HERBRAND'S THEOREM AND NON-EUCLIDEAN GEOMETRY

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Abstract. We use Herbrand's theorem to give a new proof that Euclid's parallel axiom is not derivable from the other axioms of first-order Euclidean geometry. Previous proofs involve constructing models of non-Euclidean geometry. This proof uses a very old and basic theorem of logic together with some simple properties of ruler-and-compass constructions to give a short, simple, and intuitively appealing proof.

§1. Introduction. We intend this paper to be read by mathematicians who are unfamiliar with mathematical logic and also unfamiliar with non-Euclidean geometry; therefore we ask the patience of readers who are familiar with one or both of these subjects.

We begin with a brief discussion of axioms for plane Euclidean geometry. Every such axiom system will have variables for points. Some axiom systems may have variables for other objects, such as lines or angles, but Tarski showed that these are not really necessary. For example, angles can be discussed in terms of ordered triples of points, and lines in terms of ordered pairs of points. For simplicity we focus on such a points-only axiomatization.

The primitive relations of such a theory usually include a "betweenness" relation, and an "equidistance" relation. We write $\mathbf{T}(a, b, c)$ to express that b lies (non-strictly) between a and c (on the same line), and $\mathbf{E}(a, b, c, d)$ to express that segment ab is congruent to segment cd. $\mathbf{E}$ stands for "equidistance", because in the standard model "congruent" means that the distance ab is equal to the distance cd; but there is nothing in the axioms about numbers to measure distance, or about distance itself. Sometimes it is convenient to use $\mathbf{B}(a, b, c)$ for strict betweenness, i.e., $a \neq b$ and $b \neq c$ and $\mathbf{T}(a, b, c)$.

Some of the axioms will assert the existence of "new" points that are constructed from other "given" points in various ways. For example, one axiom says that segment ab can be extended past b to a point x, lying on the line determined by ab, such that segment bx is congruent to a given segment pq. That axiom can be written formally, using the logician's symbol $\wedge$ for "and", as

$$\exists x\, (\mathbf{T}(a, b, x) \wedge \mathbf{E}(b, x, p, q)).$$

It is possible to replace the quantifier $\exists$ with a "function symbol". We denote the point x that is asserted to exist by $ext(a, b, p, q)$. Then the axiom looks like

$$\mathbf{T}(a, b, ext(a, b, p, q)) \wedge \mathbf{E}(b, ext(a, b, p, q), p, q).$$

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This transformation is called *Skolemization.* This form is called "quantifier-free", because $\exists$ and $\forall$ are called "quantifiers", and we have eliminated the quantifiers. Although the meaning of the axioms is the same as if it had $\forall a, b, p, q$ in front, the $\exists$ has been replaced by a function symbol.

When a theory has function symbols, then they can be combined. For example, $ext(a, b, ext(u, v, p, q), ext(a, b, p, q))$ is a term. The definition of "term" is given inductively: variables are terms, constants are terms, and if one substitutes terms in the argument places of function symbols, one gets another term.

In Tarski's axiomatization of geometry, there are only a few axioms that are not already quantifier-free. One of them is the segment extension axiom already discussed. Another is Pasch's axiom. Moritz Pasch originally proposed this axiom in 1852, to repair the defects of Euclid. It intuitively says that if a line meets one side of a triangle and does not pass through the endpoints of that side, then it must meet one of the other sides of the triangle. In other words, under certain circumstances, there will exist the intersection point of two lines. A quantifier-free version of Tarski's axioms will contain a function symbol for the point asserted to exist by (a version of) Pasch's axiom.

Another axiom in Tarski's theory asserts the existence of an intersection point of a circle and a line, provided the line has a point inside and a point outside the circle. Another function symbol can be introduced for that point. Then the terms of this theory correspond to certain ruler-and-compass constructions. The number of symbols in such a term corresponds to the number of "steps" required with ruler and compass to construct the point defined by the term.

The starting point for the work reported here is this: a quantifier-free theory of geometry, whose terms correspond to ruler-and-compass constructions, viewed as a special case of situation of much greater generality: some first-order, quantifier-free theory. Herbrand's theorem applies in this much greater generality, and we will simply investigate what it says when specialized to geometry.

§2. Herbrand's theorem. Herbrand's theorem is a general logical theorem about any axiom system whatsoever that is

- first-order, i.e., has variables for some kind(s) of objects, but not for sets of those objects, and
- quantifier-free, i.e., $\exists$ has been replaced by function symbols

Herbrand's theorem says that under these assumptions, if the theory proves an existential theorem $\exists y\, \phi(a, y)$, with ϕ quantifier-free, then there exist finitely many terms $t_1, \ldots, t_n$ such that the theory proves

$$\phi(a, t_1(a)) \vee \phi(a, t_2(a)) \cdots \vee \cdots \phi(a, t_n(a)).$$

The formula ϕ can, of course, have more variables that are not explicitly shown here, and a and x can each be several variables instead of just one, in which case the t_i stand for corresponding lists of terms. For a proof see [1], p. 48.

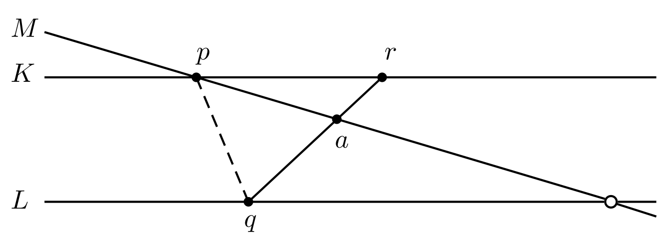
In order to illustrate the theorem, consider the example when ϕ is $\phi(a, b, c, x, y)$, and it says that $a \neq b$, and x lies on the line determined by ab, and y does not lie on that line, and xy is perpendicular to ab and c is between x and y. Collinearity can be expressed using betweenness, and the relation $xy \perp ab$ can also be expressed using betweenness and equidistance. Then $\exists x, y\, \phi(x, y)$ says that there exists a line through point c perpendicular to ab. Usually in geometry, we give two different constructions for such a line, according as c lies on line ab or not. If it does, we "erect" a perpendicular at c, and if it does not, we "drop" a perpendicular from c to line ab. When we "drop" a perpendicular, we compute $foot_1(a, b, c)$, and we can define $head_1(a, b, c) = c$. When we "erect" a perpendicular, we compute $head_2(a, b, c)$, and we can define $foot_2(a, b, c) = c$. Thus if c is not on the line, we have $\phi(a, b, c, foot_1(a, b, c), head_1(a, b, c))$, and if c is on the line, we have $\phi(a, b, c, foot_2(a, b, c), head_2(a, b, c)$. Since c either is or is not on the line we have

$$\phi(a, b, c, foot_1(a, b, c), head_1(a, b, c)) \vee \phi(a, b, c, foot_2(a, b, c), head_2(a, b, c))$$

Comparing this to Herbrand's theorem, we see that we have specifically constructed examples of two lists (of two terms each) t_1 and t_2 illustrating that Herbrand's theorem holds in this case. Herbrand's theorem, however, tells us *without doing any geometry* that if there is any proof at all of the existence of a perpendicular to ab through c, from the axioms of geometry mentioned above, then there must be a finite number of ruler-and-compass constructions such that, for every given a, b, c, one of those constructions works. We have verified, using geometry, that we can take the "finite number" of constructions to be 2 in this case, but the beauty of Herbrand's theorem lies in its generality.

§3. Non-Euclidean geometry. Euclid listed five axioms or postulates, from which, along with his "common notions", he intended to derive all his theorems. The fifth postulate, known as "Euclid 5", had to do with parallel lines, and is also known as the "parallel postulate." See Fig. 1.

FIGURE 1. Euclid 5. M and L must meet on the right side, provided $\mathbf{B}(q, a, r)$ and pq makes alternate interior angles equal with K and L.



From antiquity, mathematicians felt that Euclid 5 was less "obviously true" than the other axioms, and they attempted to derive it from the other axioms. Many false "proofs" were discovered and published. All this time, mathematicians felt that geometry was "about" some true notion of space, which was either given by the physical space in which we live, or perhaps by the nature of the human mind itself. Finally, after constructing long chains of reasoning from the assumption that the parallel postulate is false, some people came to the realization that there could be "models of the axioms" in which "lines" are interpreted as certain curves, and "distances" also have an unusual interpretation. Such models were constructed in which Euclid 5 is false, but the other axioms are true. Hence, Euclid 5 can never be proved from the other axioms. There was a good reason for all those failures! See [3] and [5] for the full history of these fascinating developments, and descriptions of the models in question.

§4. Tarski's axioms for geometry. In order to state our theorem precisely, we need to mention a specific axiomatization of geometry. For the sake of definiteness, we use the axioms (A1-A11) of Tarski, as set forth in the definitive reference [9]. We list those axioms in Table 1. Those who do not read German can consult [12].

Of these axioms, we need concern ourselves in detail only with those few that are not already quantifier-free. Axiom (A4) is the segment extension axiom discussed above; we introduce the symbol $ext(a, b, p, q)$ to express it in quantifier-free form. The lower dimension axiom (A8) states that there exists three non collinear points. We introduce three constants α, β, and γ to express it in quantifier-free form. The two modified axioms are explicitly:

4.1. Pasch's axiom. Moritz Pasch [7] (see also [8], with an historical appendix by Max Dehn) supplied (in 1882) an axiom that repaired many of the

Table 1. Tarski's axioms for geometry

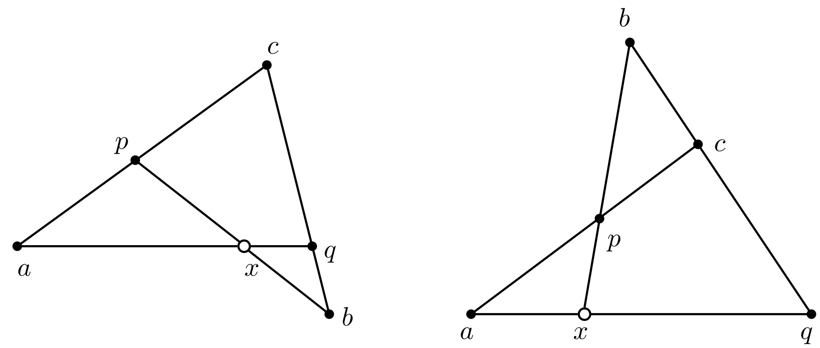
A1	Symmetry	$\mathbf{E}(a,b,b,a)$
A2	Pseudo-Transitivity	$\mathbf{E}(a,b,c,d) \wedge \mathbf{E}(a,b,e,f) \ \rightarrow\ \mathbf{E}(c,d,e,f)$
A3	Cong Identity	$\mathbf{E}(a,b,c,c) \ \rightarrow\ a=b$
A4	Segment extension	$\exists e(\mathbf{T}(a,b,e) \wedge \mathbf{E}(b,e,c,d))$
A5	Five segments	$\mathbf{E}(a,b,a',b') \wedge \mathbf{E}(b,c,b',c') \wedge$ $\mathbf{E}(a,d,a',d') \wedge \mathbf{E}(b,d,b',d') \wedge a \neq b \wedge$ $\mathbf{T}(a,b,c) \wedge \mathbf{T}(a',b',c') \ \rightarrow\ \mathbf{E}(c,d,c',d')$
A6	Between Identity	$\mathbf{T}(a,b,a) \ \rightarrow\ a=b$
A7	Inner Pasch	$\mathbf{T}(a,p,c) \wedge \mathbf{T}(b,q,c) \ \rightarrow$ $\exists x\,(\mathbf{T}(p,x,b) \wedge \mathbf{T}(q,x,a))$
A8	Lower Dimension	$\exists abc(\neg\mathbf{T}(a,b,c) \wedge \neg\mathbf{T}(b,c,a) \wedge \neg\mathbf{T}(c,a,b))$
A9	Upper Dimension	$\mathbf{E}(a,p,a,q) \wedge \mathbf{E}(b,p,b,q) \wedge \mathbf{E}(c,p,c,q) \wedge p \neq q$ $\rightarrow\ \mathbf{T}(a,b,c) \vee \mathbf{T}(b,c,a) \vee \mathbf{T}(c,a,b)$
A10	Parallel	$\exists xy(\mathbf{T}(a,d,t) \wedge \mathbf{T}(b,d,c) \wedge a \neq d \ \rightarrow$ $\mathbf{T}(a,b,x) \wedge \mathbf{T}(a,c,y) \wedge \mathbf{T}(x,t,y))$
A11	Continuity	$\forall XY((\exists a(\forall xy, x \in X \wedge y \in Y \ \rightarrow\ \mathbf{T}(a,x,y)))$ $\rightarrow\ \exists b(\forall xy, x \in X \wedge y \in Y \ \rightarrow\ \mathbf{T}(x,b,y)))$
CA	Circle axiom	$\mathbf{T}(a,x,b) \wedge \mathbf{T}(a,b,y) \wedge \mathbf{E}(a,x,a,p) \wedge$ $\mathbf{E}(a,q,a,y) \ \rightarrow\ \exists z(\mathbf{E}(a,z,a,b) \wedge \mathbf{T}(p,z,q))$

Table 2. Axioms A4 and A8 in quantifier-free form

A4′	Segment extension	$\mathbf{T}(a,b,e) \wedge \mathbf{E}(b,e,c,d))$
A8′	Lower Dimension	$\neg\mathbf{T}(\alpha,\beta,\gamma) \wedge \neg\mathbf{T}(\beta,\gamma,\alpha) \wedge \neg\mathbf{T}(\gamma,\alpha,\beta))$

defects that nineteenth-century rigor found in Euclid. Roughly, a line that enters a triangle must exit that triangle. As Pasch formulated it, it is not in $\forall\exists$ form. There are two $\forall\exists$ versions, illustrated in Fig. 2. These formulations of Pasch's axiom go back to Veblen [13], who proved outer Pasch implies inner Pasch. Tarski originally took outer Pasch as an axiom. In [4], Gupta proved both that inner Pasch implies outer Pasch, and that outer Pasch implies inner Pasch, using the other axioms of the 1959 system. In the final version [9], inner Pasch is an axiom. Here are the precise statements of the axioms illustrated in Fig. 2:

FIGURE 2. Inner Pasch (left) and Outer Pasch (right). Line pb meets triangle acq in one side. The open circles show the points asserted to exist on the other side.



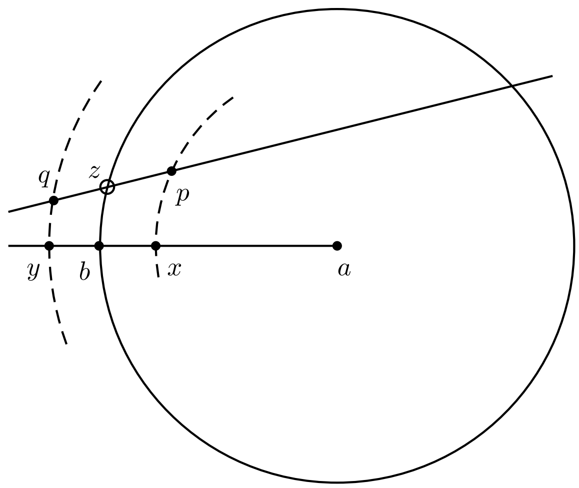
$$\mathbf{T}(a,p,c) \wedge \mathbf{T}(b,q,c) \;\to\; \exists x\, (\mathbf{T}(p,x,b) \wedge \mathbf{T}(q,x,a)) \qquad \text{(A7) inner Pasch}$$
$$\mathbf{T}(a,p,c) \wedge \mathbf{T}(q,c,b) \;\to\; \exists x\, (\mathbf{T}(a,x,q) \wedge \mathbf{T}(b,p,x)) \qquad \text{outer Pasch}$$

In order to express inner Pasch in quantifier free form, we introduce the symbol $ip(a,p,c,b,q)$ for the point x asserted to exist. This corresponds to the ruler-and-compass (actually just ruler) construction of finding the intersection point of lines aq and pb. There is a codicil to that remark, in that Tarski's axiom allows the degenerate case in which the segments aq and pb both lie on one line (so that there are many intersection points, rather than a unique one), but we do not care in this paper that in such a case the construction cannot really be carried out with ruler and compass. Also, we call the reader's attention to this fact: point c is not needed to draw the lines with a ruler, but it is needed to "witness" that the lines actually "should" intersect.

4.2. Continuity and the Circle Axiom. Axiom (A11) is the "continuity" axiom. In its full generality, it says that "first-order Dedekind cuts are filled." Closely related to (A11) is the "circle axiom" (CA), which says that if p lies inside the circle with center a and passing through b, and q lies outside that circle, then segment pq meets the circle (see Fig. 3).[1]

[1] There is no "standard" name for this axiom. Tarski did not give this axiom a name, only a number; in [9] and other German works it is called the "Kreisaxiom", which we translate literally here. In [4] it is called the "line and circle intersection axiom", which we find too long. In [3] (p. 131) it is called the "segment-circle continuity principle."

FIGURE 3. Circle Axiom(CA). Point p is inside, q is outside, so pq meets the circle.

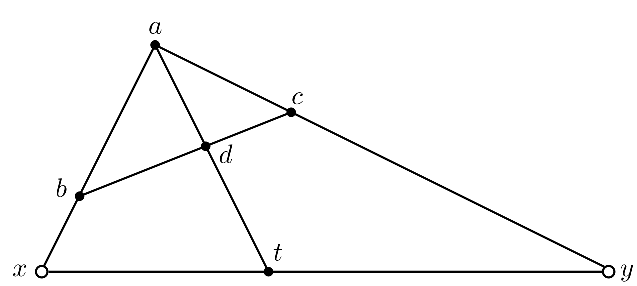


Points x and y in the figure serve as "witnesses" that p and q are inside and outside, respectively. Specifically, "p lies inside the circle" means that $ap < ab$, which in turn means that there is a point x between a and b such that $\mathbf{E}(a, x, a, p)$, i.e., segment ax is congruent to ap. Similarly, "q lies outside the circle" means there exists y with $\mathbf{B}(a, b, y)$ and $\mathbf{E}(a, q, a, y)$. In order to express segment-circle continuity in quantifier-free form, we can introduce a symbol $i\ell c(p, q, a, b, x, y)$ for the point of intersection of pq with the circle. Even though x and y are not needed for the ruler-and-compass construction of this point, they must be included as parameters of $i\ell c$.

We return below to the general axiom (A11), but first we show how to finish the proof of our main theorem if only the circle axiom is used, instead of the full schema (A11).

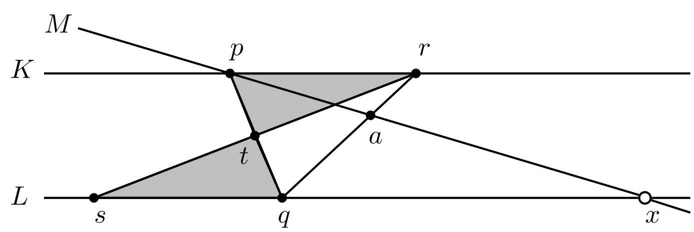
4.3. The parallel axiom. Tarski used a variant formulation (A10) of Euclid 5, illustrated in Fig. 4. One can prove the equivalence (A10) with Euclid 5, and (A10) has the advantage of being very simply expressed in a points-only language. Open circles indicate the two points asserted to exist.

FIGURE 4. Tarski's parallel axiom (A10).



For our independence proof, we work with Tarski's axiom A10 rather than with Euclid 5. Nevertheless, we include a formulation of Euclid's parallel postulate, expressed in Tarski's language. Euclid's version mentions angles, and the concept of "corresponding interior angles" made by a transversal. Fig. 5 illustrates the following points-only version of Euclid 5.

FIGURE 5. Euclid 5. Transversal pq of lines M and L makes corresponding interior angles less than two right angles, as witnessed by a. The shaded triangles are assumed congruent. Then M meets L as indicated by the open circle.



$$\mathbf{B}(q,a,r) \wedge \mathbf{B}(p,t,q) \wedge pr = qs \wedge pt = qt \wedge rt = st \qquad \text{(Euclid 5)}$$
$$\wedge \neg\, Col(s,q,p) \;\rightarrow\; \exists x\, (\mathbf{B}(p,a,x) \wedge \mathbf{B}(s,q,x))$$

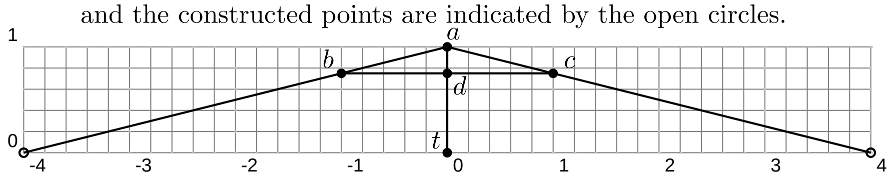
§5. Consistency of non-Euclidean geometry via Herbrand's theorem. The point of this paper is to show that one can use the very general theorem of Herbrand to prove the consistency of non-Euclidean geometry, doing extremely little actual geometry. All the geometry required is the observation that when we construct points from some given points, at each construction stage the maximum distance between the points at most doubles.

In order to state our theorem precisely, we define T to be Tarski's "neutral ruler-and-compass geometry", where "neutral" means that the parallel axiom (A10) (equivalent to Euclid 5) is not included, and "ruler-and-compass" means that (A11) is replaced by the circle axiom (CA). In addition, T uses the quantifier-free versions of the segment-extension and dimension axioms discussed above. The following lemma states precisely what we mean by, "at each construction state the maximum distance between the points at most doubles."

LEMMA 1. *The function symbols of T have the following property, when interpreted in the Euclidean plane $\mathbb{R}^2$: if f is one of those function symbols, i.e., f is ext or $i\ell c$ or ip, then the distance of $f(x_1, \ldots, x_j)$ from any of the parameters $x_1, \ldots x_j$ is bounded by twice the maximum distance between the x_j.*

Proof. When we extend a segment ab by a distance pq, the distance of the new point $ext(a,b,p,q)$ from the points a,b,p,q is at most twice the maximum of ab and pq. The point constructed by ip is between some already-constructed points, so ip does not increase the distance at all. The point constructed by $i\ell c$ is no farther from the center a of the circle than the given point b on the circle is,

Figure 6. Construction of a point too far away. Here $k = 2$ and the constructed points are indicated by the open circles.



and hence no more than ab farther from any of the other points, and hence no more than twice as far from any of the other parameters of $i\ell c$ as the maximum distance between those points.

Theorem 1. *Let T be Tarski's "neutral ruler-and-compass geometry", where "neutral" means that the parallel axiom (A10) (equivalent to Euclid 5) is not included, and "ruler-and-compass" means that (A11) is replaced by the "circle axiom" (CA). Then T does not prove the parallel axiom (A10).*

Proof. Suppose, for proof by contradiction, that T does prove (A10). There is a formula $\phi(a, b, c, d, t, x, y)$ such that axiom (A10) has the form

$$\exists x, y\, \phi(a, b, c, d, t, x, y),$$

where ϕ expresses the betweenness relations shown in the figure. Then, by Herbrand's theorem, there are finitely many terms $X_i(a, b, c, d, t)$ and $Y_i(a, b, c, d, t)$, for $i = 1, 2, \dots, n$, such that T proves

$$\bigvee_{i=1}^{n} \phi(a, b, c, d, t, X_i(a, b, c, d, t), Y_i(a, b, c, d, t)).$$

Let k be an integer greater than the maximum number of function symbols in any of those $2n$ terms. Choose points a, b, c, d and t in the ordinary plane $\mathbb{R}^2$ as follows (see Fig. 6)

$$\begin{aligned} t &= (0, 0) \\ a &= (0, 1) \\ b &= (-1, 1 - 2^{-k-2}) \\ c &= (1, 1 - 2^{-k-2}) \\ d &= (0, 1 - 2^{-k-2}) \end{aligned}$$

Suppose x and y are as in (A10); then one of them has a nonnegative second coordinate, and the other one must have a first coordinate of magnitude at least 2^{k+2}. But then, according to the lemma, it cannot be the value of one of the terms $X_i(a, b, c, d, t)$ or $Y_i(a, b, c, d, t)$, which, since they involve k symbols starting with points no more than distance 2 apart, cannot be more than 2^{k+1} from any of the starting points. This contradiction completes the proof.

§6. Full first-order continuity. In this section we show how to extend the above proof to include the full (first-order) continuity axiom (A11) instead of just the circle axiom. The difficulty is that (A11) is far from quantifier-free, but instead is an axiom schema. That means, it is actually an infinite number of axioms, one for each pair of first-order formulas (ϕ, ψ). The axiom says, if the points satisfying ϕ all lie on a line to the left of the points satisfying ψ, then there exists a point b non-strictly between any pair of points (x, y) such that $\phi(x)$ and $\psi(y)$.

The keys to extending our proof are Tarski's deep theorem on quantifier-elimination for algebra, and the work of Descartes and Hilbert on defining arithmetic in geometry. Modulo these results, which in themselves have nothing to do with non-Euclidean geometry, the proof extends easily to cover full continuity, as we shall see.

A *real-closed field* is an ordered field $\mathbb{F}$ in which every polynomial of odd degree has a root, and every positive element has a square root. Tarski proved in [11] the following fundamental facts:

- Every formula in Tarski's language is provably equivalent to a quantifier-free formula.
- Every model of Tarski's axioms has the form $\mathbb{F}^2$, where $\mathbb{F}$ is a real-closed field, and betweenness and equidistance are interpreted as you would expect.

Since Descartes and Hilbert showed how to give geometric definitions of addition, multiplication, and square root, there are formulas in Tarski's language defining the operations of multiplying and adding points on a fixed line L, with points 0 and 1 arbitrarily chosen on L, and taking square roots of points to the right of 0 (see chapter 14 and 15 of [9]). Since the existence of square roots follows from the circle axiom, the full continuity schema is equivalent to the schema that expresses that polynomials of odd degree have zeroes:

$$\exists x\,(a_0 + a_1 x + \ldots + a_{n-1}x^{n-1} + x^n = 0). \tag{1}$$

Note that without loss of generality the leading coefficient can be taken to be 1. Here the algebraic notation is an abbreviation for geometric formulas in Tarski's language. The displayed formula represents one geometric formula for each fixed odd integer n, so it still represents an infinite number of axioms, but Herbrand's theorem applies even if there are an infinite number of axioms. The essential point is that this axiom schemata is purely existential, so we can make it quantifier-free by introducing a single new function symbol $f(a_0, \ldots, a_{n-1})$ for a root of the polynomial.

THEOREM 2. *Axioms A1–A9 and axiom schema A11 together do not prove the parallel axiom A10.*

Proof. Suppose, for proof by contradiction, that A10 is provable from A1–A9 and A11. Then, the models of A1–A9 and A11 are all isomorphic to planes over real-closed fields. Then, as explained above, the full schema A11 is equivalent (in the presence of A1-A10) to the schema (1) plus the circle axiom.[2]

That is, it suffices to supplement ruler-and-compass constructions by the ability to take a root of an arbitrary polynomial. The point that allows our proof to work is simply that the roots of polynomials can be bounded in terms of their coefficients. For example, the well-known "Cauchy bound" says that any root is bounded by the maximum of $1 + |a_i|$ for $i = 0, 1, \ldots n-1$, which is at most 1 more than the max of the parameters of $f(a_0, \ldots, a_{n-1})$. Below we give, for completeness, a short proof of the Cauchy bound, but first, we finish the proof of the theorem.

We can then modify Lemma 1 to say that the distance is at most the max of 1 and double the previous distance. In the application we start with points that are 1 apart, so the previous argument applies without change. That completes the proof.

Lemma 2 (Cauchy bound). *The real roots of* $a_0 + a_1 x + \ldots + a_{n-1}x^{n-1} + x^n$ *are bounded by the maximum of* $1 + |a_i|$.

Proof. Suppose x is a root. If $|x| \le 1$ then x is bounded, hence we may assume $|x| > 1$. Let h be the max of the $|a_i|$. Then

$$-x^n = \sum_{i=0}^{n-1} a_i x^i, \quad \text{so} \quad |x|^n \le h \sum_{i=0}^{n-1} |x|^i \;=\; h\frac{|x|^n - 1}{|x| - 1}$$

Since $|x| > 1$ we have

$$|x| - 1 \le h\frac{|x|^n - 1}{|x|^n} \;\le h.$$

Therefore $|x| \le 1 + h$. That completes the proof.

§7. Related proof-theoretical work of others. Skolem [10] already in 1920 proved the independence of a form of the parallel axiom from the other axioms of projective geometry, using methods similar to Herbrand's theorem. In 1944, Ketonen invented the system of sequent calculus made famous in Kleene [6] as G3, and used it to reprove Skolem's result and extend it to affine geometry. This result was reproved using a different sequent calculus in 2001 by von Plato [14]. It should be noted that the modern proof of Herbrand's theorem also proceeds by cut-elimination in sequent calculus. Our proof of the independence of Euclid's parallel axiom improves on these past results in that (i) it works for ordinary geometry, not just for projective or affine geometry, and (ii) it depends on proof theory only for Herbrand's theorem: no direct analysis or even mention of cut-free proofs is required.

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[2] It is worth emphasizing that this equivalence depends on developing the theory of perpendiculars without any continuity axiom at all, not even the circle axiom. This was one of the main results of [4], and is presented in [9], where it serves as the foundation to the development of arithmetic in geometry. It is quite difficult even to prove the circle axiom directly from A11 without Gupta's results, although Tarski clearly believed decades earlier that the circle axiom does follow from A1–A11, or he would have included it as an axiom.

§8. Another proof via a model of Max Dehn's. Max Dehn, a student of Hilbert, gave a model of A1–A9 plus the circle axiom. Dehn's model is easily described and, like our proof, has no direct relationship to non-Euclidean geometry.

An element x in an ordered field $\mathbb{K}$ is called *finitely bounded* if it is less than some integer n, where we identify n with $\sum_{k=1}^{n} 1$. $\mathbb{K}$ is *Archimedean* if every element is finitely bounded. It is a simple exercise to construct a non-Archimedean Euclidean field, or even a non-Archimedean real-closed field. (For details about Dehn's model, see Example 18.4.3 and Exercise 18.4 of [5].) Dehn's model begins with a non-Archimedean Euclidean field $\mathbb{K}$. Then the set $\mathbb{F}$ of finitely bounded elements of $\mathbb{K}$ is a Euclidean ring, but not a Euclidean field: there are elements t such that $1/t$ is not finitely bounded. These are called "infinitesimals." Dehn's point was that $\mathbb{F}^2$ still satisfies the axioms of "Hilbert planes", which are equivalent (after [9]) to A1–A9. The reason is similar to the reason that our Herbrand's-theorem proof works: the constructions given by segment extension and Pasch's axiom can at most double the size of the configuration of constructed points, so they lead from finitely bounded points to other finitely bounded points. Since square roots of finitely bounded elements are also finitely bounded, $\mathbb{F}^2$ satisfies the circle axiom too. But $\mathbb{F}^2$ does not satisfy the parallel axiom, since there are lines with infinitesimal slope through $(0, 1)$ that do not meet the x-axis of $\mathbb{F}$. (They meet the x-axis of K, but not at a finitely bounded point.)

In this way Dehn showed that (the Hilbert-style equivalent of) A1–A9, together with the circle axiom, does not imply the parallel postulate A10. We add to Dehn's proof the extension to the full first-order continuity schema A11, by the same trick as we used for our Herbrand's-theorem proof. Namely, suppose for proof by contradiction that A10 is provable from A1–A9 and A11. Then in A1–A9 plus segment-circle continuity, A11 is equivalent to the schema (1) saying that odd-degree polynomials have roots. Now construct Dehn's model starting from a non-Archimedean real-closed field $\mathbb{K}$. Then $\mathbb{F}$ still satisfies (1), because of the Cauchy bound: if the coefficients a_i are finitely bounded, so are the roots of the polynomial. But then $\mathbb{F}^2$ satisfies A11, and hence, according to our assumption, it satisfies A10 as well; but we have seen that it does not satisfy A10, so we have reached a contradiction. That contradiction shows that A10 is not provable from A1–A9 and A11.

Note that this proof, like the proof via Herbrand's theorem, does not actually construct a model of non-Euclidean geometry, that is, a model satisfying A1–A9, A11, but not A10. That is the interest of both proofs: the consistency of non-Euclidean geometry is shown, in the one case by proof theory, and the other by algebra (or model theory if you prefer to call it that), without doing any non-Euclidean geometry at all. Moreover, the classical constructions of models of non-Euclidean geometry (the Beltrami–Klein and Poincaré models described in [3], Ch. 7), satisfy not only the first-order continuity schema but also the full second-order continuity axioms. Herbrand's theorem is about first-order logic, so it cannot replace these classical geometrical constructions; but still, we have shown here that a little logic goes a long ways.

REFERENCES

[1] Samuel R. Buss, *Introduction to proof theory*, ***Handbook of proof theory*** (Samuel R. Buss, editor), Studies in Logic and the Foundations of Mathematics, vol. 137, Elsevier, Amsterdam, 1998, pp. 1–78.

[2] B. F. Caviness and J. R. Johnson (editors), ***Quantifier elimination and cylindrical algebraic decomposition***, Springer, Wien/New York, 1998.

[3] Marvin Jay Greenberg, ***Euclidean and non-Euclidean geometries: Development and history***, fourth ed., W. H. Freeman, New York, 2008.

[4] Haragauri Narayan Gupta, *Contributions to the axiomatic foundations of geometry*, ***Ph.D. thesis***, University of California, Berkeley, 1965.

[5] Robin Hartshorne, ***Geometry: Euclid and beyond***, Springer, Berlin, 2000.

[6] Stephen C. Kleene, ***Introduction to metamathematics***, van Nostrand, Princeton, New Jersey, 1952.

[7] Moritz Pasch, ***Vorlesung über Neuere Geometrie***, Teubner, Leipzig, 1882.

[8] Moritz Pasch and Max Dehn, ***Vorlesung über Neuere Geometrie***, B. G. Teubner, Leipzig, 1926, The first edition (1882), which is the one digitized by Google Scholar, does not contain the appendix by Dehn.

[9] W. Schwabhäuser, Wanda Szmielew, and Alfred Tarski, ***Metamathematische Methoden in der Geometrie: Teil I: Ein axiomatischer Aufbau der euklidischen Geometrie. Teil II: Metamathematische Betrachtungen (Hochschultext)***, Springer–Verlag, Berlin, 1983, Reprinted 2012 by Ishi Press, with a new foreword by Michael Beeson.

[10] T. Skolem, *Logisch-kombinatorische Untersuchungen über die Erfüllbarkeit mathematischer Satzsysteme, nebst einem theoreme über dichte mengen*, ***Selected papers*** (J. Fenstad, editor), Universiteitsforlaget, Oslo, 1970, pp. 103–136.

[11] Alfred Tarski, *A Decision Method for Elementary Algebra and Geometry*, ***Technical Report R-109, second revised edition, reprinted in [2], pp. 24–84***, Rand Corporation, Santa Monica, California, 1951.

[12] Alfred Tarski and Steven Givant, *Tarski's system of geometry*, ***The Bulletin of Symbolic Logic***, vol. 5 (1999), no. 2, pp. 175–214.

[13] O. Veblen, *A system of axioms for geometry*, ***Transactions of the American Mathematical Society***, vol. 5 (1904), pp. 343–384.

[14] Jan von Plato, *Terminating Proof Search in Elementary Geometry*, ***Technical Report 43***, Institut Mittag-Leffler, Djursholm, 2000/2001.

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