case 2. That completes the proof of the theorem.

Lemma 7.7. *Suppose isosceles (and not equilateral) triangle ABC is N-tiled by a tile with angles $(\alpha, \beta, \frac{\pi}{2})$. Then the base angles of ABC are equal to α or to β.*

Remark. The lemma fails for equilateral ABC.

Proof. This is an immediate consequence of Laczkovich's work: the first and second lines of Table 1 are the only ones allowing a right-angled tile, and the first line can apply to an isosceles ABC only if ABC and the tile are both right isosceles triangles.

Theorem 7.8. *Suppose isosceles triangle ABC is N-tiled by a tile with angles $(\alpha, \beta, \frac{\pi}{2})$. Then*

(i) N is a square and $\alpha = \beta = \frac{\pi}{4}$, or

(ii) N is twice a square (possible for any such N with any right-angled tile), or

(iii) $N = 6k^2$ and $\beta = \pi/6$ or $\alpha = \pi/6$ (example with $N = 54$ in Fig. 5), or

(iv) N is an even sum of squares (so $N/2$ is also a sum of squares). (Possible for any such N with a suitable tile by a double biquadratic tiling as in Fig. 6).

Proof. By Lemma 7.7, the base angles of ABC are equal to α or to β. Since the conclusion of the theorem is insensitive to which angle is labeled α, we may assume the base angles are β. By Theorem 7.6, the conclusion is correct when α is a rational multiple of π; indeed in that case either (i), (ii), or (iii) holds. Therefore we may assume that α is not a rational multiple of π. If $N/2$ is a rational square, then $2N$ is a rational square, and an integer, hence an integer square. Hence N is

even, and $N/2$ is an integer. Since $N/2$ is a rational square and an integer, it is also an integer square, so N is twice a square, and case (ii) of the theorem holds.

Therefore we may assume that

$$\lambda = \sqrt{N/2} \text{ is irrational.}$$

Let X be the length of AB. I say that

$$X = \lambda$$

Twice the area of ABC is

$$X^2 \cos 2\alpha = 2X^2 \cos \alpha \sin \alpha = 2X^2 ab.$$

It is also Nab, since there are N tiles each of area $ab/2$. Therefore $N = 2X^2$. But $N = 2\lambda^2$ by definition of λ, and both X and λ are positive. Therefore $X = \lambda$, as claimed.

Let (a, b, c) be the sides of the tile; we may choose the scale so that $c = 1$. Since α is not a rational multiple of π, it is not equal to $\frac{\pi}{6}$. Since λ is irrational, Lemma 7.5 is applicable. Therefore, side AC is composed only of c edges. Let u be the number of those edges. Let M be the midpoint of AC (which may or may not be a vertex of the tiling). Then triangle ABM has angle α at B and a right angle at M. The length of AM is $u/2$, and the length of AB is λ. Therefore

$$\tan \alpha = \frac{u}{2\lambda}$$

Since λ is irrational, $\tan \alpha$ is irrational. It follows that there does not exist any linear relation $pa = qb$ with integers p and q, for if there were, then $\tan \alpha = b/a = p/q$ would be rational. It follows that there are no relations of the form $ja = pb + qc$, $jb = pq + qc$, or $jc = pa + qb$ with $j \neq 0$. From this it follows that every internal segment in the tiling has equal numbers of a edges on both sides, equal numbers of b edges on both sides, and equal numbers of c edges on both sides.

I say that N is even. For proof by contradiction, assume N is odd. Now the number of a edges in the interior is even, and the number of b edges in the interior is even, and there are no a or b edges on AC. Hence the number of a edges on AB and BC together is odd, and number of b edges on AB and BC together is odd. Suppose $AB = pa + qb$ and $BC = ra + sb$. Then $p \neq r$ and $q \neq s$, since $p + r$ is odd and $q + s$ is odd. We may suppose $p \geq r$ by relabeling A and C if necessary. Then

$$\begin{aligned} |AB| &= |BC| \\ (pa + qb) &= (ra + sb) \\ (p - r)a &= (s - q)b \end{aligned}$$

with $p - r$ a positive integer, and hence $s - q$ also a positive integer. Since we showed above that no such relations between a and b exist, we have reached a contradiction. Hence N is even, as claimed.

Lemma 7.5 also tells us that a and b are rational multiples of λ. Let x and z be rational numbers such that $a = x\lambda$ and $b = z\lambda$. Then the equation $1 = a^2 + b^2$ becomes

$$\begin{aligned} 1 &= (x^2 + z^2)\lambda^2 \\ &= (x^2 + z^2)N/2 \end{aligned}$$

Multiplying by $2N$ we have

$$(3) \qquad\qquad 2N \;=\; (xN)^2 + (zN)^2$$

Thus $2N$ is a sum of two rational squares. Then by Lemma 4.5, $2N$ is a sum of two integer squares. Then by Lemma 4.2, N is also a sum of two squares. That completes the proof of the theorem.

Corollary 7.9. *Suppose isosceles triangle ABC is N-tiled by a right triangle. Then N is not a prime congruent to 3 mod 4, nor is it twice such a prime, except for $N = 6$.*

Proof. The Corollary follows from Theorem 7.8 by Lemma 4.1.

Theorem 7.8 gives necessary and sufficient conditions on N for the existence of an N-tiling of *some* isosceles ABC by a right-angled tile, if N is even. It remains to specify exactly *which* isosceles ABC can be N-tiled, when N is given. The following theorem spells it out.

Theorem 7.10. *Given a positive integer $N > 1$, the possible isosceles triangles ABC that can be N-tiled by a right-angled tile are as follows. Here the sides of the tile are (a, b, c) and the angles are $(\alpha, \beta, \frac{\pi}{2})$.*

(i) if $N/2$ is a square, any isosceles triangle can be N-tiled (by a double quadratic tiling)

(ii) if $N/2$ is a sum of two squares, then isosceles triangle ABC with base angles β can be N-tiled with tile $(\alpha, \beta, \frac{\pi}{2})$ if and only if

$$\tan \beta = r/p \qquad \text{where } N/2 = r^2 + p^2.$$

(iii) if N is a square, the right isosceles ABC can be N-tiled by a quadratic tiling.

(iv) if N is six times a square, then the isosceles triangle with base angles $\frac{\pi}{6}$ can be N-tiled by the tile with $\alpha = \frac{\pi}{3}$ and $\beta = \frac{\pi}{6}$.

(v) If none of the above apply, then no isosceles triangle can be N-tiled by any tile.

Remark. Since $N/2$ may sometimes be expressible in more than one way as a sum of two squares, there can sometimes be more than one possible ABC and tile for a given N, but only finitely many. Moreover, if N is both a square and a sum of squares, there are more possibilities, as in Fig. 6. It will be very difficult to provide a full characterization of all N-tilings.

Proof. Ad (ii). Just divide ABC by its altitude BD and tile each half with a quadratic tiling.

Ad (iii). If ABC is N-tiled, and $N/2$ is not a square, then by Theorem 7.8, the tile has the form mentioned. In case $N/2 = p^2 + r^2$, there is a tiling made by combining biquadratic tilings of the two halves of ABC.

Ad (iv). By Theorem 7.8, if $N = 6k^2$ then a tiling with $\beta = \frac{\pi}{6}$ is possible; see Fig. 5. It remains to show that no other tile is possible. Let $N = 6k^2$. Then N is not a square, and N is not twice a square. Since N contains an odd power of 3, N is not a sum of two squares, and the same is true of $N/2$. Hence no other case in the theorem can apply, and $\beta = \frac{\pi}{6}$ is the only possibiity.

Ad (v). By Theorem 7.8, these cases are exhaustive. That completes the proof of the corollary.

8. Possible values of N in tilings with commensurable angles

We wish to add a third column to Laczkovich's Table 1, giving the possible forms of N if there is an N-tiling of ABC by the tile in that row. For example, when ABC is similar to the tile, then N must be a square, so we put n^2 in the third column. While we are at it, we add a fourth column with a citation to the result, and delete the rows corresponding to the tilings of the equilateral triangle that we have proved impossible. The revised and extended table is Table 2. All the entries in this table except the last one give necessary and sufficient conditions on N for the tilings to exist. The last one gives necessary conditions for certain tilings that probably do not actually exist, but since ABC is equilateral, this question is out of scope for this paper.

Table 2. N-tilings by tiles with commensurable angles, with form of N

ABC	the tile	form of N	citation
$(\beta, \beta, 2\alpha)$	similar to ABC	n^2	[13]
$(\beta, \beta, 2\alpha)$	$(\alpha, \beta, \frac{\pi}{2})$	$e^2 + f^2$	[13]
$(\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2})$	similar to ABC	$3n^2$	[13]
$(\beta, \beta, 2\alpha)$	$(\alpha, \beta, \frac{\pi}{2})$	$2n^2$	Theorem 7.6
$(\frac{\pi}{6}, \frac{\pi}{6}, \frac{2\pi}{3})$	$(\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2})$	$6n^2$	Theorem 7.6
equilateral	$(\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2})$	$6n^2$	Theorem 7.6
equilateral	$(\frac{\pi}{6}, \frac{\pi}{6}, \frac{2\pi}{3})$	$3n^2$	Theorem 7.6

Theorem 8.1. *Suppose (α, β, γ) are all rational multiples of 2π, and triangle ABC is N-tiled by a tile with angles (α, β, γ). Then ABC, (α, β, γ), and N correspond to one of the lines in Table 2.*

Proof. As discussed above, Laczkovich characterized the pairs of tiled triangle and tile, as given in Table 1.[3] It remains to characterize the possible N for each line. In several cases lines in Table 1 split into two or more lines in Table 2, which supplies the required possible forms of values of N. That table lists in its last column citations to the literature or theorems in this paper for each line. That completes the proof.

9. Tilings with $\gamma = 2\alpha$ and (a, b, c) commensurable

In this section and the next, we take up the row of Laczkovich's second table in which ABC is isosceles with base angles α and is tiled by a tile with $\gamma = 2\alpha$, and α is not a rational multiple of π. The condition $\gamma = 2\alpha$ can also be written as $3\alpha + \beta = \pi$. Unlike the similar-looking condition $3\alpha + 2\beta = \pi$, this condition does not imply $\gamma > \pi/2$. The vertex angle of ABC is then $\pi - 2\alpha = \alpha + \beta$. The tile $(4, 5, 6)$ satisfies $\gamma = 2\alpha$; this is shown below as an example of Lemma 11.2.

---

[3] Again, we remind readers who may check with [7] that there are three entries in Laczkovich's Theorem 5.1 that are shown in the subsequent Theorem 5.3 not to apply to tilings by congruent triangles, so they do not appear in our tables.

The numbers (a, b, c) are called *commensurable* if their ratios are rational. In that case we say the "tile is rational." If the edges of a triangle are commensurable, then the triangle is similar to one with integer edges. The remarkable fact is that, if ABC is isosceles with base angles α and vertex angle $\alpha + \beta$, then it can be N-tiled for some N by a tile with angles (α, β, γ) if and only if the edges of the tile are commensurable. This fact is really two different theorems:

- If the tile is rational, there is an N-tiling for some N, and
- If the tile is not rational, there is no N-tiling for any N.

. The first statement is due to Laczkovich [7]. We will explain his proof in this section. The second statement is proved in the next section.

Laczkovich [7] proves that an isosceles triangle with angles as described, can be dissected into triangles *similar* to the tile, plus one parallelogram; then using the commensurability condition, these triangles and the parallelogram can all be tiled with the same size of tile. The only problem is that the tile will have to be *really tiny*.

We illustrate this with the tile $(4, 5, 6)$. The idea of Laczkovich's construction (Fig. 3 in [7]) is shown in Fig. 11. Laczcovich's idea is to quadratically tile each

FIGURE 11. Laczkovich's dissection of isosceles ABC into triangles similar to $(4, 5, 6)$ and a parallelogram
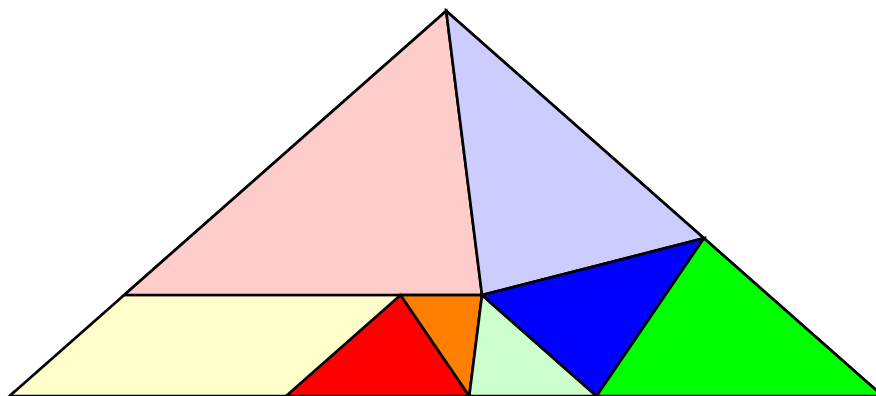
triangle, and then tile the parallelogram. As Laczcovich pointed out, the commensurability of the tile edges mean that with a small enough tile, this will succeed. We illustrate the idea in Fig. 12. But observe that the tiles shown in that figure will not work, because with that size of tile, we cannot continue the tiling into the next (blue) triangle, as if the tile is $(4, 5, 6)$, the boundary between light green and blue is five 6-edges, total 30, which cannot be made of 4-edges, as it would have to be to tile the blue triangle. Clearly we should have chosen a smaller tile, for example half that size. But with a tile half that size, we run into similar trouble at the next boundary. To choose the tile correctly, we introduce a variable for each triangle to count the number of tiles on each side of that triangle. Then there is a linear equation at each boundary. If we assume that the parallelogram will be tiled by n tiles on its diagonal side and m on its horizontal side, then these variables will satisfy the following equations. The equations show that everything is determined

FIGURE 12. We look for a tiling starting like this
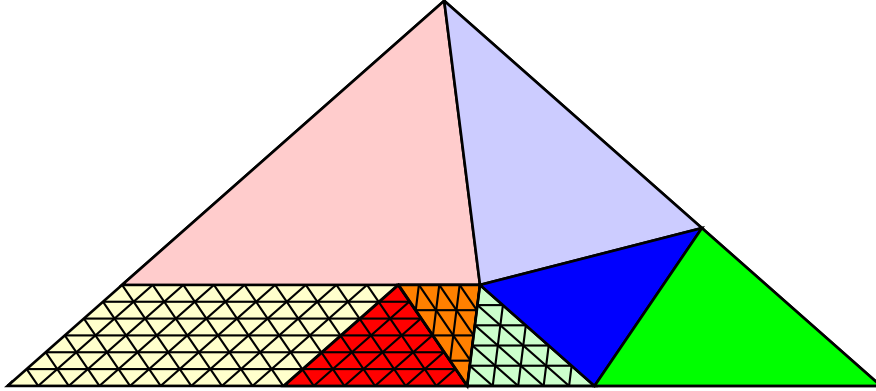
once the number of tiles on each side of the red triangle is chosen.[4]

$$
\begin{aligned}
red &= 1 \\
orange &= 4\ red/6 \\
lightgreen &= 5\ orange/4 \\
blue &= 6\ lightgreen/4 \\
green &= 5\ blue/4 \\
lightblue &= 6\ blue/4 \\
pink &= 5\ lightblue/4 \\
m &= (5\ pink - 4\ orange)/6 \\
n &= (5\ red)/5 \\
total &= red^2 + orange^2 + yellow^2 + blue^2 \\
&\quad + green^2 + lightblue^2 + pink^2 + 2mn
\end{aligned}
$$

Solving these equations, starting with $red = 1$, we get these answers (in the order of variables listed above):

$$
1 \quad \frac{2}{3} \quad \frac{5}{6} \quad \frac{5}{4} \quad \frac{25}{16} \quad \frac{15}{8} \quad \frac{75}{32} \quad \frac{869}{576} \quad 1 \quad .
$$

This reveals that Fig. 12 is misleading, in that the parallelogram of Fig. 11 is not accurately tiled–the tiled parallelogram is off by less than a pixel. To get it accurately tiled requires a *much* smaller tile. To clear those denominators we have to start with $red = 576$ instead of $red = 1$. Then we get

$$
576 \quad 384 \quad 480 \quad 720 \quad 900 \quad 1080 \quad 1350 \quad 869 \quad 576
$$

So the number of tiles required (namely the sum of the squares of the color numbers plus $2mn$) is 6028020.

Here matters stood for about eight years. Then, in 2024, Bryce Herdt found N-tilings for $N = 1125$ and then 720. These tilings are exhibited in an Appendix.

---

[4]For readers without colors: red is the triangle to the right of the parallelogram, and the other colors are in counterclockwise order from red.

10. No tilings with $\gamma = 2\alpha$ and (a, b, c) incommensurable

In this section we rule out tilings of an isosceles triangle with base angles α and vertex angle $\alpha + \beta$, in case the tile edges are incommensurable. To state the point another way, if there is a tiling of such an isosceles ABC, then the tile must be rational.

10.1. **Stars and centers.** Suppose isosceles triangle ABC is tiled by a tile with angles (α, β, γ), not a right triangle, and α is not a rational multiple of π. We will consider and analyze the possible configurations formed by tiles at a vertex of the tiling. We begin by ruling out certain possibilities.

Lemma 10.1. *Let isosceles ABC with base angles α (at A and C) be N-tiled by a tile with angles $(\alpha, \beta, 2\alpha)$, and suppose that the tile is not a right triangle and α is not a rational multiple of π. Let P be a vertex on the boundary of ABC. Then there are not two β angles of tiles at P, and there are not two γ angles of tiles at P.*

Proof. Suppose, for proof by contradiction, that two tiles have their β angles at the same boundary vertex. Then for some nonnegative integers u, v, w we have

$$
\begin{aligned}
\pi &= u\alpha + (v + 2)\beta + w\gamma \\
\pi &= (u + 2w)\alpha + (v + 2)\beta \qquad \text{since } \gamma = 2\alpha \\
0 &= (u + 2w - 3)\alpha + (v + 1)\beta \qquad \text{since } 3\alpha + \beta = \pi \\
\beta &= \left(\frac{3 - u - 2w}{v - 1} \right) \alpha
\end{aligned}
$$

Putting that value for β into $3\alpha + \beta = \pi$ we can solve for α/π:

$$
\alpha/\pi = 3 + \left(\frac{3 - u - 2w}{v - 1} \right) - 1
$$

But that is rational, contradicting the hypothesis that α is not a rational multiple of π. That completes the proof that two β angles do not occur at the same boundary vertex.

The proof for two γ angles is similar. First, if there are no β angles, then

$$
\begin{aligned}
\pi &= u\alpha + (w + 2)\gamma \\
&= (u + 4w + 4)\alpha
\end{aligned}
$$

contradiction, since α is not a rational multiple of π. And if there is one β angle, then

$$
\begin{aligned}
\pi &= u\alpha + (w + 2)\gamma + \beta \\
0 &= (u - 1)\alpha + (w + 1)\gamma \\
&= (u - 1 + 2(w + 1))\alpha \\
&= (u + 2w + 1)\alpha
\end{aligned}
$$

contradiction, since $u + 2w + 1 > 0$. Since we already proved there cannot be more than one β, that completes the proof of the lemma.

We define the angle sum of a vertex to be the sum of the angles of the tiles sharing that vertex. Except for the vertices A, B, and C, that angle sum will always be either π or 2π. It will be π if and only if the vertex lies on the interior of

the boundary of a tile or of ABC. For short, we refer to a vertex with angle sum π as a *boundary vertex*, though it need not be on the boundary of ABC.

Consider a boundary vertex. A *normal boundary vertex* has three tiles, with angles α, β, and γ. A *star* has three α angles and a β. These are the only possibilities for a boundary vertex, since α is not a rational multiple of π, as spelled out in Lemma 10.1. Next, consider an interior vertex. A *normal interior vertex* has two each of (α, β, γ) angles. A *center* has three γ and two β angles. (For example, there is a center in Fig. 11, more or less in the center of the figure.) There may also be interior vertices other than centers that are not normal; these will have either $4\alpha + 2\beta + \gamma$ or $6\alpha + 2\beta$. These vertices we call *interior stars*. The case of angles $4\alpha + 2\beta + \gamma$ we call a *single interior star* and the other case is a *double interior star*.

Lemma 10.2. *Suppose isosceles triangle ABC is tiled by a tile with angles (α, β, γ) with α not a rational multiple of π. Let $\mathcal{C}$ be the number of centers and $\mathcal{S}$ the number of stars, counting double interior stars twice. Then $\mathcal{S} + 1 = \mathcal{C}$. In particular there is at least one center.*

Example. When the tiling begun in Fig. 11 is completed, there will be one center and zero stars. All the vertices introduced by quadratic tilings will be normal vertices. If we then combine four copies of this tiling to create a quadratic tiling of a triangle twice the size, there will be four centers, balanced by four interior starts in the midpoints of the sides, where the three of the four copies have common vertices.

Remark. This lemma is about the only thing we can prove about the internal structure of tilings. We use it only for the existence of at least one center.

Proof. Each tile has one each of α, β, and γ angles. At the vertices of ABC we have three α angles and one β angle (just as we have at a star). Counting the vertex angles we have equal numbers of α, β, and γ angles at each normal vertex. At each vertex we define the "excess" or "deficit" of each of (α, β, γ) to be the difference between the number of those angles at the vertex and the number at a normal vertex. At a star we have two excess α angles and a deficit of one γ angles. At a single interior star the same applies; at a double interior star we have double that contribution. At a center we have an excess of one γ and a deficit of two α. At interior stars we have excesses of α and β and deficits of γ. Adding up the excesses and deficits from the vertices of the tiling, including A, B, and C, we must get zero. The vertices of ABC count the same as a star. One center will "balance" one star, in the sense that the deficits and excesses add to zero. (For example, in Fig. 11, we have one center, and no stars; so the center balances the vertices of ABC, which count as a star.)

If there are no interior stars, then the number of stars, plus one for ABC, will equal the number of centers. If, however, there are interior stars, those will require additional centers to balance them, one center for each single interior star and two for each double interior star. Since we defined $\mathcal{S}$ by this double-counting of double interior stars, we still have $\mathcal{C} = \mathcal{S} + 1$. That completes the proof of the lemma.

10.2. **The tile is rational.** If Q is a vertex of a tiling, and QR is an internal segment of the tiling supporting a tile on one side with its a edge on QR and a vertex at Q, and supporting a tile on the other side with its b edge on QR and a

vertex at Q, then we say QR is an *a/b edge*, or an *a/c edge at Q*. Similarly for *a/c edge* and *b/c edge*. Note that an a/b edge is also a b/a edge.

Lemma 10.3. *Let the isosceles triangle ABC with base angles α be N-tiled by a tile with angles $(\alpha, \beta, 2\alpha)$, with α not a rational multiple of π. Let Q be a center in the tiling. Then either there is an a/b edge at Q or there are both a/c and b/c edges at Q.*

Proof. Assume there is no a/b edge. We must prove there is an a/c and a b/c edge at Q. Each tile with a vertex at the center Q has an a edge ending at Q, since the angles at Q are all β or γ. At a center, five tiles meet, so that is a total of five a edges. Since five is an odd number, these edges cannot all be paired with other a edges. We have assumed there is no a/b edge; hence there is an a/c edge. Similarly, there are three b edges ending at Q, belonging to the tiles with their γ angles at Q. Since three is odd, these cannot each be paired with another b edge. Since there are no a/b edges, there is a b/c edge. That completes the proof of the lemma.

Lemma 10.4. *Let the isosceles triangle ABC with base angles α be N-tiled by a tile with angles $(\alpha, \beta, 2\alpha)$ with α not a rational multiple of π. Then the tiling contains essential segments with associated relations $jb = ua + vc$ and $Ja = Ub + Vc$.*

Proof. Suppose, for proof by contradiction, that there are no such essential segments. We consider the directed graph Γ_b defined in Definition 6.1. We wish to identify the terminal links in this graph. To that end we must consider the possible configurations that can arise when an internal segment UQ of the tiling supports on the same side a series of (one or more) tiles with their b edges, followed by a tile with an a or c edge. Let PQV be the three successive vertices, with PQ of length b and QV of length a or c. We consider all possible configurations in which Q is a vertex where three tiles on one side of a line PQ each have a vertex at Q, contributing one angle each, so the angle sum at Q is $\alpha + \beta + \gamma$ on one side of PQ. All these configurations are shown in Fig. 13. The figure shows that in each of those cases, there is a unique outgoing segment QT that is a b/c edge or a b/a edge. If this segment is extended far enough, we will come to the last b edge on the side that has a b edge at Q. Since there are no essential segments, that point cannot be a vertex of a tile on the other side of QT, so QT is an outgoing link in Γ_b.

On the other hand, if Q is a star, the possible configurations are more complicated, and some of them have zero outgoing b/c or b/a edges, while others have two. It turns out that we do not need to make use of that fact, so we do not give a diagram of these configurations.

If PQ is a link, then line PQ extends past Q as an interior segment of the tiling. Therefore no link terminates on the boundary, so certainly not at a boundary star. At an interior star Q, no interior segment of the tiling passes through Q, as if it did, two of the three γ angles would lie on one side of it, leaving an angle $\beta - \alpha$ on that side, which cannot be filled by a tile. Therefore, no link can end at an interior star, since an interior star is not located on the interior of a tile edge, but the end of a link must be on the interior of a tile edge.

Therefore the out-degree of any star is $\geq$ the in-degree, since the in-degree is zero. At a center, the out-degree is at least one, and the in-degree is zero.

At a given vertex Q (star or not) there can be at most one link ending at Q, since Q lies on the interior of a tile boundary. At a normal vertex Q, if there is an

FIGURE 13. A link PQ in Γ_b gives rise to another link through QR.
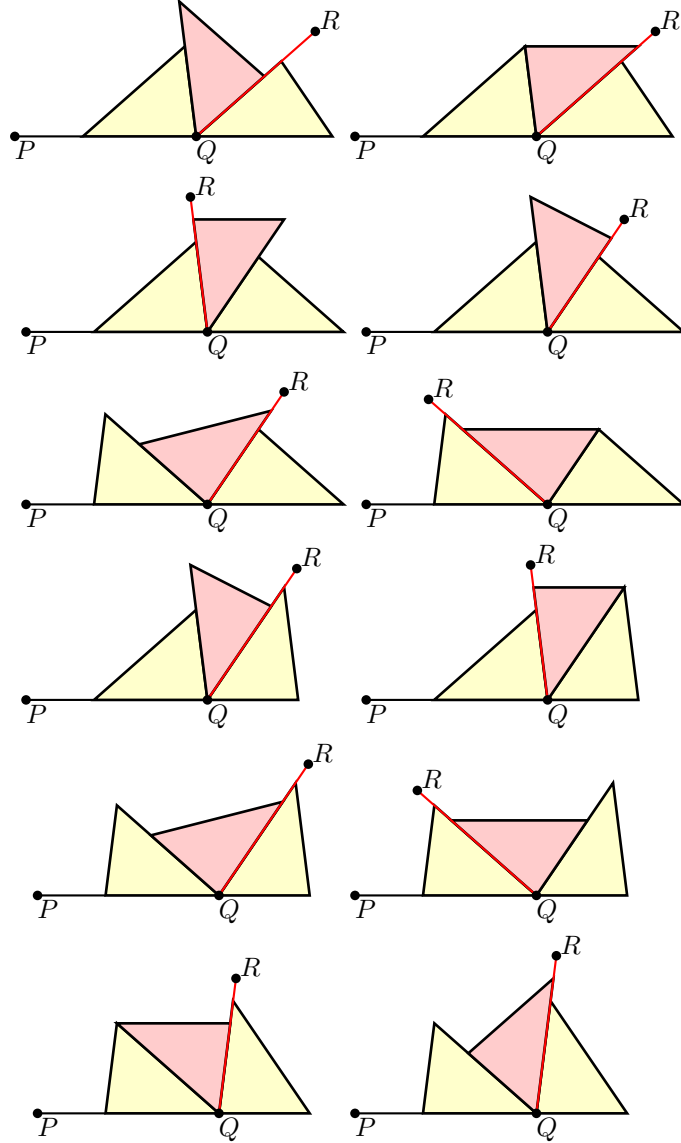
incoming link PQ, there is an outgoing link QR, as shown above. So out-degree
$\geq$ in-degree at a normal vertex. At a star or center, there are no incoming links,
and there is at least one outgoing link, so out-degree $>$ in-degree. By Lemma 10.2,
there exists at least one center. Therefore the total out-degree minus in-degree is
positive. But since each link has one head and one tail, the total out-degree equals
the total in-degree, contradiction.

We have reached a contradiction from the assumption that there is no essential
segment with associated relation of the form $jb = ua + vc$. Hence there is such a
segment.

Next we prove the existence of essential segments with relations of the forms $Ja = Ub + Vc$. This is proved in the same way, using the graph Γ_a instead of Γ_b. Again there is an outgoing link from each center, since by Lemma 10.3, there is either an a/b or an a/c edge at Q, and that edge is part of an outgoing link since there are no essential segments. Again we have to prove that at a normal boundary vertex there is a unique outgoing link. See Fig. 14. That completes the proof.

Theorem 10.5. *Let the isosceles triangle ABC with base angles α be N-tiled by a tile with angles $(\alpha, \beta, 2\alpha)$ with α not a rational multiple of π. Then the tile is rational; that is, the ratios of its sides are rational.*

Remark. If the tile is rational, then after scaling we can assume its sides are integers with no common factor.

Proof. By Lemma 10.4, there is an essential segment witnessing a relation $ja = ub + vc$, and another essential segment witnessing $Jb = Ua + Vc$. In matrix form we have

$$\begin{pmatrix} j & -u \\ U & -J \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} vc \\ -Vc \end{pmatrix}.$$

This equation can be solved for (a, b) provided $j/u \neq U/J$. We have

$$\begin{aligned}
ja &= ub + vc \leq ub \\
j/u &\leq b/a \qquad \text{with equality only when } v = 0 \\
Jb &= Ua + Vc \leq Ua \\
Jb &\leq Ua \qquad \text{with equality only when } V = 0 \\
b/a &\leq U/J \\
j/u &\leq U/J \qquad \text{with equality only when } v = V = 0
\end{aligned}$$

Thus: either a/b is rational (when $v = 0$ or $V = 0$), or a and b are both rational multiples of c (when the equation can be solved). In either case a/b is rational. Similarly, a/c is rational. That completes the proof of the theorem.

FIGURE 14. A link PQ in Γ_a gives rise to another link through QR.
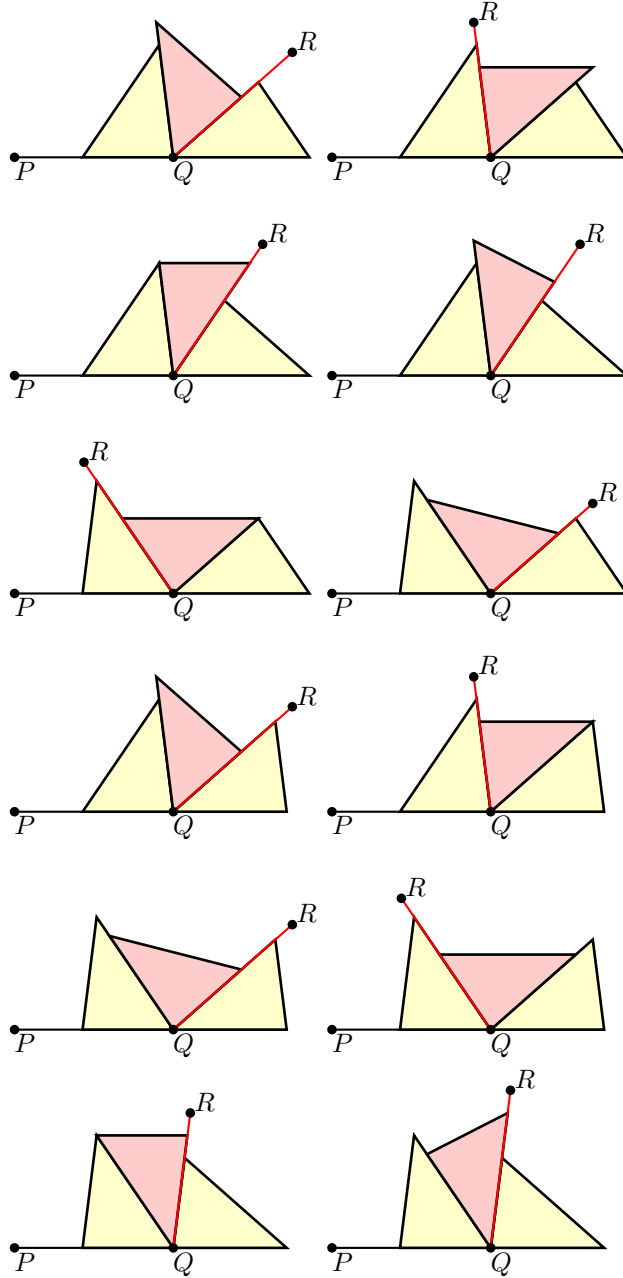
11. ON THE NUMBER OF TILES REQUIRED WHEN $\gamma = 2\alpha$

We continue to consider tilings of isosceles ABC with base angles α and vertex angle $\alpha + \beta$. In the previous two sections, we showed that the tile has to be rational, and that in that case, an N-tiling always exists, for some N. Next we will try to

show that some values of N are impossible. We have two theorems along that line: First, N cannot be a prime number. Second, N has to be "at least so big", i.e., we have a lower bound on N.

11.1. **Characterization of the tile.**

Lemma 11.1. *Suppose (a, b, c) are the integer sides of a triangle with angles $(\alpha, \beta, 2\alpha)$. Then*
$$c^2 = a^2 + ab.$$

Remark. Rational triangles with $\gamma = 2\alpha$ correspond to solutions of this equation with $c < a + b$ and $b < a + c$ and $a < b + c$. For example, $(4, 5, 6)$, and $(9, 7, 12)$.

Proof. By the law of cosines,
$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos\gamma \\ &= a^2 + b^2 - 2ab\cos 2\alpha && \text{since } \gamma = 2\alpha \\ &= a^2 + b^2 - 2ab(2\cos^2\alpha - 1) \\ &= a^2 + b^2 + 2ab - 4ab\cos^2\alpha \end{aligned}$$

By the law of sines, $\sin\alpha/a = \sin\gamma/c = \sin 2\alpha/c = 2\sin\alpha\cos\alpha/c$, so $\cos\alpha = c/(2a)$. Hence

$$\begin{aligned} (4) \qquad\qquad c^2 &= a^2 + b^2 + 2ab - bc^2/a \\ (5) \qquad\qquad &= (a+b)^2 - bc^2/a \\ (6) \qquad\qquad c^2(1 + b/a) &= (a+b)^2 \\ (7) \qquad\qquad c^2 &= a(a+b) \\ (8) \qquad\qquad c^2 &= a^2 + ab \end{aligned}$$

That completes the proof of the lemma.

The following lemma gives a more nuanced characterization of (a, b, c). It was published in [11], but we give the short proof here.[5]

Lemma 11.2. *Let (a, b, c) be integers with no common factor, and suppose the triangle with sides (a, b, c) has angles $(\alpha, \beta, 2\alpha)$. Then $(a, b, c) = (k^2, m^2 - k^2, mk)$ for some relatively prime integers (k, m), with $2k > m > k$.*

Conversely, let (a, b, c) be a triple of integers $(a, b, c) = (k^2, m^2 - k^2, mk)$ with $2k > m > k$ and m and k relatively prime. Then (a, b, c) form a triangle, and it has angles $(\alpha, \beta, 2\alpha)$.

Examples: $(4, 5, 6)$ satisfies this equation with $k = 2$ and $m = 3$. Therefore it is an example of a tile satisfying $\gamma = 2\alpha$. Although $(1, 3, 2)$ satisfies this equation with $k = 1$ and $m = 2$, it does not correspond to a triangle.

Remarks. Thus b and c are relatively prime, but a and c have a common factor k (if $k \neq 1$). Also, $c > a$ but not necessarily $c > b$, and γ can be more or less than a right angle.

Proof. By Lemma 11.1, we have
$$c^2 = a^2 + ab$$

---

[5] I am indebted to Gerry Myerson for pointing out this representation of (a, b, c) to me on MathOverflow.

Luthar observed that this can be written as

$$b^2 + (2c)^2 \;\; = \;\; (2a+b)^2$$

as is apparent upon expanding the right side. But now we can apply Lemma 4.4, according to which there are integers (m,k) such that

$$
\begin{aligned}
b &= m^2 - k^2 \\
2c &= 2mk \\
2a + b &= m^2 + k^2
\end{aligned}
$$

These equations imply the equations to be proved. That completes the proof of the first claim of the lemma.

Conversely, suppose $(a,b,c) = (k^2, m^2 - k^2, mk)$, and (a,b,c) form a triangle. Then one can check that

$$
\begin{aligned}
c^2 &= a^2 + ab \\
&= a(a+b)
\end{aligned}
$$

and we showed above that this equation characterizes $\gamma = 2\alpha$.

Finally, if $(a,b,c) = (k^2, m^2 - k^2, mk)$, then $b+c > a$ becomes $m^2 - k^2 + mk > k^2$, or $m^2 + mk > 2k^2$, which follows from $m > k$.

$a + c > b$ is $k^2 + mk > m^2 - k^2$, or $k(k+m) > (m+k)(m-k)$, or $k > m - k$, which follows from $2k > m$.

$a + b > c$ is $k^2 + (m^2 - k^2) > mk$, which follows from $m > k$. That completes the proof of the lemma.

11.2. **Possible shapes of the tile.** In this section, we consider the possible shapes of a tile (a,b,c) with $\gamma = 2\alpha$. We begin by observing that $a < c$ is the only obvious restriction on the ordering of the edges. As well as $a < b$, we can have $b < a$, as in $(9, 7, 12)$, or $a < c < b$, as in $(9, 16, 15)$.

One way of describing the shape of a triangle is by the ratios of its sides. Here we give lower bounds on some of those ratios. Actually, we use only the bound on c/b, and that only once, but it does seem necessary. We present all the bounds anyway, as they improve the reader's mental picture of these (possible) tilings.

Empirically, these bounds are tight, though we have not proved that they are best-possible. There are apparently no positive lower bounds on b/a or b/c, although again we have not proved that. When b/a is very small, an isosceles triangle ABC with $\gamma = 2\alpha$ will be very close to equilateral, and by Laczkovich's method (see Fig. 12), it can be tiled with trillions of tiny needle-shaped tiles. For example, $(a,b,c) = (61504, 497, 61752)$ has $a/b > 123$, and we get $N = 7227976999088825426993$, more than a billion trillion, and the side and base of ABC are 470043721566328 and 471939059153289.

Lemma 11.3. *Suppose (a,b,c) are the integer sides of a triangle with angles $(\alpha, \beta, 2\alpha)$. Then*

$$
\begin{aligned}
a/c &> 1/2 \\
a/b &> 1/3 \\
c/b &> 2/3
\end{aligned}
$$

and if $b < a$ we also have

$$a/c \;\; > \;\; 1/\sqrt{2}$$

Proof. By Lemma 11.2, there are relatively prime integers k, m with $m/2 < k < m$, such that $a = k^2$, $b = m^2 - k^2$, and $c = mk$. Then

$$a/c = k^2/mk = k/m > 1/2,$$

proving the first claim of the lemma.

We have

$$
\begin{aligned}
a/b \;\; &= \;\; \frac{k^2}{m^2 - k^2} \\
&> \;\; \frac{k^2}{4k^2 - k^2} \qquad \text{since } 4k^2 > m^2 \\
&= \;\; 1/3
\end{aligned}
$$

proving the second claim.

To prove the third claim, we consider the function

$$f(m, x) = \frac{mx}{m^2 - x^2}.$$

Then f is monotone increasing for $x < m$. Since $k > m/2$, and $c/b = f(m, k)$, a lower bound is $f(m, m/2)$, namely

$$
\begin{aligned}
c/b \;\; &> \;\; \frac{m^2/2}{m^2 - (m/2)^2} \\
&= \;\; \frac{2m^2}{4m^2 - m^2} \\
&= \;\; 2/3
\end{aligned}
$$

That is the third claim.

Now suppose $b < a$. That is, $k^2 - m^2 < k^2$. Then $2k^2 < m^2$, so $a/c = k/m < 1/\sqrt{2}$. That completes the proof of the lemma.

11.3. The area equation.

Lemma 11.4 (Area equation)**.** *Suppose isosceles triangle ABC, with base angles α, is N-tiled by a tile with sides (a, b, c) and angles $(\alpha, \beta, 2\alpha)$. Let X be the length of the equal sides AB and BC. Then $X^2 = Nab$.*

Proof. Let $\gamma = 2\alpha$. The base angles of ABC are α, so $\pi = 2\alpha + \angle B = \gamma + \angle B$. But also $\pi = \alpha + \beta + \gamma$, so $\angle B = \alpha + \beta$. Twice the area of ABC is given by the magnitude of the cross product of BA and BC, namely $X^2 \sin(\alpha + \beta)$. Twice the area of the tile is given by $ab \sin \gamma$. Since $\gamma = \pi - (\alpha + \beta)$, twice the area of the tile is also $ab \sin(\alpha + \beta)$. But the area of ABC is N times the area of the tile. Hence

$$X^2 \sin(\alpha + \beta) \;\; = \;\; Nab \sin(\alpha + \beta)$$

Dividing both sides by $\sin(\alpha + \beta)$, we have the area equation of the lemma. That completes the proof of the lemma.

Lemma 11.5 (Area equation, second form)**.** *Suppose isosceles triangle ABC, with base angles α, is N-tiled by a tile with sides (a, b, c) and angles $(\alpha, \beta, 2\alpha)$. Let X be the length of the equal sides AB and BC, and let Y be the length of AC. Then $XY = Nbc$, and $Y = (c/a)X$.*

Proof. Twice the area of ABC is given by the magnitude of the cross product of AB and AC, namely $XY \sin \alpha$. Twice the area of the tile is $bc \sin \alpha$. But the area of ABC is N times the area of the tile. Hence

$$XY \sin \alpha \;=\; Nbc \sin \alpha$$
$$XY \;=\; Nbc$$

which proves the first claim of the lemma. By Lemma 11.4, we have $X^2 = Nab$. Dividing $XY = Nbc$ by $X^2 = Nab$ we have

$$\frac{Y}{X} = \frac{Nbc}{Nab} = \frac{c}{a}.$$

That completes the proof of the lemma.

11.4. **The non-primality of** N**.** In this section, we will show that N cannot be a prime number. What is more, N cannot even be squarefree.

Lemma 11.6. *Let ABC be isosceles with base angles α, and α not a rational multiple of π. Suppose ABC is N-tiled by a tile with $\gamma = 2\alpha$ and integer sides (a, b, c) with no common factor. Then the squarefree part of N divides b and the squarefree part of b divides N. If N is squarefree, then N divides b, and b/N is a square, i.e., $b = N\ell^2$ for some integer ℓ.*

Example. In Fig. 12, we indicated a tiling with tile $(4, 5, 6)$ and $N = 669780$. Factoring that number, we find $N = 2^2 \cdot 3^2 \cdot 5 \cdot 61^2$. So the squarefree part of N is $b = 5$, in accordance with this lemma. This provides a check on the computation of the value of N, since it is not at all apparent from the construction of the tiling that 5 has to be the squarefree part of N.

Proof. By Lemma 11.2, there exist relatively prime integers m, k with $0 < m/2 < k < m$ such that

$$a = k^2, b = m^2 - k^2, c = km.$$

Let X be the length of the equal sides AB and BC, and Y the length of AC. Then

(9) $X^2 \;=\; Nab$ by Lemma 11.4
(10) $XY \;=\; Nab$ by Lemma 11.5

Squaring both sides of (10), we have

$$X^2 Y^2 \;=\; N^2 b^2 c^2$$

Dividing by (9),

$$Y^2 \;=\; \frac{X^2 Y^2}{X^2} \;=\; Nb\frac{c^2}{a}$$

We know a divides c^2, since $c = km$ and $a = k^2$, so $c^2/a = m^2$. Then

$$Y^2 = Nbm^2.$$

Then N divides Y^2. Then the squarefree part of N divides bm^2. But I say that actually the squarefree part of N divides b, not just bm^2.

Let p be a prime dividing N to an odd power p^{2k+1}, and let p^j be the highest power dividing m. Then $p^{2j+2k+1}$ divides Y^2, so p^{j+k+1} divides Y, so $p^{2j+2k+2}$ divides Y^2, so p^{2j+1} divides bm^2. If p does not divide b, then p^{j+1} divides m,

contradiction. Therefore p divides b. Since p was any prime dividing N to an odd power, it follows that the squarefree part of N divides b, as claimed.

Now let p be a prime dividing b to an odd power p^{2j+1}. Then p^{2j+1} divides Y^2, so p^{2j} divides Y, so p divides Nc^2/a. But b is relatively prime to $c^2/a = m^2$, so p divides N. Therefore the squarefree part of b divides N.

If N is squarefree, then b/N is an integer. Since $Y^2 = Nbm^2$, we have

$$b/N = \left(\frac{Y}{Nm}\right)^2.$$

Therefore b/N is a rational square. Since b/N is an integer, it is also an integer square.

That completes the proof of the lemma.

Theorem 11.7. *Let ABC be isosceles with base angles α, and α not a rational multiple of π. Suppose ABC is N-tiled by a tile with angles $(\alpha, \beta, 2\alpha)$ and sides (a, b, c). Then N is not squarefree. In particular, N is not prime.*

Proof. By Theorem 10.5, the tile is rational, so we can assume without loss of generality that (a, b, c) are integers with no common factor. By Lemma 11.2, there exist relatively prime integers m, k with $0 < m/2 < k < m$ such that

$$a = k^2, b = m^2 - k^2, c = km.$$

Let X be the length of the equal sides AB and BC, and Y the length of AC. Then by Lemma 11.4,

$$\begin{aligned} X^2 &= Nab \\ &= N^2a \qquad \text{since } b = N\ell^2, \text{ by Lemma 11.6} \\ &= N^2k^2\ell^2 \qquad \text{since } a = k^2 \end{aligned}$$

Taking the square root of both sides, we have

$$(11) \qquad\qquad X = Nk\ell$$

The tiling gives rise to a relation

$$(12) \qquad X = pa + qb + rc$$
$$(13) \qquad Nk\ell = pk^2 + q(m-k)(m+k) + rkm \qquad \text{since } X = Nk\ell \text{ by (11)}$$

By (11), $X \equiv 0 \bmod k$. Taking the last equation mod k we find

$$0 \equiv qm^2 \bmod k$$

Since k and m are relatively prime, we can divide by m^2:

$$(14) \qquad\qquad q \equiv 0 \bmod k$$

Putting $N\ell^2 = b = (m-k)(m+k)$ into (13), we have

$$\begin{aligned} k(m-k)(m+k)/\ell &= pk^2 + q(m-k)(m+k) + rkm \\ 0 &= pk^2 + (q - k/\ell)(m-k)(m+k) + rkm \end{aligned}$$

Then k/ℓ is necessarily an integer. Since $m - k > 0$, we have $q - k/\ell \leq 0$, and either $q - k/\ell < 0$ or $p = r = 0$. We argue by cases:

Case 1, $q - k/\ell < 0$. Then $q < k/\ell \leq k$. Then by (14), we have $q = 0$. Therefore, no tile supported by AB or BC has its b edge on AB or BC, since a relation $X = pa + qb + rc$ would arise from each of AB or BC (although perhaps

with different coefficients (p, q, r)). However, at B there are two tiles, one with an α angle and one with a β angle at B. Renaming A and C if necessary, we may assume the tile with α at B is supported by AB. Since each tile supported by AB has its a or c edge on AB, each of those tiles has a β angle at one of its vertices on AB. But there are α angles at A and at B. Then by the pigeonhole principle, one of the vertices on AB is a vertex of two tiles with β angles there. But that contradicts Lemma 10.1. That disposes of Case 1.

Case 2, $p = r = 0$. Then every tile supported by the side $Z = AB$ or BC that gave rise to (12) has its b edge on Z. Hence every tile supported by Z has a γ angle on Z. There are no γ angles at A, B, or C, so by the pigeonhole principle, there must be a boundary vertex with two γ angles. But there cannot be two γ angles at the same boundary vertex, since the only integer relations between the angles are $\alpha + \beta + \gamma = \pi$ and $3\alpha + \beta = \pi$. This contradiction shows that Case 2 is impossible. That completes the proof of the theorem.

11.5. The number of tiles on a side of ABC.

We wish to show that, given N, we can calculate a finite set of triangles and a finite set of possible tiles (a, b, c), such that if there is an N-tiling of some isosceles ABC with base angles α by some tile with $\gamma = 2\alpha$ and α not a rational multiple of π, then ABC and the tile are in those finite sets.

It will be important for that proof to have an upper bound on the number of tiles on the sides AB and BC of isosceles triangle ABC, in terms of N. This section is devoted to that problem.

We can count either the tiles supported by AB, or the tiles with an edge or a vertex on AB. At a given boundary vertex, there can be three tiles or there can be four tiles, as $\pi = \alpha + \beta + \gamma = 3\alpha + \beta$. So the two ways of counting tiles "on a side" differ, but by a bounded factor.

One might initially think that such a bound should be on the order of $\sqrt{N}$, but that idea is based on the picture in which ABC is not long and narrow. If we consider the case when α is tiny, so AB and BC are almost half as long as BC and the triangle has comparatively little interior, maybe most of the tiles touch the boundary! In that case, neglecting for the moment the fact that some tile edges may be a lot larger than others, we would expect almost a quarter of the tiles to have an edge or vertex on AB, and a quarter on BC, and half on AC. The bound we actually prove is that one of AB or BC must support fewer than $(N-1)/4$ tiles. The number of tiles supported by AB should be about half the number of tiles with edges or vertices on AB, neglecting the vertices with four instead of three tiles.

One illustration of the difficulty is the case when b is tiny, and a and c are almost equal. We already mentioned the example $(a, b, c) = (61504, 497, 61752)$. Then angle β is tiny, and α and β are both close to $\pi/3$, so ABC is nearly equilateral, and the tile is needle-shaped, long and narrow. Note that this is not at all the situation considered above when ABC itself is long and narrow. But in this situation, AB might be tiled by millions of tiles with their tiny b edges on AB, while BC might be tiled with relatively fewer tiles with their long c or a edges on BC. So there is no obvious relation between the number of tiles supported by one side and the number supported by another.

The difference between $N/4$ and $N/2$ and $(N-1)/4$ may not seem important at first, but $(N-1)/4$ enables us to prove that N cannot be twice a prime, while the

others mentioned do not, though they might suffice for N not being a prime. Furthermore, the better the bound, the more candidate values of N can be ruled out because they violate certain simple conditions. We conjecture that all N that correspond to tilings are not squarefree; but there are certainly not-squarefree numbers N that we cannot yet rule out.

We need to "get off the ground" by a close analysis of the case when N is very small. In particular, what is the smallest number of tiles that can be supported by the base AC? We show that at least four tiles are required. (That already shows $N > 7$.) A few of our lemmas will be proved also for tilings with $\gamma = 2\pi/3$; for example at least *three* tiles must be supported by AC in that case.

After these preliminary remarks, we plunge into the technical lemmas.

Lemma 11.8. *Let isosceles ABC with base angles α (at A and C) be N-tiled by a tile with angles $(\alpha, \beta, 2\alpha)$, and suppose that the tile is not a right triangle and α is not a rational multiple of π. Then no tile has one vertex on AB and another on BC.*

Proof. By Theorem 3.1, α is not a rational multiple of π, and π cannot be expressed as a linear combination of α, β, and γ, except in the way determined by the vertex angles of ABC. (That is, $\pi = 3\alpha + \beta$ if $\gamma = 2\alpha$, or $\pi = 3\alpha + 3\beta$ if $\gamma = 2\pi/3$). By Theorem 10.5, the tile is rational, so we may assume its sides are integers (a, b, c) with no common factor, with a opposite angle α and b opposite β.

Suppose some tile has an edge EF with E on AC and F on BC. Consider the triangle BEF. Since it has the same angle at B as triangle ABC, namely $\alpha + \beta$, its angles at E and F must each be α. Then the north side of EF cannot be covered by a single tile, since if it were, that tile would have two α angles, one at E and another at F. Therefore the north side of EF supports at least two tiles. By hypothesis, EF is an edge of a single; that tile, say Tile 1, must lie on the south side of EF.

Since the tile is rational by Theorem 10.5, we may assume without loss of generality that (a, b, c) are integers with no common factor. In particular, none of (a, b, c) is an integer multiple of another. Since the south side of EF is equal to one tile edge, the north side cannot be composed of all a edges, or all b edges, or all c edges, since then the edge on the south would be an integer multiple of the edge on the north.

Suppose Tile 1 has its a edge on EF. Since $a < c$, there are no c edges on the north side of EF. Hence north of EF are only b edges, so a is an integer multiple of b, contradicting the fact that a and b are relatively prime (by Lemma 11.2).

Similarly, if Tile 1 has its b edge on EF, then b is an integer multiple of a, contradiction. Finally, if Tile 1 has its c edge on EF, then since $a + b > c$, either all the tiles on the north of EF supported by EF have their a edges on EF, or they all have their b edges on EF. Then c is an integer multiple of a or c is an integer multiple of b. We argue by cases:

Case 1, c is a multiple of a, say $c = pa$. Then

$$
\begin{aligned}
c^2 &= p^2 a^2 \\
&= a^2 + ab \qquad \text{by Lemma 11.1} \\
p^2 a &= a + b \qquad \text{by the previous two lines} \\
a &= 1 \qquad \text{since a and b are relatively prime} \\
k &= 1 \qquad \text{where $a = k^2$ and $b = m^2 - k^2$, by Lemma 11.2}
\end{aligned}
$$

Then $a = 1$ and $c = mk = m$ and $b = m^2 - 1 = c^2 - 1$, so $a + b = m^2 = c^2 \geq c$, so (a, b, c) do not form a triangle, contradiction. That completes Case 1.

Case 2, c is a multiple of b, say $c = pb$. Then

$$
\begin{aligned}
pb^2 &= c^2 \\
c^2 &= a^2 + ab \qquad \text{by Lemma 11.1} \\
p^2 b^2 &= a^2 + ab \qquad \text{by the previous two lines} \\
a^2 &\equiv 0 \bmod b \\
a &\equiv 0 \bmod b \qquad \text{since a and b are relatively prime}
\end{aligned}
$$

But then b divides a. Since a and b are relatively prime, that implies $b = 1$. By Lemma 11.2, there are relatively prime (k, m) such that $b = m^2 - k^2$. Then $m^2 = k^2 + b = k^2 + 1$, which is impossible, since $k > 0$. That completes Case 2.

These contradictions complete the proof of the lemma.

Lemma 11.9. *Let isosceles ABC with base angles α (at A and C) be N-tiled by a tile with angles $(\alpha, \beta, 2\alpha)$, with α not a rational multiple of π. Let T be a tile supported by AC but not having a vertex at A or C. Then T does not have a vertex on AB or BC.*

Proof. Let PQ be the edge of T that lies on AC. Let R be the third vertex of T. We must show R does not lie on AB or AC. Suppose, for proof by contradiction that it does. By renaming A and C if necessary, we can assume that R lies on AB. Since α is not a rational multiple of π, there is only one tile, say T_1, with a vertex at A. The interior edge of T_1 connects AB with AC. That is not a shared edge with T, since at least three tiles meet at P. See Fig. 15.

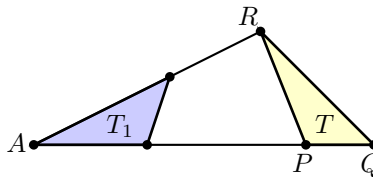FIGURE 15. RPQ cannot be a single tile as RP is too long.

Fig. 15, although already counterfactual, is not counterfactual enough, as it shows a gap between T_1 and T, which we must prove must be there, before we can prove that "RP would be too long." That is, T might share a vertex with T_1. We will begin by showing that cannot happen.

I say that P does not share a vertex on AC with T_1. Suppose, for proof by contradiction, that it does. Then P, the western vertex of T on AC, is also the

eastern vertex of T_1 on AC. Let S be the third vertex of T_1, so S lies on AB. See Fig. 16.

FIGURE 16. What if T_1 and T share a vertex on AC?
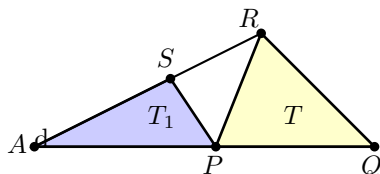
Consider triangle SPR. The side SP has length a, because that is the edge of T_1 opposite its α angle. Since $a = mk$ and $b = m^2 - k^2$, a and b are relatively prime, so a cannot be expressed as a sum of b edges; since $a < c$ that means that SP supports only one tile on the east, sharing an a edge with T_1. Call that tile T_2. Then T_2 and T_1 each have a β or γ angle at S. At P, T_1 has a β or γ angle, since its α angle is at A. By Lemma 10.1, T_2 and T_1 do not both have either a β or γ angle at P. T_2 does not have its α angle at P, since PS is its a edge. So one of T_2 and T_1 has a β angle at P, and the other has a γ. Then T has its α angle at P, and exactly those three tiles have a vertex at P. Since T has its α angle at P, its side PS is equal to b or c. Now triangle PSR has one side equal to a (namely PS), side PR equal to b or c (since T has its α angle at P), and its angle at P is either β or γ, since T has α at P and T_1 has β or γ. Therefore PSR is congruent to the tile, by the ASA congruence theorem. But T_1 has angle β or γ at S, so angle PSR is $\alpha + \gamma$ or $\alpha + \beta$, contradicting the fact that PSR is congruent to the tile. That proves that T and T_1 do not share a vertex on AC.

Now I say that T does not share a vertex with T_1 on AB either. Suppose to the contrary that it does. Then R is the eastern vertex on AB of T_1 as well as the northern vertex of T. See Fig. 17.

FIGURE 17. What if T_1 and T share a vertex on AB?
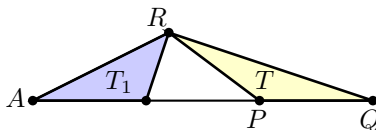
Then triangles T_1 and T have the same height (measured from AC) and since they are each one tile, they are congruent; so they have the same area. Hence they have the same base. The base of T_1 is b or c, since it has its α angle at A. Then the base PQ of T is also b or c.

Since the edge RP is not shared with T_1, as we have already proved, there is another tile between T_1 and T with a vertex at R. Since the third vertex of T, namely Q, does not lie on AB, there are at least four tiles meeting at R. Therefore there are three α angles and one β angle at R. The β angle must belong to T_1, since it has its α angle at A. Then T has its α angle at R. Hence its base PQ is opposite its α angle. Hence $PQ = a$. But we showed above that PQ is b or c. That is a contradiction; that proves (as claimed) that T does not share a vertex with T_1 on AB.

Let X be the length of AR and Y the length of RP. Since R is not a vertex of T_1, AR is composed of at least two tile edges. One of those edges is not a, since T_1 has its α angle at A. I say that also one of them is not b. For suppose AR supports only b edges of tiles. Then for some integer ℓ, $X = \ell b$. Each of those tiles has its α angle to the west and its γ angle to the east. Then at R, the tile to the west of R has its γ angle at R. Then there are exactly three tiles with vertices at R. Since RQ is an interior segment, T is the middle one of those three. Then triangle ARP is similar to the tile, since it has one α and one γ angle. Triangle ARP has α at A, and γ at R, and therefore β at P. But angle ARP is the supplement of angle of T, which is impossible, as neither α, β, nor γ is the supplement of β. Hence, as claimed, one of the edges supported by AR is not b.

We now intend to reach a contradiction by showing that the length of RP is more than the length of any tile edge a, b, or c. That will be a contradiction, because RPQ is a tile, so RP has to be equal to a, b, or c.

Let x and y be two of (a, b, c), not necessarily distinct, but one of x and y is not a, and one is not b. Then I say that $x + y > a$, $x + y > b$, and $x + y > c$. If we prove that, we can take x and y to be two of the edges supported by RP, and since Y is a tile edge, and thus equal to a, b, or c, we will have $RP > Y$ as claimed.

Since $x + y > x$, and $x + x > x$, we are done unless the third edge is distinct from the two to be added. If the two to be added are distinct, we are done, because two sides of a triangle are together greater than the third. That leaves the cases $a + a > b$, $b + b > c$, $a + a > c$, $b + b > a$, $c + c > b$, $c + c > a$. The cases $c + c > a$ and $b + b > a$ follow from $c > a$ and $b > a$. We can drop $a + a > b$ and $a + a > c$ because we know one of two edges to be added is not a. We can drop $b + b > a$ and $b + b > c$ because we know that one of two edges to be added is not b. That leaves only $c + c > b$ still to prove. By Lemma 11.3, we have $c/b > 2/3$. Hence $c + c > (4/3)b > b$. That completes the proof of the lemma.

Remark. $b + b > c$ is not generally true. We have seen that b can be much smaller than c. Hence we had to show that RP could not support only b edges.

Lemma 11.10. *Let isosceles ABC with base angles α (at A and C) be N-tiled by a tile with angles $(\alpha, \beta, 2\alpha)$, with α not a rational multiple of π. Then there are at least four tiles supported by AC. If the tile instead has angles $(\alpha, \beta, 2\pi/3)$ instead of $(\alpha, \beta, 2\alpha)$, then there are at least three tiles supported by AC.*

Proof. First assume $\gamma = 2\alpha$. We start by proving that at least three tiles are supported by the base AC. Suppose, for proof by contradiction, that only two tiles are supported. Those two tiles have their α angles at A and C, and their a edges both end at the shared vertex P on AC. Without loss of generality we may assume that Tile 1, with edge AP, has its β angle at P, and Tile 2, with edge PC, has its γ angle at P, since two β angles or two γ angles at P is impossible. Then the remaining angle to be filled at vertex P is α. By Theorem 3.1, α is not a rational multiple of π, and hence not a multiple of β. Since α is not a multiple of β, the gap must be filled by a single tile, Tile 3, with its α angle at P. Then Tile 3 has its c side either against Tile 1 or Tile 2, but that is impossible since those edges are of length a and terminate at the boundary, and $c > a$. Hence there are indeed at least three tiles supported by AC. Note that this argument works for both cases, $\gamma = 2\alpha$ and $\gamma = 2\pi/3$.

Now suppose there are exactly three tiles supported by AC. Not all three tiles on AC can have their c edges on AC, since then each would have a β angle on AC,

and by the pigeonhole principle, there would be a vertex on AC with two β angles. If the tiles at A and C both have their b edges on AC, then they have their γ angles on AC, so the middle tile on AC cannot have a γ angle on AC, so it has its c edge on AC. Therefore the possible values of Y are exactly $2c + b$, $2c + a$, $2b + c$, and $b + c + a$. So far, γ could be 2α or $2\pi/3$.

Now we assume $\gamma = 2\alpha$. Then Lemma 11.5 applies, yielding $X = (a/c)Y$. Therefore, the possible values of X are exactly

$$2a + (ba/c), 2a + a^2/c, 2ab/c + a, (a + b + c)(a/c)$$

By Lemma 11.2, $a = k^2$, $b = m^2 - k^2$, $c = mk$, where k and m are relatively prime. Note that

$$(a + b + c)(a/c) = (k^2 + (m^2 - k^2) + mk)(k/m) = (m + k)k = a + c.$$

Then the possible values of X are

$$2k^2 + k(m^2 - k^2)/m, 2k^2 + k^3/m, 2k(m^2 - k^2)/m, (m + k)k$$

Since $m > k \geq 1$, none of the first three can be an integer. Therefore

$$X = (m + k)k = a + c.$$

The question now is, what are the tiles supported by AB, the sum of whose edges on AB is $a + c$? One possibility is that there are exactly two tiles supported by AB, one with its a edge on AB and the other with its c edge. However, that is impossible, since the tiles at A and C both have their α angles at A or C, and hence their a edges do not lie on AC. So X has some other configuration of tiles supported. Then $X = ua + vb + wc = a + c$ for some nonnegative integers u, v, w. Then u or w must be zero.

If $u = w = 0$ then $vb = a + c$, i.e., $v(m^2 - k^2) = k^2 + km$. Dividing by $m + k$ we have $v(m - k) = k$. Since k is relatively prime to $m - k$, k divides v. But also v divides k. Hence $v = k$. Then AC is composed of k tile edges of length b. Then each edge has a γ angle on AC. Since there are α angles at A and C, then by the pigeonhole principle there is a vertex with two γ angles. But that is impossible, since the only relations between the angles are $\alpha + 3\beta = \pi$ and $\alpha + \beta + \gamma = \pi$. Thus we have ruled out the case $u = w = 0$.

Now suppose $v = w = 0$. Then $ua = a + c$, i.e., $(u - 1)a = c$, or $(u - 1)k^2 = km$, so $u - 1 = m$. Then $ma = c = km$, so $m = k$, contradiction. Hence not both v and w are zero.

I say that $w \neq 0$. To prove that, suppose $w = 0$. Then $u \neq 0$ and $v \neq 0$ and $ua + vb = a + c$. Then

$$
\begin{aligned}
uk^2 + v(m^2 - k^2) &= k^2 + km \\
(u - 1)k^2 + vm^2 &= km
\end{aligned}
$$

Then k divides v, since k is relatively prime to m. Since $v \neq 0$, we have $v \geq k$.

$$
\begin{aligned}
km &= (u - 1)k^2 + vm^2 \\
&\geq vm^2 && \text{since } u - 1 \geq 0 \\
&\geq km^2 && \text{since } v \geq k \\
&> km && \text{since } m > k \geq 1 \\
km &> km
\end{aligned}
$$

But that is a contradiction, reached on the assumption $w = 0$. Therefore $w \neq 0$, as claimed.

Then

$$
\begin{aligned}
ua + vb + wc &= a + c \\
uk^2 + v(m^2 - k^2) + wkm &= k^2 + km \\
uk^2 + vm^2 + (w - 1)km &= (v + 1)k^2 \\
(u - 1)k^2 + vm^2 + (w - 1)km &= vk^2
\end{aligned}
$$

I say that $u = 0$. If $u \neq 0$, then all the terms on the left are non-negative, and $vm^2 > vk^2$ since $m > k$, so the equation is impossible. Hence $u = 0$, as claimed. Then

$$
\begin{aligned}
v(m^2 - k^2) + wkm &= k^2 + km
\end{aligned}
$$

(15)
$$
\begin{aligned}
vm^2 + (w - 1)km &= (v + 1)k^2
\end{aligned}
$$

Since $m > k$, we have $vm^2 > vk^2$, and since $w \neq 0$ we have $(w-1)km > (w-1)k^2$. If $w \neq 1$ then the left side of (15) is greater than the right, contradiction. Hence $w = 1$. Then

$$a + c = ua + vb + wc = vb + c$$

since $w = 1$ and $u = 0$. Then $a = vb$, which is impossible since a and b are relatively prime and not zero. This contradiction depends only on the assumption that there are exactly three tiles supported by AC. Since we already proved there are at least three tiles supported by AC, we have now proved there are at least four tiles supported by AC. That completes the proof of the lemma.

Lemma 11.11. *Let isosceles ABC with base angles α (at A and C) be N-tiled by a tile with angles $(\alpha, \beta, 2\alpha)$, with α not a rational multiple of π. Then one of the sides AB or BC supports strictly less than $(N - 1)/4$ tiles. If the tile instead has angles $(\alpha, \beta, 2\pi/3)$, then one of the sides supports strictly less than $N/4$ tiles.*

Remarks. This bound is not used in our tiling non-existence theorem that N cannot be squarefree. But it is crucial to the theorem that, given N, there is an explicitly computable set of possible ABC and tiles.

Proof. Let P be a vertex of the tiling lying on the interior of a side of ABC. If only two tiles meet at P then they cannot have different angles, since any two of (α, β, γ) make together less than π. But if they have the same angle at P, that angle would be a right angle, contrary to hypothesis. Therefore at least three tiles meet at each such vertex P.

Let n and m be, respectively, the total number of tiles with an edge or vertex on AB, and the total number of tiles with an edge or vertex on BC. First suppose the tile has $\gamma = 2\alpha$. By Lemma 11.10, AC supports at least four tiles. The middle two do not touch AB or BC even in a vertex, by Lemma 11.9. and Lemma 11.8, and between the tiles that have no vertex at A or C there is (at least) a fifth tile, having only a vertex on AC, which also does not touch AB or BC, by Lemma 11.9. Then $n + m \leq N - 3$, since N is the total number of tiles, but at least three have a side or vertex on AC and do not touch AB or BC, and by Lemma 11.8, no tile contributes to both n and m. Therefore either $n \leq (N - 3)/2$ or $m \leq (N - 3)/2$. Relabeling A and C if necessary, we can assume without loss of generality that $n \leq (N - 3)/2$. Now let p and q, respectively, be the number of tiles supported by AB, and the number of tiles with one and only one vertex on AB. Then $q \geq p - 1$ (it might

be strictly greater if some vertices have more than three tiles sharing that vertex). Therefore

$$
\begin{aligned}
p - 1 &\leq q \\
p + q = n &\leq (N-3)/2 \\
p &\leq (N-3)/2 - q \\
&\leq (N-3)/2 - (p-1) \\
2p &\leq (N-3)/2 + 1 = (N-1)/2 \\
2p &\leq (N-1)/2 \\
p &\leq (N-1)/4
\end{aligned}
$$

We now will establish strict inequality in place of $\leq$. Suppose that $p = (N-1)/4$. Then there are exactly four tiles supported by AC, and at all vertices on AC except A and C, there are just three tiles meeting, since any more would introduce a strict inequality $p - 1 < q$, instead of $p - 1 \leq q$. Therefore each boundary vertex has one α, one β, and one γ angle. The corner tiles have their α angles at A and C. The two tiles adjacent to the corner tiles, with only a vertex on AC, do not have their α angles on AC, since their a sides must match the a sides of the corner tiles. (It is impossible that a is an integer multiple of b, since a is relatively prime to b, by Lemma 11.2.) Therefore the middle vertex on AC must have two α angles, contradiction. (In other words, if AC supports only four tiles, there must be a "boundary star" on AC.) Hence, as claimed, $p < N/4$. That completes the proof in case the tile is $(\alpha, \beta, 2\alpha)$.

Now suppose the tile is $(\alpha, \beta, 2\pi/3)$ and $p = N/4$. Then we start with $n + m \leq N - 1$ instead of $N - 3$, since Lemma 11.10 gives us only three tiles supported by AC instead of four. The result is $p \leq N/4$ instead of $p \leq (N-1)/4$. Then there must be exactly three tiles supported by AC and each boundary vertex has one α, one β, and one γ angle, or else there would be strict inequality in the computation. That completes the proof of the lemma.

11.6. **What is the least N permitting a tiling?** The smallest explicitly-known such tiling has $N = 1125$, with tile $(4, 5, 6)$, as shown in Fig. 23. We will show below that $N \geq 45$. There is a swath of ignorance between 45 and 1125.

Until now, we have no *a priori* estimate on the size of (a, b, c). For example, there is no *a priori* reason why we could not have $N < 100$ and (a, b, c) each greater than a million. We now provide such a bound.

Lemma 11.12. *Suppose isosceles triangle ABC is N-tiled by tile (a, b, c) with $\gamma = 2\alpha$. Then $a, b,$ and c are less than m^2, where m is as in Lemma 11.2 and satisfies*

$$
m < N + \frac{(N+1)^2}{16}.
$$

Remark. The form of the bound is not very beautiful, but we want to use it for fairly small N, and its asymptotic value is not of interest. We only care that there is *some* explicit and reasonably-sized bound.

Proof. By Lemma 11.2, there exist relatively prime integers m and $k \leq m$ such that $a = k^2$, $b = m^2 - k^2$, and $c = mk$. Let $X = |AB|$ be the length of the two

equal sides of ABC. The tiling provides integers p, q, r such that $X = pa + qb + rc$. Then

$$
\begin{aligned}
X^2 &= Nab \\
X^2 &= N(k^2)(m^2 - k^2) \\
X &= pk^2 + q(m^2 - k^2) + rmk \\
X &\equiv (p - q)k^2 \quad\quad \mathrm{mod}\ m \\
X^2 &\equiv (p - q)^2 k^4 \quad\quad \mathrm{mod}\ m \\
N(k^2)(m^2 - k^2) &\equiv (p - q)^2 k^4 \quad\quad \mathrm{mod}\ m \\
-Nk^4 &\equiv (p - q)^2 k^4 \quad\quad \mathrm{mod}\ m \\
-N &\equiv (p - q)^2 \quad\quad \mathrm{mod}\ m,\ \text{since } \gcd(k, m) = 1 \\
N + (p - q)^2 &\equiv 0 \quad\quad \mathrm{mod}\ m
\end{aligned}
$$

Therefore m divides $N + (p - q)^2$. Since $N + (p - q)^2$ is positive, that implies

(16) $$ m \leq N + (p - q)^2 $$

By Lemma 11.11, we may assume $p + q + r < (N + 1)/4$, since that is true on one of the two sides of ABC of length X. In particular, each of p, q, r is $< (N + 1)/4$. Hence

$$
\begin{aligned}
|p - q| &< (N + 1)/4 \\
(p - q)^2 &< \frac{(N + 1)^2}{16}
\end{aligned}
$$

By (16), we have

$$ m < N + \frac{(N + 1)^2}{16}. $$

Since $k \leq m$, both a and b are $\leq m^2$. Hence both a and b are bounded by

$$ \left(N + \frac{(N + 1)^2}{16} \right)^2. $$

That completes the proof of the lemma.

By a *boundary tiling* of ABC by the tile (a, b, c), we mean a placement of tiles supported by the boundary of ABC touching every point of the boundary of ABC. Let X be the side AB, and Y the base AC, of isosceles ABC. A boundary tiling of ABC provides integers (p, q, r) and (u, v, w) with

$$
\begin{aligned}
X &= pa + qb + rc \\
X^2 &= Nab \quad\quad \text{the area equation} \\
Y &= ua + vb + wc
\end{aligned}
$$

We use the phrase *possible boundary tiling* to mean a way of writing X and Y in this form, with integers (p, q, r) and (u, v, w) satisfying Lemma 11.11, and (a, b, c) satisfying Lemma 11.12. Of course, a tiling gives rise to a boundary tiling, but not every boundary tiling can be completed to a tiling, let alone every "possible" boundary tiling. Each "possible boundary tiling" might correspond to many different ways of arranging the tiles (in different orders) on the boundary, but there will be only finitely many ways. We note that a boundary tiling might use different (p, q, r) on the two sides AB and BC; we shall return to that point below.

Lemma 11.13. *Let isosceles triangle ABC with base AC and vertex B have base angles α, not a rational multiple of π. Let (a, b, c) be integers with no common factor forming a triangle with angles $(\alpha, \beta, 2\alpha)$. Let X be the length of side AB Suppose there is an N-tiling of ABC by (a, b, c) whose boundary tiling on AB or BC corresponds to $X = pa + qb + c$. Then $q \neq 0$ and $r \neq 0$.*

Proof. First suppose $q = 0$. That means there are no b edges on AC, so every tile supported by AC has a β angle on AC. But since the top and bottom tiles have their α angles at C and A, and no vertex has two β angles, this violates the pigeonhole principle. Now suppose $r = 0$. That means there are no c edges on AC, so every tile supported by AC has a γ angle on AC. But since the top and bottom tiles have their α angles at C and A, and no vertex has two γ angles, this violates the pigeonhole principle. That completes the proof of the lemma.

Lemma 11.14. *Given N, there is a finite set Δ of tiles (a, b, c) having integer sides with no common factor and angles (α, β, γ), and for each tile in Δ, a finite number of representations*

$$X = pa + qb + rc$$
$$Y = ua + vb + wc$$

such that if isosceles triangle ABC with base angle α can be N-tiled with some tile, then the tile belongs to Δ and the boundary representations determined by the tiling are among those allowed for that tile.

Example. With $N = 36$, there are just two possible tiles: $(9, 16, 15)$ and $(16, 9, 20)$, as we will show below, and the possible boundary tilings are given in Table 3.

Proof. Let N be given. Then the number of possible boundary tilings is finite (and one can easily loop through them), by definition of "boundary tiling." By Lemma 11.2 and Lemma 11.12, every N-tiling gives rise to a possible boundary tiling.

We spell out the algorithm implicit in the preceding lemmas: Given N, we loop through all (k, m) satisfying Lemma 11.12 and with k and m relatively prime.[6] There are only finitely many (k, m) to loop through, because according to Lemma 11.12, we have an explicit bound on m in terms of N. Since $k < m$, both m and k are bounded in terms of N.

For each such (k, m), we compute the tile

$$(a, b, c) = (k^2, m^2 - k^2, mk).$$

We reject triples (a, b, c) that either

- cannot form triangles because one side is greater than or equal to the sum of the other two, or
- the squarefree part of N is not equal to the squarefree part of b

Then X is defined by the area equation $X^2 = Nab$, and Y is defined by the area equation $XY = Nbc$. We reject triples (a, b, c) if either

- X is not an integer, or
- Y is not an integer

---

[6] The condition k and m relatively prime is important, because it results in a and b being relatively prime, which is assumed in Lemma 11.11. Without it, we got some spurious possible boundary tilings with tiles like $(46, 45, 54)$, which cannot correspond to real tilings because Lemma 11.11 would be violated.

Then we loop through all triples (p, q, r) satisfying the bound of Lemma 11.11 such that

$$X \quad = \quad pa + qb + rc$$

We can immediately reject (p, q, r) if $q = 0$ or $r = 0$, by Lemma 11.13. Now $Y = (c/a)X$. We need to check whether Y can be expressed in the form $ua+vb+wc$. There are only finitely many possibilities for (u, v, w), so we can check that. If Y cannot be so expressed, then we can reject this (a, b, c).

Otherwise, we output the possible boundary tiling given by

$$X \quad = \quad pa + qb + rc$$
$$Y \quad = \quad ua + vb + wc.$$

That completes the proof of the lemma.

The algorithm as described above eliminates $N < 20$, but finds possible boundary tilings for $N = 20, 28, 36, 44, 45$. Below we will discuss improvements to the algorithm and eliminate some of these values.[7]

Theorem 11.15. *Given N, it is decidable by a computation whether there exists an N-tiling of some (any) triangle ABC by a tile with $\gamma = 2\alpha$, (where ABC has base angles α).*

Proof. For a fixed ABC and tile, it is (in principle) computationally decidable whether there is a tiling: By Lemma 11.14, there are finitely many possible boundary tilings so in principle you can all the possible ways of arranging tiles on the boundary, and check by backtracking search whether the boundary tiling can be completed to an N-tiling, just like solving a jigsaw puzzle. That completes the proof.

11.7. **Ruling out more values of N.** The problem of constructing an N-tiling divides into two parts: first construct an ABC and a tile (a, b, c), and a possible boundary tiling

$$X \quad = \quad pa + qb + rc$$
$$Y \quad = \quad ua + vb + wc$$

Then, use backtracking search to either find an N-tiling, or show that there is none.

The second part of this (the part involving backtracking) is not a trivial program, and even if coded, it would probably take too long when N is large. But the first algorithm (searching for a possible boundary tiling) is very easy to implement, as no geometry is involved, just some simple linear equations. We have already described that algorithm in Lemma 11.14.

Lemma 11.16. *Let isosceles triangle ABC with base angles α be N-tiled by an integer-sided triangle with angles $(\alpha, \beta, 2\alpha)$. Then $N \neq 20$.*

Remark. It seems that $N = 20$ is the *only* value of N (among those passed by Lemma 11.14) that can be rejected this way and is not rejected by simpler arguments; at least, it's the only one less than 1000.

---

[7]We coded the algorithm twice, once in SageMath, which offers unlimited precision integers, and once in C, taking care to use 64-bit integers in C. We got the same results from both implementations.

Proof. Let n be the minimum possible number n of tiles supported on the boundary; then there must be at least $n-1$ tiles with a side or vertex on the boundary, thanks to there being that many gaps between the tiles, and no double-counting of tiles filling those gaps, because of Lemma 11.8 and Lemma 11.9. Then if $n + (n - 2)$ exceeds N, we can reject that possible boundary tiling.

When $N = 20$, the tile $(4, 5, 6)$ leads to $X = 20 = a + 2b + c$, so X supports 4 tiles, and 20 cannot be written as a sum of fewer tiles. Then $Y = 30 = 5c$, $n = 11$, $n + (n - 1) = 21 > 20$. So this possible tile is rejected. Since that is the only possible tile for $N = 20$, there is no 20-tiling. That completes the proof.

Lemma 11.17. *Let isosceles triangle ABC with base AC and vertex B have base angles α, not a rational multiple of π. Let (a, b, c) be integers with no common factor forming a triangle with angles $(\alpha, \beta, 2\alpha)$. Let X and Y be the lengths of side AB and base AC respectively. Suppose*

$$X = pa + qb + c$$

and

$$Y = ua + vb + c \text{ or } ua + vb + 2c$$

Then there is no N-tiling of ABC by (a, b, c) whose boundary tilings correspond to those representations.

Remark. So, if the only possible boundary tilings are as in the lemma, then there is no N-tiling of ABC by (a, b, c) at all.

Example. Consider $N = 28$. Consider $(a, b, c) = (9, 7, 12)$. By the area equations (9) and (10), if there were a 28-tiling of an isosceles ABC with base angles α, we would have $X = 42$ and $Y = 56$. Then $X = a + 3b + c$ and $Y = 2a + 2b + 2c$. The lemma shows we cannot have a tiling corresponding to these representations of X and Y. Below we will show that these are the only possible decompositions of X and Y realizable in a tiling, thus ruling out a 28-tiling.

Proof. Suppose that such a tiling exists. By hypothesis, the decomposition $Y = ua + vb + wc$, with $w = 1$ or 2, corresponds to the tiles supported by AC, and the decomposition $X = pa + qb + c$ corresponds to the tiles supported by AB and the tiles supported by BC. That is, there is only one c edge on AB, only one c edge on BC, and at most two c edges on the base AC.

The tile T_1 at A has its α angle at A. Suppose, for proof by contradiction, that T_1 has its c edge on AB. Since there is only one c edge on AB, all the rest of the tiles supported by AB have a γ angle on AB. No tile has its γ angle at B, since the total angle at B is $\alpha + \beta$, which cannot be written in any other way as a rational linear combination of α and β.

Therefore the top tile on AB, with a vertex at B, has its γ angle to the south on AB. Since there cannot be two γ angles at any boundary vertex, all the tiles on AB above T_1 have their γ angles are to the south. Let the next tile (that is, next to T_1) supported by AB be T_3, and let the one between T_1 and T_3 be T_2; and let P be the shared vertex on AB of these three tiles. Then T_1 has its β angle at P, and T_3 has its γ angle at P. Hence T_2 has its α angle there. Then the a edge of T_2 is not shared with T_1. Since $a < c$, and the a edge of T_1 has both endpoints on the boundary of ABC, the c edge of T_2 is not shared with T_1. Hence it must be the b edge of T_2 that is shared with T_1. Then $b < a$. The remaining part of the a edge of T_1, namely $a - b$, must be composed of b edges, since its length is less than a and

less than b. Then a is a multiple of b, say $a = kb$. By Lemma 11.1, $c^2 = a^2 + ab$. Then $c^2 = k^2 b^2 + kb^2$, so b divides c as well as a, contradiction, since without loss of generality we may assume a, b, and c have no common divisor. This contradiction shows that T_1 does not have its c edge on AB (as was assumed at the beginning of this paragraph). Therefore, T_1 has its c edge on the base AC, rather than on AB.

Now let R be the eastern vertex of T_1, lying on AC, and as before let T_2 be the tile sharing the northeast edge of T_1, which is the a edge. As before T_2 cannot share its b or c edge with T_1, and therefore has its a edge there, so R is a vertex of both T_1 and T_2. Then the angle of T_2 at R is not α, and the angle of T_1 at R is β. At R the tile angles are either (β, γ, α) or $(\beta, \alpha, \alpha, \alpha)$. In either case the next tile on AC, with western vertex at R, has angle α at R.

Now this whole argument can be repeated, using vertex C instead of A. Therefore the tile on AC at C has its c edge on AC, and the next tile west of that has its α angle to the east; call that vertex S.

Then the two tiles at A and C account for both c edges that can occur on AC. Then every tile supported by RS does not have its c edge on AC, and therefore has a γ angle on AC. Since these γ angles do not occur at the endpoints of RS, the number of γ angles exceeds the number of vertices where they can occur. By the pigeonhole principle, there must be a vertex with two γ angles. But that is a contradiction. That completes the proof of the lemma.

Theorem 11.18. *Let isosceles triangle ABC with base angles α be N-tiled by a triangle with angles $(\alpha, \beta, 2\alpha)$, and α not a rational multiple of π. Then $N \geq 45$.*

Proof. $N = 20$ is eliminated by Lemma 11.16. By Lemma 11.14, the only possible tiles are the ones shown in Table 3.

TABLE 3. Possible boundary tilings

N	(a, b, c)	X	Y
28	(9,7,12)	$42 = a + 3b + c$	$56 = 2a + 2b + 2c$
36	(9,16,15)	$72 = a + 3b + c$	$120 = a + 6b + c$
36	(16, 9, 20)	$72 = a + 4b + c$	$90 = 2a + 2b + 2c$
44	(25, 11, 30)	$110 = a + 5b + c$	$132 = 2a + 2b + 2c$
45	(4,5,6)	$30 = a + 4b + c$	$45 = a + b + 6c$
45	(4,5,6)	$30 = 2a + 2b + 2c$	$45 = a + b + 6c$

It remains to eliminate 28, 36, and 44. According to Lemma 11.17, given N and (a, b, c), if the representations of X and Y as combinations of (a, b, c) involve only one c edge in X and one or two in Y, then there is no corresponding N-tiling. Adding that easily-checkable condition to the algorithm in Lemma 11.14, we find that $N = 28, 36$, and 44 are eliminated, leaving $N = 45$ as the least value for which possible boundary tilings are found.

For the benefit of the reader who prefers pencil and paper to computer code, we give a direct argument for each of those three values of N, with a few steps left to the reader's pencil:

We first take up the case $N = 28$, which was discussed in the example following the statement of Lemma 11.17. That lemma shows that no tiling is possible corresponding to the representations of X and Y given in Table 3. There cannot be other representations with more c edges, since no multiple of 12 cannot be made

from (some of) one 9 and three 7s, and no multiple of 12 can be made from (some of) two 9s and two 7s, and a and b are relatively prime.

It remains to consider representations with fewer c edges. In this case that would mean zero c edges; but that would contradict Lemma 11.13. Hence no 28-tiling of this isosceles triangle ABC by this (a, b, c) exists.

We turn to the case $N = 36$. Suppose, for proof by contradiction, there is a 36-tiling by $(9, 16, 15)$. By Table 3, $(X, Y) = (72, 120)$. I say that $72 = a + 3b + c$ is the only possible representation of 72 that could occur in the tiling. Suppose $72 = pa + qb + rc$. By Lemma 11.13, q and r are not zero. Suppose $r \geq 3$. Then $11 = pa + (q-1)b + (r-3)c$, which is impossible. If $r = 2$, then $26 = pa + (q-1)b$; but that is impossible. By Lemma 11.13, $r \neq 0$. Then r must be 1, so $57 = pa + qb$. We have $p \leq 6$ since $6 \cdot 9 > 57$; so $57 - pa$ is a multiple of 16. One can check that the only possibility is $p = 1$, $q = 3$, which gives the known decomposition $X = a + 3b + c$. (Another way of looking at this is that, if $r > 1$, we must be able to "trade in" some of one a and three bs for a number of cs, i.e., make a multiple of 15 out of (some of) one 9 and three 16s. But that is impossible.)

We also must show that $Y = a + 6b + c$ is the only decomposition of $Y = 120$ that could occur in the tiling.

Then by Lemma 11.17, those representations for X and Y in Table 3 do correspond to the tiling, Therefore there is no 36-tiling by the tile $(9, 16, 15)$.

Table 3 also has a second entry for $N = 36$, namely $(a, b, c) = (16, 9, 20)$, with $X = 72 = a + 4b + c$ and $Y = 90 = 2a + 2b + 2c$. One can check with pencil and paper that it is not possible to make a multiple of 20 with up to two a edges and up to four b edges. Then by Lemma 11.17, no such tiling exists. Since Table 3 represents the results of Lemma 11.14, there is no 36-tiling of any isosceles ABC.

Turning to $N = 44$, the tile would have to be $(25, 11, 30)$. To apply Lemma 11.17, given the representations of X and Y in the table, it suffices to check (with pencil and paper) that no multiple of 30 can be made of up to one a edges and up to five b edges, or up to two a edges and up to two b edges. That completes the proof of the theorem.

We note that the technique does not extend to $N = 45$. After that the next possibilities left open are $63, 64, 72$.

12. TILINGS OF AN ISOSCELES TRIANGLE BY A TILE $(\alpha, \beta, 2\pi/3)$

In this section we take up the tilings of an isosceles triangle (and not equilateral) ABC with base angles α or β, by a tiling with angles $(\alpha, \beta, 2\pi/3)$, where α is not a rational multiple of π. Let $\gamma = 2\pi/3$. We could insist that the base angles are called α, but then we may have to speak of tilings by $(5, 3, 7)$ instead of $(3, 5, 7)$; so it is convenient to allow the base angles to be β sometimes. If the base angles are α, then the vertex angle is $\pi - 2\alpha = \alpha + 3\beta$.

Laczkovich proved [7, Theorem 2.5] that there exist tiles that can be used to tile *some* such ABC, but N constructed by his method can be large. He proceeds by first constructing a dissection of ABC into *similar* rational triangles and parallelograms. Fig. 18 shows such a preliminary dissection. Then to get a tiling by congruent triangles, we have to choose a very small tile such that if each of the visible triangles is tiled quadratically, then every shared edge is an integer multiple of the tile edges. For example if the red triangle will get p^2 tiles and the light

blue triangle will get q^2 tiles, then we must satisfy $pb = qa$, in this case $5p = 3q$. There will be another such equation on every shared boundary. To clear all the denominators we will have to use a large number of tiles.

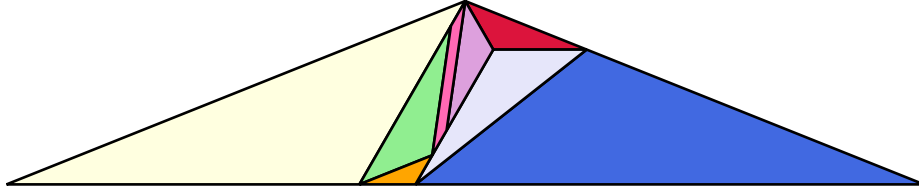FIGURE 18. With tile $(3, 5, 7)$, N would be 1878500, too large to draw.

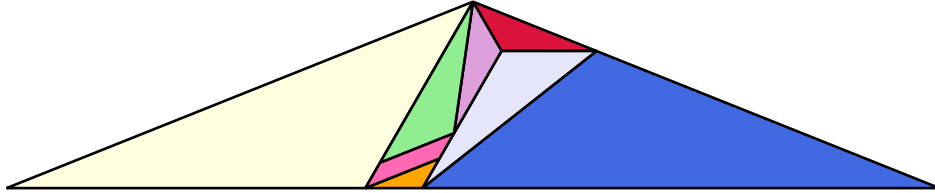FIGURE 19. With this configuration, N is 3681860–even larger.

Fig. 19 shows that a slightly different arrangement of the similar triangles and parallelograms can make a difference in the resulting number of tiles. In one figure, the parallelogram is tiled using a and c edges on the boundary; in the other figure, the parallelogram is tiled using a and b edges on the boundary. That makes the equations at the boundary different, even though the other boundary conditions are the same and the areas of the two parallelograms are equal.

Nevertheless, these tilings are too large to draw. In 2024, Bryce Herdt discovered a 2673-tiling, which is exhibited in the Appendix. This is presently the smallest known tiling of an isosceles triangle by a tile with $\gamma = 2\pi/3$.

12.1. **The tile is rational.** Suppose given an N-tiling of some triangle ABC by a tile with angles (α, β, γ) and sides (a, b, c). In this section we will prove that the tile has to be rational, i.e., the ratios of the sides are all rational, so after a suitable scaling, they will be integers. The proof uses the graphs Γ_c introduced by Laczkovich and described above; several preparatory lemmas will be developed first.

Definition 12.1. *A c/a segment is a left-terminated interior segment PQ of the tiling supporting two tiles on opposite sides of PQ, each with a vertex at P, one with its c edge on PQ and one with its a or b edge on PQ. The segment is said to "emanate from P." Similarly for c/b segment and a/b segment.*

Remarks. The point Q serves only to indicate the direction of the segment; it can be any point on that ray.

Lemma 12.2. *Let ABC be an isosceles triangle with base angles α, tiled by tile (α, β, γ) with $\gamma = 2\pi/3$. Let PQR be an internal segment of the tiling with only c edges on one side of PQ, such that the tile on that side supported by QR with a vertex at Q has an a or b edge on PQ. Then there is a c/a or c/b segment emanating from Q.*

Proof. We only consider tiles on the one side of PQR mentioned in the lemma. There are two cases: Either there are three tiles with a vertex at Q, or there are six.

Case 1, three tiles at Q. Then one of them has its γ angle at Q, and hence no c edge at Q. The other two have a c edge ending at Q. One of those lies on PQ. The other does *not* lie on QR, by hypothesis. Since there is no other c edge ending at Q, that third c edge forms either a c/b segment or a c/a segment.

Case 2, six tiles at Q. Then all six angles are α or β. Each of the six tiles has a c edge ending at Q. One lies on PQ and five lie on interior segments. Since five is odd, one of those c edges is not paired with another c edge, and hence constitutes a c/a segment or a c/b segment. That completes the proof of the lemma.

Lemma 12.3. *Suppose isosceles triangle ABC with base angles α is N-tiled by $(\alpha, \beta, 2\pi/3)$, with α not a rational multiple of π. Then there is a relation*

$$jc = pa + qb$$

with nonnegative integers p, q, j and $j > 0$.

Proof. Suppose, for proof by contradiction, that there is no such relation. Then, by Lemma 12.2, if PQ is a link in the graph Γ_c, then there is a c/b or c/a segment emanating from Q. Extend that segment to the maximal segment QR supporting only tiles with c edges on QR. Since there is no relation $jc = pa + qb$, R cannot be the vertex of a tile on the other side of QR. Therefore QR is a link in Γ_c.

Therefore the out-degree of every node Q in Γ_c is at least one. But the in-degree of Γ_c is always at most one. Since the total out-degree is equal to the total in-degree, it follows that every node of Γ_c has both in-degree and out-degree equal to 1. Since no link of Γ_c can terminate on the boundary of ABC, there can be no links of Γ_c emanating from a vertex on the boundary of ABC.

I say there is at least one c edge on AC. For if not, every tile supported by AC has its γ angle at a vertex on AC. Since $\gamma > \pi/2$, there cannot be two γ angles at any one vertex. But γ angles do not occur at A or C, where there are only α angles. Then there is one more γ angle on AC than possible vertices to receive them, so by the pigeonhole principle, some vertex on AC has two γ angles, contradiction. Therefore, as claimed, there is at least one c edge on AC. Similarly, there is at least one c edge on AB and at least one c edge on BC. Since there is a single tile at A with its α angle at A, the b edge of that tile lies on AC or on AB. Then there exists a segment PQ lying on AB or on AC supporting only tiles with c edges on PQ, and with a b edge beyond Q. Then by Lemma 12.2, there is a c/a segment or a c/b segment emanating from the boundary point Q, say QR. Choose R as far as possible from Q such that QR bounds only c tiles on one side. Then R is not a vertex of a tile on the other side of QR, since that would give rise to a relation $jc = pa + qb$ with $j > 0$. Hence QR is a link in Γ_c. But that is a contradiction, since Q is on the boundary of ABC. That completes the proof of the lemma.

Lemma 12.4. *Suppose isosceles triangle ABC with base angles α is N-tiled by $(\alpha, \beta, 2\pi/3)$, with α not a rational multiple of π. Then there is a relation*

$$ja = pb + qc$$

with nonnegative integers p, q, j and $j > 0$ and $p > 0$.

Proof. Suppose, for proof by contradiction, that there is no such relation. A **center** is a vertex of the tiling where three tiles meet, each having its γ angle at that vertex. A **star** is a vertex P where six tiles lying on one side of a line through P have three α and three β angles at P. A star can occur on the boundary of ABC or in the interior. A **double star** is a vertex where twelve tiles meet, six with α angles and six with β angles. The three vertices of ABC together have six angles, three β and three α, the same count as a star. Let $\mathcal{S}$ be the number of stars (counting a double star as two), and $\mathcal{C}$ the number of centers. Now let us calculate the number of α angles, plus the number of β angles, minus twice the number of γ angles. At each vertex other than stars, centers, and A, B, and C, we get zero. At each center we get -6. At each star we get 6 (and 12 at double stars). Adding them up we get $6\,\mathcal{S} - 6\,\mathcal{C} + 6$, where the final 6 is for A, B, and C together. Since the total number of α is N, the total number of β is N, and the total number of γ is N, we get zero for the grand total. That is, $6\,\mathcal{S} - 6\,\mathcal{C} + 6 = 0$. Then $\mathcal{S} = \mathcal{C} - 1$. (For example, in Fig. 19, we see one center and no stars.)

Now we consider the graph Γ_a. Every center has an out-link, since at a center P there are three tiles, each with an a edge and a b edge at P. Since 3 is odd, one of the a edges shares a segment with one of the b edges, i.e., an a/b edge emanates from P. (For example, note the center in Fig. 19.) Let Q be the farthest point from P along that segment such that PQ supports only a tiles on one side, say the "left" side. If Q were vertex of a tile on the other side, we would have a relation $ja = pb + qc$, and p would be positive since there is a b edge on the "right" side of PQ. Since by hypothesis, there is no such relation, Q is not a vertex of a tile on the other side. Then PQ is a link in Γ_a. On the the other hand, the in-degree of a center is zero, since no segment of the tiling passes through P.

At a star Q on an internal segment PQ, six tiles meet, providing six c edges, three a edges, and three b edges. There could be an incoming link at Q, if the tile on PQ at Q has its a edge there, the tile past Q does not have its a edge on PQ extended, and the other two a edges are not on the same segment. The in-degree of Γ_a can never exceed 1, since it is impossible for two lines of the tiling to cross at Q when a link ends at Q. At the vertices A, B, and C the in-degree is zero, since a link cannot terminate on the boundary. At A and C the out-degree is zero since there are no interior edges. At B there might be outgoing links, or not.

Now we calculate the out-degree minus the in-degree vertex by vertex. At centers it is 1. At stars it is 0 or -1 (or -2 possibly at double stars). Let t be the total out-degree minus in-degree at stars; then $0 \geq t \geq -S$. At A and C it is zero. At B it is non-negative, say n_B. At all other vertices it is zero. The total of out-degree minus in-degree is then $\mathcal{C} + t + n_B \geq \mathcal{C} - \mathcal{S}$. Since $\mathcal{S} = \mathcal{C} - 1$, the total out-degree minus in-degree is ≥ 1. On the other hand, it is zero since every link has a head and a tail. This contradiction completes the proof of the lemma.

Theorem 12.5. *Let ABC be an isosceles (and not equilateral) triangle with base angles α. Suppose ABC is tiled by a tile $(\alpha, \beta, 2\pi/3)$ with α not a rational multiple of π. Then the tile is rational.*

Proof. Suppose ABC is tiled as in the lemma. By Lemma 12.3 and Lemma 12.4, there are relations

$$jc = pa + qb \qquad \text{with nonnegative integers } j, p, q \text{ and } j > 0$$
$$Ja = Pb + Qc \qquad \text{with nonnegative integers } J, P, Q \text{ and } J > 0 \text{ and } P > 0$$

Dividing by c we have

$$j = p(a/c) + q(b/c)$$
$$-Q = -J(a/c) + P(b/c)$$

Case 1, $q \neq 0$. The equations can be solved for (a/c) and (b/c), provided the determinant $pP + Jq \neq 0$. Since $q \neq 0$ and $J \neq 0$, and $p \geq 0$ and $q \geq 0$, the determinant is not zero.

Case 2, $q = 0$. Then $a/c = j/p$ is rational by the first equation, and b/c is rational by the second equation, since $P \neq 0$.

That completes the proof of the theorem.

12.2. **The Diophantine equation** $c^2 = a^2 + b^2 + ab$**.** Let (a, b, c) be the sides of a triangle with angles $(\alpha, \beta, 2\pi/3)$. According to the law of cosines, we have

$$c^2 = a^2 + b^2 - 2ab\cos(2\pi/3)$$
$$= a^2 + b^2 + ab \qquad \text{since } \cos(2\pi/3) = -1/2$$

Therefore this Diophantine equation determines the possible rational triangles with a $2\pi/3$ angle.

Lemma 12.6. *Suppose $c^2 = a^2 + b^2 + ab$, and (a, b, c) are integers with no common factor. Then (a, b, c) are pairwise relatively prime.*

Proof. If prime p divides any two of (a, b, c) then it also divides the third one.

Lemma 12.7. *Suppose (a, b, c) are integers with no common factor that are the sides of a triangle with angles $(\alpha, \beta, 2\pi/3)$. Then $2b + a$ is relatively prime to each of a, b, and c, except that if a is even, 2 divides both a and $2b + a$.*

Proof. By Lemma 12.6, $2b + a$ is relatively prime to b and a, with the exception mentioned in the statement. It remains to prove $2b + a$ is relatively prime to c. Suppose, for proof by contradiction, that p is a prime that divides both c and $2b + a$. Then p is not 2, since then c and a would both be even, contradicting Lemma 12.6.

Suppose, for proof by contradiction, that $p = 3$. Then mod 3 we have $2b + a \equiv 0$. Adding b to both sides we have $3b + a \equiv b$. But $3b \equiv 0$, so $a \equiv b$. Now $c = a^2 + b^2 + ab = (a + b)^2 - ab \equiv a^2 \bmod 3$, since $a \equiv b$. Since $p|c$ we have $p|a^2$ and hence $p|a$. Hence a and b are both divisible by 3, contradiction, since (a, b) are relatively prime. Hence $p \neq 3$.

Then we have, mod p,

$$c \equiv 0$$
$$c^2 \equiv a^2 + b^2 + ab$$
$$2b + a \equiv 0$$

Substituting $c = 0$ in the last two equations we have

$$(17) \qquad\qquad 0 \equiv a^2 + b^2 + ab$$
$$(18) \qquad\qquad 2b + a \equiv 0$$

From the second equation we have $a \equiv -2b$. Since a and b are relatively prime, and $p \neq 2$, this implies that neither a nor b is divisible by p. Substituting $a = -2b$ in (17), we have

$$
\begin{aligned}
0 &\equiv 4b^2 + b^2 - 2b^2 \\
&\equiv 3b^2 \\
0 &\equiv b^2 \qquad \text{since } p \neq 3 \\
0 &\equiv b
\end{aligned}
$$

Then $a \equiv -2b \equiv 0$. Hence p divides both a and b, contradiction, since a and b are relatively prime. That completes the proof of the lemma.

Remark. Using the techniques of [3], Corollary 6.3.15, p. 353, we are able to parametrize the solutions of $c^2 = a^2 + b^2 + ab$ by two integer parameters (s,t) or one rational parameter s/t. Having worked this out, and used it in preliminary versions, in the end I found simpler proofs without it. Nevertheless I mention the reference in case it may be useful to somebody.

12.3. The area equation for an isosceles tiling with $\gamma = 2\pi/3$.

Lemma 12.8. *Let isosceles triangle ABC with base angles α be N-tiled by a tile with angles $(\alpha, \beta, 2\pi/3)$. Suppose α is not a rational multiple of π. Let X be the length of the equal sides AB and BC, and Y the length of the base AC. Then the area equation is*

$$
X^2(2b + a) = Nbc^2
$$

and another form of the area equation is

$$
XY = Nbc
$$

Proof. By the law of cosines,

$$
\begin{aligned}
a^2 &= b^2 + c^2 - 2bc\cos\alpha \\
\cos\alpha &= \frac{b^2 + c^2 - a^2}{2bc} \\
&= \frac{b^2 + (a^2 + b^2 + ab) - a^2}{2bc} \\
&= \frac{2b^2 + ab}{2bc} \\
(19) \qquad \cos\alpha &= \frac{2b + a}{2c}
\end{aligned}
$$

Twice the area of ABC is $X^2 \sin(\pi - 2\alpha) = X^2 \sin 2\alpha$. Twice the area of the tile is $bc\sin\alpha$. Equating the area of ABC to N times the area of the tile, we have

$$
\begin{aligned}
X^2 \sin 2\alpha &= Nbc\sin\alpha \\
2X^2 \sin\alpha \cos\alpha &= Nbc\sin\alpha \\
2X^2 \cos\alpha &= Nbc
\end{aligned}
$$

Substituting for $\cos\alpha$ the value from (19),

$$
\begin{aligned}
2X^2 \left(\frac{2b + a}{2c}\right) &= Nbc \\
X^2(2b + a) &= Nbc^2
\end{aligned}
$$

That completes the proof of the first formula of the lemma.

To prove the second form: twice the area of ABC is $XY\sin\alpha$. Twice the area of the tile is $bc\sin\alpha$. Therefore $XY = Nbc$. That completes the proof of the lemma.

12.4. A necessary condition.

Lemma 12.9. *Let isosceles triangle ABC with base angles α be N-tiled by a tile with angles $(\alpha, \beta, 2\pi/3)$ and sides (a, b, c). Suppose α is not a rational multiple of π. Then*

(i) $2b + a$ *divides* N*, and* $Nb/(2b+a)$ *is a square, say* m^2.

(ii) The side and base of ABC are given by

$$X = mc$$
$$Y = m(2b+a)$$

Remarks. This lemma gives us an *a priori* bound on (a, b, c), namely $2N$, since $c^2 = a^2 + b^2 + ab \le (a+b)^2 \le (2b+a)^2 \le (2N)^2$. Also, if N is prime, $N = 2b + a$, and $b = m^2$. It is unknown if this actually can happen.

Proof. Let X be the length of the equal sides AB and BC. According to Lemma 12.8,

$$X^2(2b+a) = Nbc^2.$$

By Lemma 12.6, a, b, and c are pairwise relatively prime. By Lemma 12.7, if a is odd, then $2b + a$ is relatively prime to each of a, b, and c. On the other hand, if a is even, then b and c are odd, so $2b + a$ is relatively prime to c and b. Thus, whatever the parity of a, $2b + a$ is relatively prime to b and c. Then by the area equation, $2b + a$ divides N.

According to the area equation,

$$\left(\frac{X}{c}\right)^2 = \left(\frac{Nb}{2b+a}\right)$$

Therefore $Nb/(2b+a)$ is a rational square, and since it is an integer, it is an integer square, say m^2. That completes the proof of part (i).

Ad (ii). Since X/c and m are positive and have equal squares, they are equal, so $X = cm$ as claimed. We compute Y:

$$
\begin{aligned}
\cos\alpha &= \frac{b^2 + c^2 - a^2}{2bc} && \text{by the law of cosines} \\
&= \frac{2b^2 + ab}{2bc} && \text{since } c^2 = a^2 + b^2 + ab \\
&= \frac{2b + a}{2c} \\
Y &= 2X\cos\alpha && \text{where } X = |AB| \\
&= \frac{X}{c}(2b+a)
\end{aligned}
$$

By part (i), c divides X, so the right-hand side is an integer. That completes the proof of the lemma.

Example 1. In the tiling whose construction begins with Fig. 19, we have $N = 75140$, $(a, b, c) = (3, 5, 7)$, so $2b + a = 13$, and $Nb/(2b+a) = 170^2$, so $m = 170$, $X = mc = 1190$, and $Y = m(2b+a) = 2210$.

Example 2. With $N = 33$, $(a, b, c) = (5, 3, 7)$, $2b + a = 11$, $Nb/(2b + a) = 15$, $m = 3$, $X = mc = 21$, and $Y = m(2b + a) = 33$. We think that no such tiling exists, although the present lemma does not rule it out. In principle one can "just check all the possibilities", but that is easier said than done.

Example 3. With $N = 37$, $(a, b, c) = (5, 16, 19)$, $2b + a = 37$, $Nb/(2b + a) = 16$, $m = 4$, $X = mc = 76$, and $Y = m(2b + a) = 148$. See Fig. 20. We prove in Theorem 12.11 that no such tiling exists. The method of proof does not depend on 37 being prime and does not extend to $N = 71$.

12.5. **Ruling out small values of N.**

Theorem 12.10. *If there is an N-tiling of some isosceles triangle ABC with base angles α by a tile with angles $(\alpha, \beta, 2\pi/3)$, then N is at least 33. If $N \leq 200$ then N is one of the values shown in Table 4, and the side and base of ABC must be as given in the table.*

TABLE 4. $N < 200$ and (a, b, c) not ruled out by Lemma 12.9.

N	(a, b, c)	(X, Y)
33	(5, 3, 7)	(21, 33)
37	(5, 16, 19)	(76, 148)
46	(7, 8, 13)	(52, 92)
65	(3, 5, 7)	(35, 65)
71	(39, 16, 49)	(196, 284)
74	(56, 9, 61)	(183, 222)
130	(16, 5, 19)	(95, 130)
132	(5, 3, 7)	(42, 66)
148	(5, 16, 19)	(152, 296)
154	(8, 7, 13)	(91, 154)
184	(7, 8, 13)	(104, 184)
193	(143, 25, 157)	(785, 965)

Remarks. We do not suggest that tilings for N in the table do, or do not, exist, only that they are not ruled out by the simple considerations of area and boundary tiling.

The prime numbers 37, 71, and 193 are not ruled out immediately, and two of those are congruent to 3 mod 4. Hence the possibility of N prime for this kind of tiling is not ruled out by the area equation and boundary-tiling conditions; but at least the cases 7, 11, 19 are eliminated, which is required for a proof that there are no N-tilings of any triangle for those values of N.

Actually, we are able to rule out $N = 37$; see Theorem 12.11 below. But the argument is special to $N = 37$, and does not appear to have anything to do with the primality of 37.

Proof. Let the positive integer N be given, and suppose there is an N-tiling of some isosceles ABC by a tile $(\alpha, \beta, 2\pi/3)$. By Theorem 12.5, the tile is rational, so we may suppose its sides are integers (a, b, c) with no common divisor. According to Lemma 12.9, $2b + a$ divides N (so a and b are at most N), and $Nb/(2b + a)$ is a square, say m^2. Then, since the tile has a $2\pi/3$ angle, c is determined by the

equation $c^2 = a^2 + b^2 + ab$. If c is not an integer, then we do not consider (a, b, c) further. Also if (a, b, c) is not a triangle, because the sum of two of its sides is less than the third, we do not consider it further. Table 4 was computed by running this algorithm for $N \leq 200$. There are no entries for $N < 33$. That completes the proof of the theorem.

We note that it would be a waste of time to compute the length of the base Y and reject (a, b, c) in case Y is not an integer, because Y always *has* to be an integer $m(a+2b)$, by Lemma 12.9. Similarly, it would be a waste of time to look for possible boundary tilings in the hope of rejecting some tiles, since by Lemma 12.9, with $m = X/3$ we always have $X = mc$ and $Y = ma + 2mb$.

Theorem 12.11. *There is no 37-tiling of an isosceles triangle with base angles α, using a tile with $\gamma = 2\pi/3$.*

Proof. By Theorem 12.10, the tile would have to be $(a, b, c) = (5, 16, 19)$. Then the area equation can be used to show that $(X, Y) = (76, 148)$. That makes the altitude of ABC equal to $10\sqrt{3} = 17.32$. If there is a tiling, there must be a four tiles at B, three of which have their β angles at B, and the other its α angle there. Number those tiles 1 to 4 starting from AB and ending at BC. Renaming A and C if necessary, we may assume that the α angle at B belongs to Tile 1 or Tile 2, so Tile 3 and Tile 4 have their β angle at B. Those two tiles each have a c edge. The case when Tile 4 has its c edge in the interior is shown in Fig. 20. In that position,

FIGURE 20. The tile is inside ABC, but just barely.
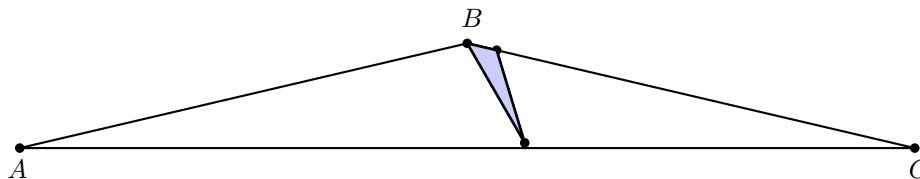
a tile barely fits into triangle ABC, and its eastern b edge cannot be matched by another tile's b edge, for that tile would not be inside ABC. Nor can tiles be laid there with a edges; so this case is impossible. Therefore Tile 4 has its c edge on BC, and shares its a edge with Tile 3. See Fig. 21.

FIGURE 21. No 37-tiling: Tile 4 with its c edge on the boundary.
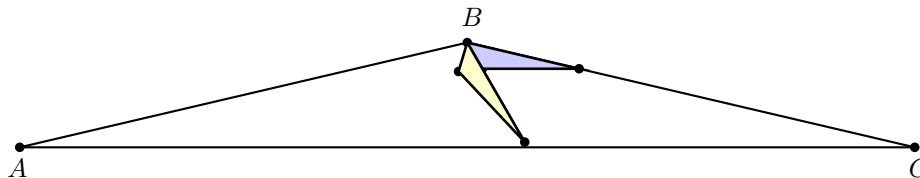
Tile 3 cannot have its c edge on the west, as a c edge emanating from B at that angle would extend past AC. (Its y-coordinate, with AC on the x-axis, would be -0.866.) Therefore it has its c edge on the east, next to the a edge of Tile 4, leaving an impossible situation, as a b or c edge will not fit inside ABC on the east of Tile 3, nor will any number of a edges. That contradiction completes the proof.

Remark. The next case of prime N to consider would be $N = 71$. It does not seem fruitful to continue this game by hand; and in this paper, we abstain from the attempt to establish non-existence results by computer search, because of the difficulty of establishing the correctness of such results beyond a shadow of doubt. It is true that we used a computer in Theorem 12.10, but only in the most trivial way: a doubtful reader could easily replicate Table 4, perhaps even by hand.

12.6. **Given N, find the possible tiles and ABC.**

Theorem 12.12. *Given N, we can efficiently compute a finite set Δ of (a, b, c, X), such that if there is an N-tiling of some isosceles triangle ABC with base angles α by a tile with $\gamma = 2\pi/3$, then the tile is (a, b, c) and the side of ABC is X, for some (a, b, c) in Δ.*

Remark. Then by backtracking search, applied to each tile (a, b, c) and isosceles triangle ABC with base angles α and side X with (a, b, c, X) in Δ, we can determine (in principle) if any N-tiling of any isosceles ABC exists. But we do not undertake that in this paper; see the previous remark.

Proof. Let N be given. The algorithm given in the proof of Theorem 12.10 determines the possible tiles (a, b, c), in such a way that $N/(2b + 1)$ is a square, say m^2. Then $X = mc$ must be the side of triangle ABC, if there is any N-tiling of isosceles ABC by (a, b, c), and $Y = m(2b + a)$ is the base, by Lemma 12.9. That completes the proof of the theorem.

13. Open problems

The methods and results of this paper leave us still unable to answer some interesting questions. Here we list several. In the following, as elsewhere in this paper, "isosceles" means "isosceles and not equilateral."

(i) What is the smallest N such that some isosceles triangle with base angles α can be N-tiled by a tile of the form $\gamma = 2\alpha$? The smallest such tiling so far explicitly constructed has 1125 tiles, but for all we know there is a 45-tiling. In fact, we do not even know the smallest N such that some isosceles triangle can be tiled by the tile with sides $(4, 5, 6)$ (which is the tile used in the 1125-tiling).

(ii) Is it possible to N-tile some isosceles triangle with N a prime number, when the tile has $\gamma = 2\pi/3$? If it is possible, N has to be at least 71. (For right-angled tiles, it is possible when N is congruent to 1 mod 4, but not when N is congruent to 3 mod 4; when $\gamma = 2\alpha$ it is never possible.)

(iii) Find easily checkable necessary and sufficient conditions on N for the existence of N-tilings of some isosceles ABC with $\gamma = 2\alpha$ or $\gamma = 2\pi/3$. Or, determine the existence or non-existence of such tilings, one N at a time, using exhaustive computer search. You can use Table 4 for your initial test data.

14. Conclusions

We have studied the possible tilings of an isosceles (and not equilateral) triangle ABC by a tile that is a right triangle, or by a tile of the form $(\alpha, \beta, 2\alpha)$ where the base angles of ABC are equal to α. In the case of a tile $(\alpha, \beta, 2\alpha)$, we derived a necessary condition from the area equation, and we made use of directed graphs inspired by Laczkovich to prove that the tile is necessarily rational.

We analyzed the case of a right-angled tile thoroughly enough to give a complete characterization of the possible values of N for which some isosceles ABC can be N-tiled. Namely, Theorem 7.8 says N is twice a square or twice an even sum of squares, except of course for the right isosceles triangle, which can be quadratically N-tiled for any square N, including odd squares.

In the case of a tile with $\gamma = 2\alpha$, we gave a necessary condition, using the area equation and the law of cosines for the tile. That this necessary condition is not trivial is shown by our proof that N cannot be prime. $N = 45$ is the least number for which we do not know whether a tiling exists, and 1125 is the smallest N for which we are certain that there does exist a tiling.

Finally, in the case of a tile with $\gamma = 2\pi/3$, we gave a necessary condition and an algorithm to check it. There are no such tilings for $N < 33$. There is one for $N = 75140$. Between those two values of N, there are many values of N satisfying our necessary conditions, for which we do not know whether tilings exist.

In all the possible cases of Laczkovich's tables, we have been able to show (either in this paper or in unpublished work) that given N, there is a finite set Δ of tiles (a, b, c) and triangles ABC such that either there is no N-tiling falling under that line of the table, or one of the finite set permits an N-tiling. Hence, there is (in principle) an algorithm, albeit inefficient, for determining if there is an N-tiling. The inefficiency arises from the exponentially large number of ways of trying to place N tiles of a specific shape into a specific ABC.

See the previous section for a list of open problems.

15. Appendix: Tilings found by Bryce Herdt

In 2024, Bryce Herdt found several new tilings with tiles $(4, 5, 6)$ (for which $\gamma = 2\alpha$) and $(3, 5, 7)$ (for which $\gamma = 2\pi/3$). These tilings dramatically lowered the N for the "smallest known tiling", both of isosceles triangles and of equilateral triangles. Here we exhibit Herdt's tilings.

Consider Fig. 11. The calculations after that figure give the number of tiles required on each edge in that figure. All those numbers are divisible by 6, except $m = 869$. Therefore, if we shrink the diagram by a (linear) factor of 6, we can still tile all the triangles in the figure with $(4, 5, 6)$, and the yellow parallelogram will be 869 by 480 (since it was $576 \cdot 5$, after shrinking it is $576 \cdot 5/6 = 480$. That 869 by 480 parallelogram cannot be tiled with tiles all in the same orientation. But, as Herdt pointed out to me (in 2024), often a parallelogram can be divided into two smaller parallelograms, which can be tiled with tiles in different orientations. Fig. 22 illustrates the technique. In the case at hand, $869 = 425 + 444 = 85b + 74c$, while 480 is divisible by b and c (5 and 6). The final value of N will be $6028020/6^2 = 167445$. That is, unfortunately, still too large to draw in a space smaller than several meters.

FIGURE 22. Decomposing a parallelogram with top $pc + qb$ and side bc
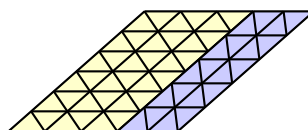
Starting from a different dissection of the triangle into triangles and parallelograms, Herdt was able to construct the 1125-tiling shown in Fig. 23. Observe how the parallelogram-dissection technique has been applied twice, in both the lower left and the lower right corners of the tiling.

FIGURE 23. Herdt's 1125-tiling by $(4, 5, 6)$
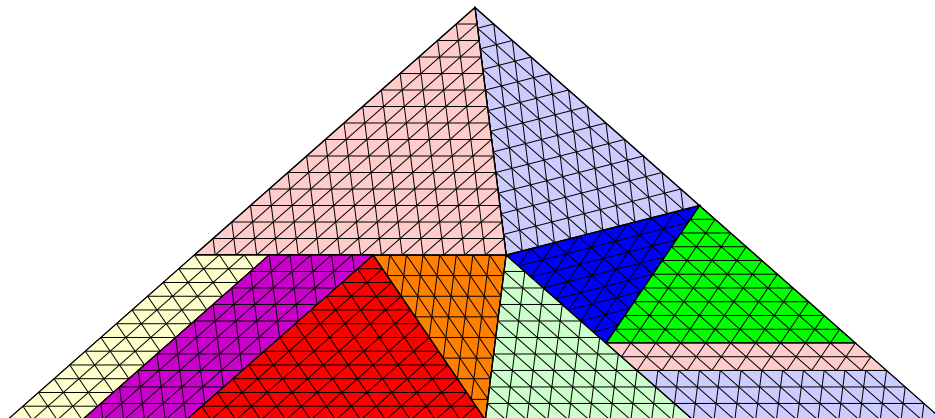
Dissecting ABC into similar triangles and parallelograms in a different way, Herdt was able to reduce N still further, to 720, as shown in Fig. 24.

FIGURE 24. Herdt's 720-tiling by $(4, 5, 6)$
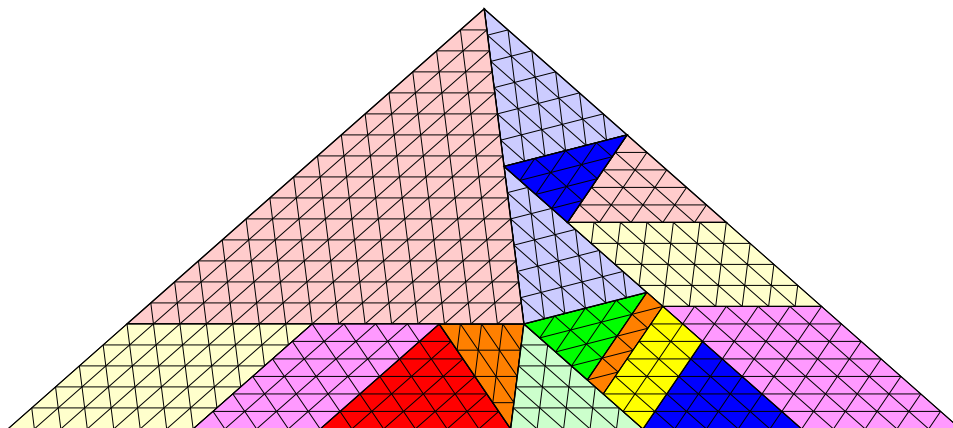
Herdt also found new tilings with the tile $(3, 5, 7)$, which has $\gamma = 2\pi/3$. He began by finding a 1215-tiling of an equilateral triangle of side 135; this was also done by his technique of decomposing parallelograms, starting with a known 10935-tiling. Then he flanked that equilateral triangle by two triangles similar to $(3, 5, 7)$, producing the tiling shown in Fig. 25. Here $N = 1215 + 2 \cdot 27^2 = 2673$. This is presently the smallest known tiling of an isosceles triangle by a tile with $\gamma = 2\pi/3$.

FIGURE 25. Herdt's 2673-tiling by $(3, 5, 7)$. Here $\gamma = 2\pi/3$.
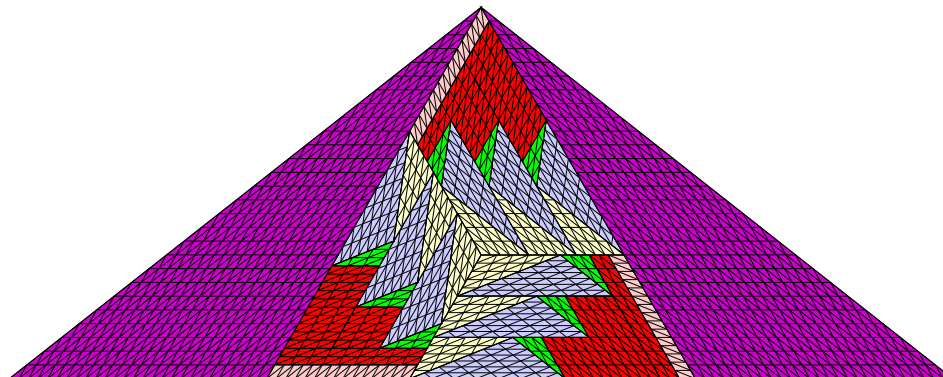
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