ON THE NOTION OF EQUAL FIGURES IN EUCLID

MICHAEL BEESON

Abstract. Euclid uses an undefined notion of "equal figures", to which he applies the common notions about equals added to equals or subtracted from equals. This notion does not occur in modern geometrical theories such as those of Hilbert or Tarski. Therefore to account for Euclid in modern geometry, one must somehow replace Euclid's "equal figures" with a defined notion. In this paper we present a new solution to this problem, and moreover we argue that "Euclid could have done it". That is, it is based on mathematics that was available in Euclid's time, including ideas related to Euclid's Proposition I.44. The proof uses the theory of proportions. Hence we also discuss the "early theory of proportions", which has a long history.

1. Introduction

The word *area* almost never occurs in Euclid's *Elements*, despite the fact that area is clearly a fundamental notion in geometry.[1] Instead, Euclid speaks of "equal figures." Apparently a "figure" is a simply connected polygon, or perhaps its interior. The notion is neither defined nor illustrated by a series of examples; for example, it is never made clear whether a figure has to be convex, or even whether a circle is a figure, or whether a figure has an interior, or is just made of lines.

The notion of "equal figures" plays a central role in Euclid. For example, the culmination of Book I is the Pythagorean theorem. Nowadays we would, if required to express the theorem without algebraic formulas, say that given a right triangle, the area of the square on the hypotenuse is the sum of the areas of the squares on the sides. But Euclid said instead, that the square on the hypotenuse is equal to the squares on the sides, taken together. His proof shows how the two squares can be cut up into pieces that can be rearranged to make this equality of figures evident, given earlier propositions about equal figures.

Euclid did not define "figure", and neither shall we. For purposes of Euclid Book I, we can think of triangles and convex quadrilaterals, as these are the only figures mentioned, but presumably Euclid did not mean *only* these, but meant to include combinations of these as well. Euclid also did not define "equal figures", but simply treated it as a primitive (undefined) notion. The main point of this paper is that he *could* have given a precise definition of "equal figures", at least for equal triangles and convex quadrilaterals, and in that way proved all the theorems of Book I without needing an undefined notion of "equal figures."

Date: April 2, 2022.

[1]It does occur in English translation in Prop. I.35, but in the context *parallelogrammic areas*, which according to Heath's commentary, is intended mainly to emphasize that only four-sided figures are meant, i.e., regular polygons of more than four sides with opposite sides parallel are not meant.

That Euclid did have area in mind when speaking of equal figures seems clear from Book II, in which the whole thrust of the book is towards showing how, given any rectilineal figure, to find a square equal to the given figure; one might interpret that as giving a method to calculate the area of any rectilinear figure.

Nor was Euclid alone in avoiding the word "area." A century later, when Archimedes calculated the area of a circle, he did not express his result by saying that the area of the circle is π times the square of the radius. Instead, he said that circle is equal to the rectangle whose sides are the radius and half the circumference. (See Fig. 1). (So a circle did count as a figure for Archimedes!) [2]

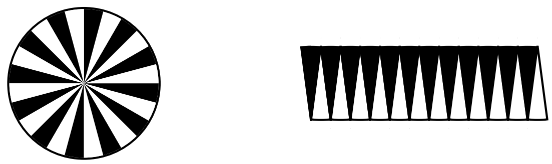

FIGURE 1. Archimedes proved the circle is equal to a certain rectangle, but he didn't use the word "area."

Why did Euclid avoid the word area? Not because he did not know that area can be measured; it must have been for more abstract, mathematical reasons. Let us consider his problem: if he were to use the word, he would either have to *define* it, or put down some *postulates* about it. Both choices offer some difficulties. Area involves assigning a *number* to each figure, to measure its area. It is therefore not a purely geometric concept. Moreover, even if one is willing to introduce numbers, that just pushes the problem back one step: one must then define or axiomatize numbers. Euclid knew that he did not know how to define area in general, so that choice was out. The other choice was to write down some axioms that area obeys. The most obvious one is additivity: if a figure can be cut into two pieces, then its area is the sum of the areas of the pieces. But then there are delicate questions about the meaning of "cut" and "piece." Euclid (or one of his unknown predecessors) discovered that it would possible to avoid all these complications by replacing "area" by the concept of *equal figures.* He noticed that if he used the word "equal", and also re-interpreted "taken together" and "taken from" as if these operations were applicable to figures as well as to lines and angles, then the additivity properties would look like special cases of the common notions 2 and 3, namely "if equals be added to equals, the wholes are equal", and "if equals be subtracted from equals, the remainders are equal."[3] Common notion 5, "the whole is greater than the part", could be taken to imply that a figure cannot be equal to a part of itself, and common notion 4, "things which coincide with one another

[2]But Heron, in his *Metrica* of 50 CE [1], did use the word ἐμβαδόν, which is translated as "area", and explains it this way: "A cubit area is called when a square plot has each side of one cubit." The tile of *Metrica* shows Heron's concern with techniques for actually calculating areas; he even gives a numerical procedure for computing approximate square roots. He also has no problem multiplying four lengths and then taking the square root to get an area. But that answer was a number, not a geometric length. The identification of line segments with numbers was not a part of Greek mathematics.

[3]This could now be explained using set theory, as set-theoretic union and difference, but that is a development only of the past century, and the verifications of Euclid's common notions for this notion still involve real numbers as well as sets.

are equal to one another", could be interpreted to imply that congruent figures are equal. Using these interpretations of the common notions, Euclid (thought he) could avoid all the complications mentioned above.[4]

We will give an example of how Euclid reasoned about equal figures, namely Euclid I.35. See Fig. 2.

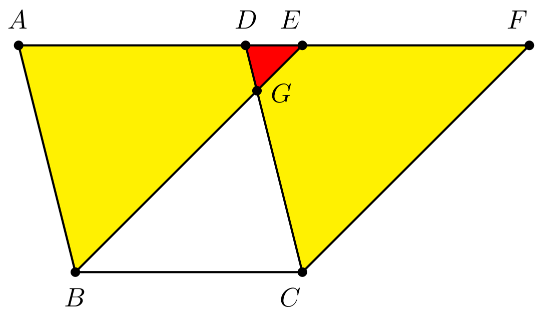

FIGURE 2. Euclid's proof of I.35

Euclid wants to prove the parallelograms $ABCD$ and $BCFE$ are equal. He proves the triangles ABE and DCF are congruent. Implicitly, he assumes DEG and DGE are equal figures (that is, the order of listing the vertices does not matter). Then "subtracting equals from equals", the yellow quadrilaterals are equal. Then, "adding equals to equals", he adds triangle BCG (implicitly assuming BCG is equal to BGC) to arrive at the desired conclusion. To formalize this proof, we needed so-called cut-and-paste axioms, as well as the axiom that ABC and ACB are equal triangles. In this paper, we will show how to define "equal figures" so these propositions can be proved instead of assumed.

Although it appears to the modern eye (e.g. [10]) that Euclid meant "figures with equal area" when he said "equal figures", it is worth noting that not only does he never mention the word "area", but he also never speaks of one figure being greater than another, although certainly areas can be compared. He never applied the common notions that mention "greater than" to figures. Perhaps he thought "A greater than B" generally means that B is equal to a part of A; that definition, if applied to figures, does not lead to the same laws that "greater than" enjoys for lines. This line of thought casts a shadow of doubt on the theory that "equal figures" meant "equal area", without suggesting another interpretation.

Later generations of mathematicians were not willing to accept Euclid's over-liberal interpretation of the common notions in support of "equal figures." See the summary discussion with many references on pp. 327–328 of [7]. In particular, once mathematicians had some experience with axiomatization, it became obvious that "equal figures" is not a special case of equality, since equal figures cannot be substituted for each other in every property. Instead, it is a new relation, and the original choice that Euclid finessed faces us directly: we must either define or axiomatize the notion.

[4]Actually, Euclid needed one more property: halves of equal figures are equal, used in Prop. I.39. The step that (implicitly) uses that property occurs in Euclid's text without justification.

Euclid does not mention "equal figures" until Prop. I.35. The reader is urged to look at the proofs of Euclid's Propositions I.35, 42, 43, 47, and 48, to identify the lines of Euclid's proofs where common notions about equal figures are used. They are relatively few in number, but crucial to Euclid's development. For example, Euclid Book I culminates in the Pythagorean theorem, which cannot even be stated without the notion of equal figures.

We mention a few matters of notation. Euclid usually used upper-case letters for points. We use both upper-case and lower-case letters for points; this allows the notation to suggest correspondences, as in "Triangle ABC is congruent to triangle abc." Since the force of tradition in geometry is strong, we justify this choice: One might also choose to use primes or stars or circumflexes, as in "ABC is congruent to $A'B'C'$." Such notations are hard to fit into diagrams, and cannot be cut-and-pasted into computer systems for formal proofs, leading to errors of transcription.

A second matter of notation is "betweenness". Starting with Pasch [13], geometers have used the relation "B is between A and C"; Hilbert [11] introduced the notation $\mathbf{B}(A, B, C)$, which we shall use in a few places. Following Hilbert we take it to mean "strict betweenness"; that is, it means B lies on the line (segment) AC and is not equal to A or C.

Acknowledgements. I would like to thank Victor Pambuccian and Vincenzo De Risi for their help with the history of this subject. I would like to thank the anonymous referee for an extremely careful reading and many helpful suggestions.

2. Possible ways to define area and equal figures

The problem we are trying to solve is to eliminate the need for a primitive notion of "equal figures." In this section we explain the principal approaches to this problem that have been tried.

2.1. **Hilbert's equidecomposition.** When Hilbert wrote his influential book [11], he chose to define the notion. His definition was still used in the much more modern book [10], p. 200. The problem with that definition is that it mentions the notion of natural number, in speaking of cutting a figure into "a finite number" of triangles. It is therefore not a purely geometric notion. More technically, it cannot be expressed in a first-order language with only geometric variables. This concept does not permit us to define *area*, but only *equal area* for polygons. That would, in itself, not be a problem, as Book I needs only triangles and convex quadrilaterals, but Hilbert himself showed that (even for triangles) the claim that equal figures are equidecomposable requires Archimedes's axiom [11, §19].

2.2. **Defining area by calculus.** Since the eighteenth century, we have had the option to define area using integrals, an option that was not available to Euclid, who wrote even before Archimedes's work on the circle pioneered the use of limits. But that too is not a purely geometric definition, since it involves real numbers and functions as well as limits. However, it is the *only* workable definition of area that mathematicians have found.

2.3. **Descartes's geometric arithmetic.** Descartes and later Hilbert defined *geometric arithmetic*. That is, we fix a certain line (the x-axis) and two points on that line (0 and 1). Then certain geometric constructions exist for defining the addition and multiplication of line segments with 0 as one endpoint, and one can give geometric proofs of the laws of arithmetic, such as the associative, commutative, and distributive laws.

When Hilbert, and later Tarski, worked on the formalization of elementary geometry, their principal aim was to develop this theory of "geometric arithmetic." They did not follow Euclid and did not, for example, prove Prop. I.35 and subsequent propositions that depend on equal figures, but proceeded by the most direct route to geometric arithmetic. That route led through the theorems of Pappus and Desargues.

It is not the case that just because we can multiply we can define areas, even of triangles. The problem of interpreting what Euclid meant by "equal area" is not automatically solved by defining geometric arithmetic.

Besides, segment arithmetic was conceptually alien to Greek thought. The Greeks never multiplied lengths to get lengths. A length times a length produced a rectangle; multiplying three lengths produced a solid; four lengths could not be multiplied. Even Heron, who had no problem multiplying numbers to compute areas, did not think he was multiplying lengths to get lengths. That conceptual step was not taken until Descartes, in 1637. Even Vieta, who introduced using letters for quantities in algebra, still adhered to the "principle of homogeneity" in which all algebraic terms in an equation had to have the same degree.[5]

2.4. **Axiomatize the properties of area.** We could try to axiomatize the properties of an "area function" from sets of points to numbers. To avoid the complications of "numbers", Hartshorne tried this approach using any ordered Abelian group for the values of area. See pp. 205ff. of [10] , where it is shown how to do this using a minimum of assumptions. But this is also not a geometric notion, as the area function has to take values *somewhere*. Hartshorne uses any abelian group.

2.5. **Direct axiomatization of the equal-figure axioms.** The definitions of "equal figure" discussed above all are unsatisfactory, since they require the concepts of real number, or natural number, or both; or else, in the case of segment arithmetic, do not actually lead to a definition of equal area.

When we set out, in previous work [4], to formalize Euclid Book I, we saw no other alternative. Following Sherlock Holmes's maxim that when the impossible is eliminated, what remains is the truth, we chose the remaining alternative: to axiomatize the notion. That was also implicitly the choice in [10], where Hilbert's definition is only used for long enough to establish the properties that Euclid used, and to it is added "de Zolt's axiom", that if Q is a figure contained in another figure P, and $P - Q$ has a nonempty interior, than P and Q are not equal figures. Hartshorne says that he does not know a purely geometrical proof of this from Hilbert's definition of equal figures; probably he means that to prove it, you must prove that two figures are equal in Hilbert's sense if and only if they have equal area, in the sense of area defined by integrals.

The common notions of Euclid include

- CN2: If equals are added to equals, then the wholes are equal.
- CN3: If equals are subtracted from equals, then the remainders are equal.

[5]For example, see [15], p. 86, where Vieta writes A cubus + B quad. in A, equetur B quad.in Z, or in modern symbols, $A^3 + B^2A = B^2Z$ instead of $x^3 + px = q$. See [9] for further discussion. Incidentally, one sees both Viète and Vieta, the French and Latin spellings of the name.

- CN5: The whole is greater than the part.

Euclid meant these to apply to various sorts of "things"; in Book I he used them for triangles and quadrilaterals, which are the only "figures" in Book I. Our plan in [4], and Hartshorne's plan in [10], was to translate the "common notions" mentioned above into first-order axioms. Since the variables range over points, triangles are just triples of points, so "equal triangles" is a 6-ary relation, $ET(A, B, C, a, b, c)$. Then we need an 8-ary relation EF for "equal quadrilaterals". (EQ was already in use for equality, so we used EF for "equal figures.") In Euclid Book I, only triangles and quadrilaterals are used, so we stopped there. The parts of a figure are smaller figures. New figures are constructed, in Euclid's proofs, by cutting off triangles from larger triangles or quadrilaterals, and also by pasting on figures to other figures along a common edge.

Euclid also used two principles about equal figures without ever formulating them as axioms or common notions: halves of equals are equal, and doubles of equals are equal. These also correspond to axioms in our list of equal-figures axioms.

Since there are several ways to do this cutting and pasting, we get several "equal-figures axioms" this way. The last common notion, CN5, becomes "de Zolt's axiom", after the person who first formulated it. These axioms express the ways in which Euclid used the "common notions" as applied to figures.

This approach to the notion of equal-figures by direct axiomatization was successful: we constructed proofs "faithful to Euclid" that were verifiably logically correct. Nevertheless, the addition of fifteen new axioms to Euclid is a bit unsatisfying, even if they do correspond well to Euclid's actual proofs. What we really want is a treatment of geometry that

(i) Corresponds as well as possible to Euclid (but corrects the errors), and

(ii) Has axioms that correspond well to basic geometric intuitions, and

(iii) meets modern standards of rigor.

The system of axioms that we used in [4], or the similar system in [10], Euclid meets these standards, if you think that the fifteen equal-figure axioms correspond to a basic geometric intuition. But it seems that they do not really: they are justified by an appeal to our intuitions about area, and area cannot be expressed purely geometrically, as it fundamentally involves using numbers to measure area.

3. Aim and methods of this paper

The contribution of this paper is to eliminate the "equal figures" axioms by *defining* the notions of "equal triangles" and "equal quadrilaterals", by a definition that Euclid could have given, and proving the properties expressed in the "equal figures" axioms, so that Euclid Books I to IV could be developed without the equal figures axioms.

This aim would not be met by following Hilbert and Tarski, who first define segment arithmetic and then use it to define area. That route would take us far from Euclid, who never thought of multiplying two line segments. While it would meet the requirement to eliminate the "equal figures" axioms by defining that notion, it would not meet the requirement that "Euclid could have done it."

Our method to achieve this aim is to define "equal rectangles" using a figure much like the one Euclid uses for Prop. I.44. and use that to define "equal triangles" and "equal quadrilaterals" and use those defined notions to prove the propositions of Euclid Book I. To reiterate: we will

- define "equal rectangles" and "equal triangles", and
- prove the propositions of Euclid Book I using those defined notions (rather than extra axioms), and
- use proofs "in the spirit of Euclid"

It follows that the equal-figures axioms are actually superfluous, in the sense that, using the new definition of "equal figures", we could formalize Euclid Book I directly, without adding any equal-figures axioms. But we then take one step more, and show that the equal-figures axioms can in fact all be proved.

Our proof that the defined notion of "equal figures" has the required properties uses some theorems about similar triangles and proportion. In Euclid, those theorems are present, but only in Book V, after the development in Book IV of Eudoxes's theory of "magnitudes." To carry out our program thus requires the demonstration that Euclid could have developed the necessary theory of proportion without using the Axiom of Archimedes (on which Book V depends), and preferably without using Book III (theorems about circles), and of course without using "equal figures," so that it would be available in the last third of Book I. It turns out that we are not the first to seek an earlier development of the theory of proportion; in §9.3 below, we discuss this subject at length, with historical notes.

The proofs that we give in this paper are *informal* in the sense that they have been written for humans to read, but *rigorous* in the sense that they can be carried out in any formal system adequate for elementary geometry. For example in Hilbert's system, or Tarski's system, or the system used in [4], or the textbook [10], or (apart from the errors corrected in [4]) in Euclid's own system, minus steps about equal figures that we are trying to justify here.[6] This paper could have been written and read in the nineteenth century, before the invention of modern logic and computers. It would, of course, have needed to be after the theorems on proportionality in § 9.

4. Similar triangles and proportion

One of the principles about "equal figures" that Euclid uses in at least two crucial places is that ABC is equal to BCA. Under the definition of "equal triangles" that we give below, this principle turns out to require the basic theorems about similar triangles and proportion for its proof. Specifically, if two right triangles have their hypotenuses and one leg proportional, then they have corresponding angles equal. Also, the proof that "equal rectangles" is a transitive notion seems to require properties of proportion.

In this section, we state the definitions and properties of proportion that we will use. Euclid proved these theorems in Book VI, using the results of Book V, which is based on the axiom of Archimedes. Because of the reliance on Archimedes's axiom and possibly the use of equal figures, this is not useful for us. But Paul Bernays proved these results without using Archimedes's axiom, and by means acceptable for our purposes, in his Supplement II to [11]. In this section, we will give the

[6]It is therefore *not* necessary to fix a particular first-order version of Euclid to check this paper, unless of course, one wants to check the proofs by computer. In that case, refer to [4] or the perhaps more accessible [3].

definition of proportion, prove some easy lemmas, and make it clear that there are exactly two non-trivial theorems in the subject, namely the "interchange theorem" and the "fundamental theorem", which are stated below, and proved by Bernays and Kupffer [12, 14]. Their proofs and some relevant history will be discussed in §9.

Definition 4.1. *$AB : AC = Ab : Ac$ if BAC is a right triangle, b is on ray AB, c is on ray AC, and BC is parallel to bc (or $BC = bc$), as shown in Fig. 3. More generally $PQ : RS = pq : rs$ if PQ, RS, pq, and rs are congruent to such segments AB, AC, Ab, and Ac.*

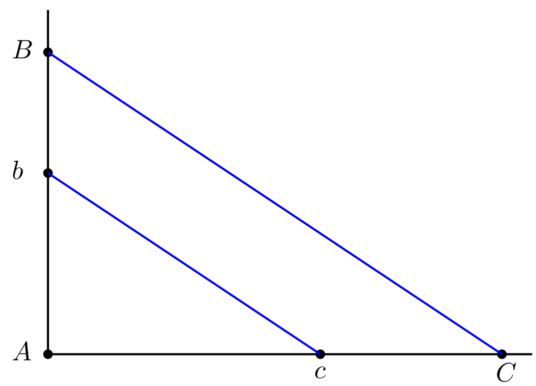

FIGURE 3. Here $AB : AC = Ab : Ac$ because $BC \parallel bc$.

Formally, we have defined a relation taking eight arguments of type "point".[7]

It follows from the symmetry of "parallel" and congruence that $pq : rs = PQ : RS$ if and only if $PQ : RS = pq : rs$, which we will use without further explicit mention. We also have the following simple property:

Lemma 4.2. *$pq : rs = PQ : RS$ if and only if $rs : pq = RS : PQ$.*

Proof. Immediate from the definition of proportion and the symmetry of "parallel".

The following definition is Euclid's wording, but the meaning he attached to "proportional" is different.

Definition 4.3. *Two triangles ABC and abc are similar if their corresponding angles are equal and their corresponding sides are proportional, that is, $AB : AC = ab : ac$.*

Notation. If we write $a : b = c : d$, the letters a, b, c, d stand for pairs of point variables. We may think of these pairs as segments, but formally the relation $a : b = c : d$ is an 8-ary relation on points. That the abbreviated notation is convenient is illustrated by shortening the statement of Lemma 4.2: $u : v = p : q$

[7]We have not defined $AB : AC$ as a function taking four points, or two segments; the use of the equality symbol and colon in informal writing is just an abbreviation for the 8-argument relation. Bernays in his Supplement II to [11] does define $a : b$ for segments a, b to be, in effect, the angle whose tangent is b/a; but he never makes any use of that definition other than to verify that the equality of such angles implies the definition of proportionality we give here.

if and only if $v : u = q : p$. Variables occurring next to a colon cannot be points; they must be pairs of points, so there is no ambiguity in using lower-case letters in this way, as well as for points.

Lemma 4.4 (transitivity)**.** *If $a : b = c : d$ and $c : d = e : f$ then $a : b = e : f$.*

Proof. Immediate from the transitivity of "parallel."

Bernays proves the existence and uniqueness of the fourth proportional; we present versions of his proofs, to check that they work in the present framework, that is, without relying on parts of Euclid past I.35.

Lemma 4.5 (existence of the fourth proportional)**.** *For each p, q, r, there exists a segment x such that $p : q = r : x$*

Proof. Given segments p, q, r, we let A and B be the endpoints of p. Erect a perpendicular AC to AB at A, with $AC = q$. Then construct point b on ray AB so that $r = Ab$. We may assume $b \neq B$, since if $b = B$ then $p = q$, so we may take $x = r$. Then we may construct line ℓ through b parallel to BC. (See Fig. 3; we have to prove point c in the figure exists.) Since BAC is a right angle, A does not lie on BC, and angle ACB is less than a right angle. Then by Euclid 5, line AC (possibly extended) meets line ℓ in a point c. Then by the definition of proportionality, we have $AB : AC = Ab : Ac$; taking $x = Ac$ we have $p : q = r : x$. That completes the proof.

Lemma 4.6 (uniqueness of fourth proportional)**.** *For each a, b, c, there is exactly one segment x such that $a : b = c : x$ (up to congruence).*

Proof. This is an easy consequence of the definition. We spell out the details to facilitate formalization. Suppose $a : b = c : d$ and $a : b = c : x$. By Lemma 4.4, $c : d = c : x$. Since proportion is defined up to congruence, we may assume that in Fig. 4, AB and Ab are both equal to c, while AC is d and Ac is x. But then, $b = B$, so by the definition of proportion, BC and bc coincide. Hence $c = C$, as both are the intersection point of line BC with line AC. Hence $Ac = AC$; but that is $x = d$. That completes the proof.

The following theorem is Euclid V.16, except that Euclid's definition of proportion is not the same.

Theorem 4.7 (Interchange theorem)**.** *If $a : b = p : q$, then $a : p = b : q$.*

This theorem will be proved in §9.

Lemma 4.8. *If two right triangles have their legs proportional, then their corresponding angles are equal.*

Proof. Since proportionality is defined up to congruence, we may assume without loss of generality that the two right triangles ABC and aBc have their right angle at B in common, and a lies on ray BA and c lies on ray BC. Then by the definition of proportionality, AC is parallel to ac (or coincident). Then the corresponding angles are either identical, or are corresponding angles of the traversals BA and BC or the parallel lines AC and ac. Hence the corresponding angles are equal. That completes the proof of the lemma.

The following theorem was named by Bernays in his Supplement II to [11].

Theorem 4.9 (fundamental theorem of proportion)**.** *If two parallels delineate the segments AC and Ac on one side of an angle and AB, Ab on the other side of the angle, then $AB : Ab = AC : Ac$.*

This theorem will be proved in §9. Fig. 4 illustrates the fundamental theorem. In the figure, $AD = AC$ and $Ad = Ac$. Modulo the interchange theorem, we could write the conclusion $AB : AC = Ab : Ac$, which is equivalent to $DC \parallel dc$ in the figure, by the definition of proportionality, where

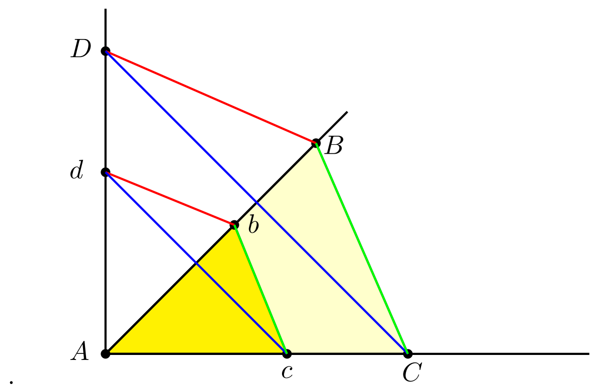

FIGURE 4. If $BC \parallel bc$ then $AB : Ab = AC : Ac$

The figure makes it clear that the fundamental theorem is a consequence of Desargues's theorem. Whether it implies Desargues's theorem in some simple way we do not know. Bernays proved it by much more elementary means, discussed in §9.

Corollary 4.10. *If two triangles ABC and abc have corresponding angles equal, then $AB : AC = ab : ac$.*

Proof. Since proportionality is defined up to congruence, we may assume $a = A$ and AB and Ab are collinear and AC and Ac are collinear. Then the fundamental theorem applies. That completes the proof.

Corollary 4.11. *If two right triangles have their hypotenuses and one leg proportional, then their corresponding angles are equal.*

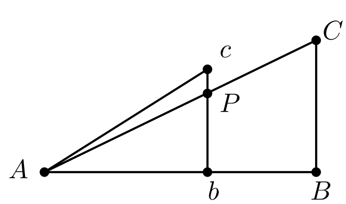

FIGURE 5. $AB : ab = AC : ac$ implies angles CAB and cAb are equal.

Proof. Please refer to Fig. 5. Since proportionality is defined up to congruence, we may assume without loss of generality that the two right triangles are ABC and

Abc, with vertex A in common and b on ray AB, and right angles at b and B, and $AB : ab = AC : ac$. We must show that c lies on ray AC.

We have two straight lines Ac and Bc, and line AB falling on those two straight lines. The interior angles cAb and Abc are together less than two right angles, since Abc is right and cAb is less than right, since Abc is right. Therefore, by Euclid 5, AC meets bc in a point P on the same side of AB as c. Then we have

$$\begin{array}{rcll} AB : ab & = & AC : ac & \text{by hypothesis} \\ AB : ab & = & AC : AP & \text{by the fundamental theorem, since } BC \mid\mid bc \\ Ac & = & AP & \text{by the uniqueness of the fourth proportional} \end{array}$$

It remains to show $c = P$, which seems visually obvious, but is not trivial to prove. Assume, for proof by contradiction, that $\mathbf{B}(b, P, c)$. Then triangle AcP is isosceles, so angle AcP is equal to angle aPc. Since the angles of triangle AcP make together two right angles, angle APc is less than a right angle. But also angle APb is less than a right angle, since angle AbP is a right angle. But angles APb and APc are supplements, since $\mathbf{B}(b, P, c)$, so together they make two right angles, contradiction. Hence P is not between b and c.

Now assume, for proof by contradiction, that $\mathbf{B}(b, c, P)$. Again triangle AcP is isosceles, so angle AcP is less than a right angle. Also angle Abc is less than a right angle. But now angles Abc and AcP are supplements, contradiction.

Hence neither $\mathbf{B}(b, c, P)$ nor $\mathbf{B}(b, P, c)$. But P lies on line bc; therefore $P = c$. Then c lies on ray AC. Hence angle cAb is equal to angle CAB. Hence also angle Acb equals angle ACB. That completes the proof of the lemma.

We next give two "combination rules" introduced and proved by Bernays, [11, p. 204]. We repeat the proofs given by Bernays, as they are short. The reader can check that they use only the interchange theorem and the existence of the fourth proportional.

Lemma 4.12 (Bernays's "first combination rule")**.**
If $a : b = a' : b'$ *and* $b : c = b' : c'$ *then* $a : c = a' : c'$.

Proof. Suppose If $a : b = a' : b'$ and $b : c = b' : c'$. Then

$$\begin{array}{rl} a : a' = b : b' & \text{by the interchange theorem} \\ b : b' = c : c' & \text{by the interchange theorem} \\ a : a' = c : c' & \text{by Lemma 4.4} \\ a : c = a' : c' & \text{by the interchange theorem} \end{array}$$

That completes the proof.

Lemma 4.13 (Bernays's "second combination rule")**.**
If $a : b = b' : a'$ *and* $b : c = c' : b'$ *then* $a : c = c' : a'$.

Proof. By Lemma 4.5, let u be the fourth proportional to a, b, c'.

$a : b = c' : u$	since u is the fourth proportional
$a : b = b' : a'$	by hypothesis
$c' : u = b' : a'$	by the preceding two lines
$c' : b' = u : a'$	by the interchange theorem
$b : c = c' : b'$	by hypothesis
$u : a' = b : c$	by Lemma 4.4
$c' : u = a : b$	since $a : b = b' : a' = c' : u$
$c' : a' = a : c$	by Lemma 4.12

That completes the proof.

We have now derived the theory of proportionality from just two theorems, the interchange theorem and the "fundamental theorem or proportionality. In §9, we will discuss Bernays's and Kupffer's proofs of those theorems, with due attention to the extensive history of this subject. In the meantime, we move on to develop our theory of equal figures, using these facts about proportion.

5. Defining equal rectangles and equal triangles

5.1. **Order matters.** Is "triangle ABC" the same triangle as "triangle BCA"? If you are willing to accept the answer "no", you may skip this subsection; otherwise read on.

Neither Euclid nor his modern successors defined "triangle"; triangles are always introduced and referred to by triples of points ABC. In Proposition I.35, where equal figures are used, it is taken for granted that DGE is equal to EGD, so that "subtracting" these two triangles from larger ones is "subtracting equals." But Euclid never commits himself to saying that DGE and EGD are the *same* triangle, or are *different but equal* triangles.

We can't tell which he meant, since in Euclid, triangles are just given by *ordered* triples of points. Since we do not actually have triangles in our ontology, perhaps it doesn't matter much. But we do define a relation of "triangle congruence." This is a 6-ary relation on points, defined in terms of segment congruence (more formally, the "equidistance relation"). Namely, ABC is congruent to abc if AB and ab are congruent, BC and bc are congruent, and AC and ac are congruent. According to this definition, "order matters": ABC will be congruent to BAC only when BC and AC are congruent.

Actually, Euclid never defines "congruent triangles." For example, look up his statement of I.4 (the SAS criterion). But our definition is in accordance with a long tradition since Euclid: when we say that ABC and abc are congruent triangles, we intend to imply that AB and ab are corresponding sides, BC and bc are corresponding sides, ABC and abc are corresponding angles, etc. That is, the order in which we mention the vertices of triangle ABC does matter, if we are to speak of congruent triangles and have the names of the triangles convey which are the corresponding sides and angles in the congruence.In other words: this is not a deep philosophical issue about the nature of triangles. It is a convention concerning how we describe triangles by mentioning their vertices. Whether ABC is "the same triangle as BAC" or "a different triangle from BAC" simply never comes up, because we don't say what a triangle is.

We define the *base of triangle ABC* to be AB. This is just a way of selecting the first two of the three points.

5.2. **Equal rectangles.** We begin by defining the 8-ary relation `ER`, "equal rectangles." Two rectangles are *congruent* if their sides are pairwise equal. Explicitly, $ABCD$ is congruent to $abcd$ if $AB = ab$ and $BC = bc$.

Definition 5.1. *Any two given rectangles R and S are congruent to rectangles placed like $FEBG$ and $BMLA$ in Fig. 6, where $\mathbf{B}(A, B, E)$ and $\mathbf{B}(G, B, M)$, and $ABGH$ and $BMKE$ are rectangles. By definition the two rectangles $BEFG$ and $BMLA$ are* **equal rectangles** *if and only if $\mathbf{B}(H, B, K)$, that is, the line HK passes through the common vertex B of the two rectangles. We then say that the two original rectangles R and S are equal.*

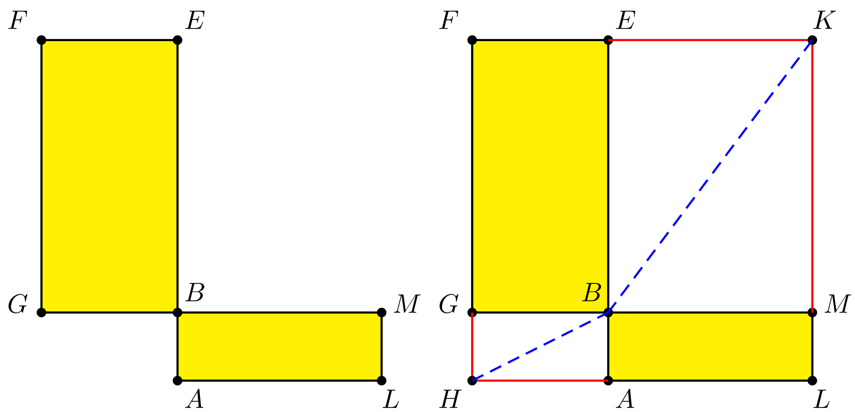

FIGURE 6. (Left) Place copies of two given rectangles as $BEFG$ and $BMLA$ with two sides collinear. (Right) The other sides (extended) meet by Euclid 5 forming a large rectangle. Then $BEFG$ and $BMLA$ are defined to be equal if $\mathbf{B}(H, B, K)$. In the case shown, they are not equal.

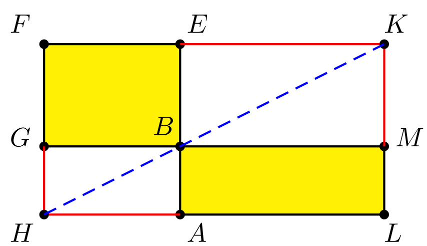

FIGURE 7. In this case the two rectangles are equal, since B is between H and K. That is, the two dashed lines form one straight line.

The connection between equal rectangles and the theory of proportion is given in the following lemma:

Lemma 5.2. *The rectangle with base b and height a is equal to the rectangle with base c and height d if and only if $b : c = d : a$, and also if and only if $b : d = c : a$.*

Proof. The two proportionality statements are equivalent, by Theorem 4.7. Suppose rectangle 1 has base b and height a, and rectangle 2 has base c and height d.

Please refer to Fig. 7, in which rectangle 1 is $FEBG$ and rectangle 2 is $BMLA$.

Suppose the two rectangles are equal. Then B is between H and K, so HAB and BMK have their corresponding angles equal. By Corollary 4.10, $KM : BM = AB : AH$. By the interchange theorem, $KM : AB = BM : AH$. Now $KM = EB = a$, $AB = d$, $BM = AL = c$ and $AH = GB = b$. Thus $a : d = c : b$. By Lemma 4.2 then $d : a = b : c$. That completes the left-to-right direction of the proof.

Now suppose $d : a = b : c$. The last few steps are reversible, leading to $KM : BM = AB : AH$. Then by Corollary 4.8, the right angles HAB and BMK have their corresponding angles equal. I say that implies B is between H and K. Indeed, since GM is parallel to HL, the extension of line HB through B makes an angle with BM equal to angle KBM. If we knew that K is on the opposite side of GM from H, we could conclude by Euclid I.7 that K lies on that extension; that is, B is between H and K.[8]

To prove that K is on the opposite side of GM from H, we note first that H is on the opposite side of GM from F since $\mathbf{B}(H, G, F)$. So it suffices to prove that H is on the same side of GM as K. For that it suffices to show that FA and AK both meet GM. Those assertions are applications of a lemma called `parallelpasch` that we proved during our formalization of Euclid Book I. The lemma says that if a point A lies on the extension of one side EB of a parallelogram $EBGF$, then AF meets BG; that is exactly what we need here.

That completes the proof of the lemma.

Lemma 5.3. *"Equal rectangles" is an equivalence relation.*

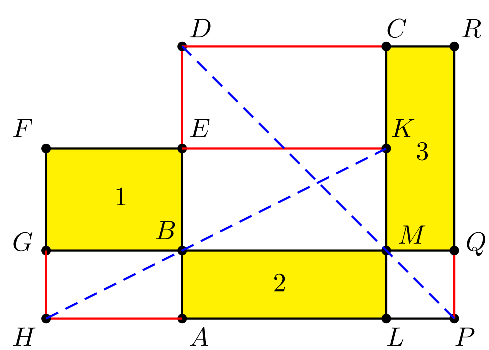

FIGURE 8. If rectangle 1 equals rectangle 2 equals rectangle 3, then rectangle 1 equals rectangle 3.

Proof. We omit the proofs of reflexivity and symmetry. The difficult part is transitivity. Suppose rectangle 1 is equal to rectangle 2, and rectangle 2 is equal to

[8]Euclid I.7 is quite a bit more difficult to prove than Euclid thought; it is hard to prove that an angle cannot be both equal to and less than another angle. Hilbert avoided the difficulty by including uniqueness in his angle-copying axiom.

rectangle 3. We have to prove that rectangle 1 is equal to rectangle 3. See Fig. 8. We have

$$\begin{aligned} MQ : BA &= BM : MC && \text{since } 3 = 2 \\ BA : BG &= BE : BM && \text{since } 2 = 1 \end{aligned}$$

We apply Bernays's second combination rule (Lemma 4.13); with

$$a = MQ, b = BA, b' = BM, a' = MC, c = BG, c' = BE$$

The hypotheses of the combination rule become the above equations. The conclusion is $a : c = c' : a'$; that is,

$$MQ : BG = BE : MC.$$

Then rectangle 3 is equal to rectangle 1, by Lemma 5.2. That completes the proof of the lemma.

Lemma 5.4. *Rectangles $ABCD$ and $BCDA$ are equal; also for any permutation of the vertices that is still a rectangle.*

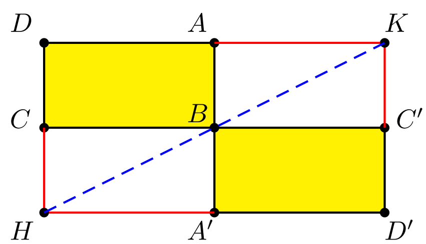

FIGURE 9. $ABCD$ is equal to $BCDA$

Proof. Refer to Fig. 9. To test whether $ABCD$ is equal to $BCDA$, we place AB horizontal and make $BC' = BC$ and $BA' = AB$. Then all four of the smaller rectangles in the figure are congruent (in the orientations shown). Hence their diagonals are congruent; hence B is the common midpoint of the diagonals of the large rectangle $DKD'H$. Hence B is between H and K. Hence $ABCD$ is equal to $BCDA$. That completes the proof.

Lemma 5.5. *Two rectangles with the same height are equal if and only if they have the same width. Two rectangles with the same width are equal if and only if they have the same height.*

Proof. First observe that $a : b = a : c$ if and only if $b = c$, as follows from the definition of proportionality. Then the theorem follows from Lemma 5.2. That completes the proof.

Lemma 5.6. *If one rectangle has both width and height less than a second rectangle, the two rectangles are not equal.*

Proof. By Lemma 5.2, it suffices to show that if $b < c$ and $a < d$, then we do not have $b : c = d : a$. To prove this, let AOB be a right angle with $AO = c$ and $BO = d$. By the definition of $<$ for line segments, there are points P and Q with $\mathbf{B}(0, P, A)$ and $\mathbf{B}(0, Q, B)$ and $OP = b$ and $OQ = a$. By inner Pasch, the lines PB

and AQ meet, and hence they are not parallel. Then by definition, we do not have $b : c = d : a$. That completes the proof.

Lemma 5.7. *If equal rectangles are cut off from equal rectangles (each cut off by one dividing line) then the remaining rectangles are equal.*

Remark. Euclid would have justified this by his common notion, "if equals be subtracted from equals, the remainders are equal."

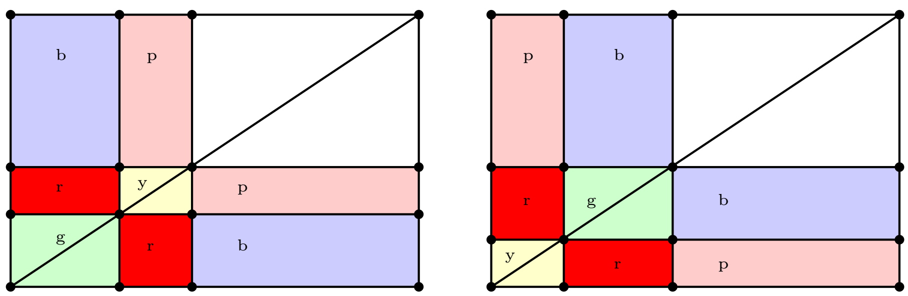

Figure 10. Cutting off equal rectangles (pink) from equal rectangles (pink and blue) leaves equal rectangles (blue).

Proof. Refer to Fig. 10. Rectangles in the figure are labeled p for pink, b for blue, y for yellow, and r for red, in case colors are not visible in the copy you are reading. The hypothesis is illustrated in the left part of the figure, where the equal rectangles being cut off are pink, and the equal rectangles from which they are cut off are blue-and-pink. Because these rectangles are equal, the points shown on the diagonal are in fact on the diagonal. Now we have to show that the blue rectangles are also equal. By Lemma 5.4, then pink-and-blue rectangles in the second figure are equal, so their common vertex lies on the diagonal. The two green rectangles are congruent, since their widths and heights are equal to the width and height of the blue rectangles. Similarly, the two yellow rectangles are congruent. We have to show that the common point of the yellow and green rectangles lies on the diagonal. One easy proof of that is this: the left figure shows that the two red rectangles are equal. But after permuting their vertices, those red rectangles are equal to the the corresponding red rectangles in the right figure. Therefore, the common point of the yellow and green rectangles lies on the diagonal. That completes the proof.

Lemma 5.8. *If equal rectangles are pasted on to equal rectangles, making larger rectangles, then the resulting larger rectangles are equal.*

Remark. Euclid would have justified this by his common notion, "if equals be added to equals, the wholes are equal." This can be taken literally if we define the sum of two rectangles having a common side to be their union.

Proof. The theory of proportion will not be used. We refer again to Fig. 10. The rectangles assumed equal are the pink rectangles. The rectangles pasted on are the blue rectangles. The lower left corner of the rectangle in the left figure is determined by the blue rectangles. We must show that if the blue rectangles are equal, that point lies on the diagonal of the white (and yellow) rectangles. The equality of the blue rectangles is witnessed in the right figure by the fact that the white and green rectangles have a common diagonal. The green rectangles are congruent, and the yellow rectangles are congruent, and the white rectangles are congruent. From the left figure, the yellow and white triangles have corresponding angles equal. From the right figure, the green and white triangles have corresponding angles equal. Hence the yellow and green triangles have corresponding angles equal. Hence the yellow and green rectangles have a common diagonal (in both figures). That completes the proof.

5.3. **Equal triangles.**

Definition 5.9. *The* **first circumscribed rectangle** $ABDK$ *of triangle* ABC *is the rectangle such that* C *lies on the line through* D *and* K.

(The word "circumscribed" in this context does not imply that the triangle lies within the rectangle.) To justify the definition, we have to show how to construct the circumscribed rectangle. That is done as follows: By I.29 there is a parallel line L to AB through C; then by I.12 there are perpendiculars BD from B to L and AK from A to L. Then AK is parallel to BD, since they are both perpendicular to L, and DK is parallel to AB by construction, so $ABDK$ is a parallelogram. Since $ABDK$ has a right angle, it is a rectangle.

Definition 5.10. *Triangles* ABC *and* abc *are* **equal triangles** *if their first circumscribed rectangles are equal. See Fig. 11.*

In this definition, the order of the vertices of ABC matters. To be precise (as we must when formalizing!) a triangle is an ordered triple of points. Lemma `ETpermutation` says that triangles with the same vertices (in any order) are equal. We will prove that below.

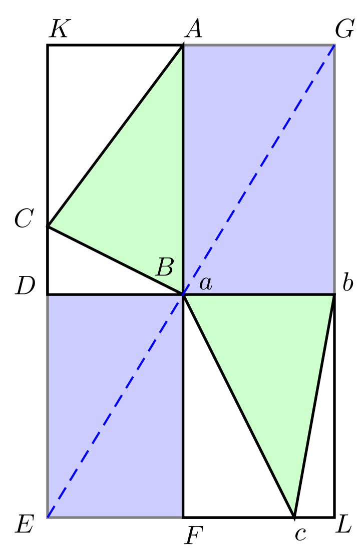

FIGURE 11. ABC is equal to abc if EBG is a straight line

Lemma 5.11. *Equal triangles is an equivalence relation.*

Proof. The required properties of equal-triangles follow from the definition and the fact that equal-rectangles is an equivalence relation (Lemma 5.3).

Lemma 5.12 (Euclid's I.37)**.** *Triangles which are on the same base and in the same parallels are equal to one another. Specifically, they are equal triangles.*

Remarks. Remember that the base of ABC is AB. The base depends on the order in which the vertices are listed. When applied to a segment, the word "equal" means the same as "congruent", and there is an $PQ = QP$, expressing the idea that order is not important. Strict equality (identity) is never used, so it is safe to use "equal" for "congruent", as Euclid does.

Proof. The base and "parallels" determine the circumscribed rectangle up to congruence, so two triangles with equal (congruent) base and altitude have congruent circumscribed rectangles, and congruent rectangles are equal. That completes the proof.

6. Verification of the equal-triangles axioms

6.1. **Remarks about the principle that ABC is equal to BCA.** This is one of the equal-figure axioms. Euclid assumed this principle without proof (for the first time) at the penultimate line of the proof of I.35, where triangle DGE has to be equal to triangle GED, in order for the justification Euclid gives for that line (subtracting equals from equals) to apply.

The principle is used again, and again without explicit mention, in Euclid's Prop. VI.2. That proposition is equivalent to Corollary 4.10, so it shows that the principle that ABC is equal to BCA not only is implied by, but implies, the basic theory of similar triangles. Specifically, for those with their copy of Euclid at hand, in the proof of VI.2, triangle BDE is first shown equal to CDE since both have base DE and the same height, and then is considered as having base BD in order to apply V.1. In the proof of V.1, applied here, BDE is proved equal to other triangles with base BD and the same height.

The referee offered to prove the equality of ABC and BAC as follows: ABC and BAC are congruent by SAS (that is, I.4); and by one of our equal-figures axioms, congruent triangles are equal. But we have already discussed in §5.1 that "order matters", and non-isosceles ABC is not congruent to BAC. For this proof to be valid, triangle congruence would have to be defined more generally, allowing the vertices to be mentioned in any order. But that is neither the usage in Euclid, nor the tradition in the intervening millenia, nor the modern formal usage. So if we want EBG and BEG to be equal triangles in Euclid's proof of I.35, we will have to justify that step somehow. In [4] we introduced an axiom to do that.[9]

In this paper, we take a different approach: we supply a definition of "equal triangles". Then we must prove that ABC is equal to BCA. We evidently cannot use Euclid's own theory of similar triangles to justify the proof that ABC is equal to BCA, since that would make the proof of VI.2 circular. But we never

[9]There is a related precise question: Is the equal-figure axiom that ABC and BAC are equal figures redundant? That is, can it be derived from the rest of the axioms in the system of [4]? We do not know the answer.

intended to use Euclid's theory of similarity anyway, as we wish to avoid the reliance on Archimedes's axiom that is introduced in Book V. We will instead use the nineteenth-century theory of proportionality.

Turning to modern times, the principle that ABC is equal to BCA is also implicitly assumed and used in textbooks, as I will now explain. Consider the usual formula for calculating the area of a triangle: base times height divided by 2. Here we arbitrarily select one side as the base. But if we select another side, will we necessarily get the same answer? "Of course we will", because the area depends only on the *unordered* vertices. But in stating that, we have imported some knowledge that is not in Euclid's axioms or the school curriculum. If we were to try to *define* the area by the base times height over 2 formula, we would then have to *prove* that we get the same answer no matter which side is taken as the base. That is closely related to the problem at hand, of verifying that ABC is equal to BCA. It is not exactly the same problem, as area is not involved in our definition of equal figures, but it addresses the same underlying issue.

6.2. **Verification that ABC and BCA are equal.**

Theorem 6.1. *The triangles ABC and BCA are equal.*

Proof. Applying the definition of equal triangles, we get the situation shown in Fig. 12.

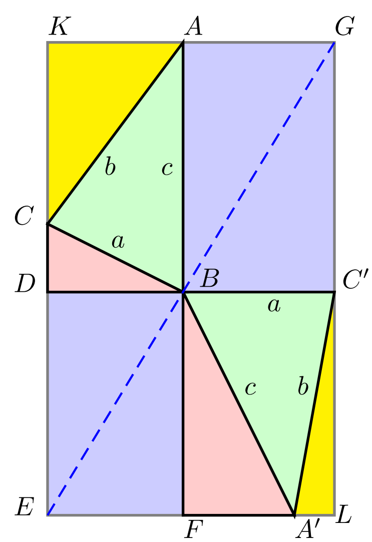

FIGURE 12. ABC is equal to BCA if EBG is a straight line

In the figure, the upper left and lower right rectangles are the circumscribed rectangles of the triangles, and the light blue rectangles are constructed from them. Their diagonals are BG and EB. It has to be proved that these diagonals lie on one straight line EBG.

We have

$$\begin{array}{rcll} BA' &=& c \;=\; AB & \\ BC &=& a \;=\; BC' & \\ AB &=& b \;=\; AC' & \\ ABC &=& A'BC' & \text{definition of triangle congruence} \\ \angle ABC &=& \angle A'BC' \;=\; \beta \text{ (the angle opposite } b) & \\ \angle CBD &=& \angle A'BF \;=\; \pi/2 - \beta & \end{array}$$

Triangles CDB and $A'FB$ (shown pink in the figure, if the copy you are reading is in color) are similar, since they are both right triangles and have equal angles at B, as just shown. Therefore $BF : BA' = BD : BC$, or $BF : c = BD : a$. Since $BD = EF$, also $BF : c = EF : a$. Then by Theorem 4.7,

$$BF : EF = c : a.$$

But $GC' = AB = c$, and $BC' = a$, so $GC' : BC' = c : a$. Hence

$$FB : EF = GC' : BC',$$

by Lemma 4.4. Then by Corollary 4.11, triangles EBF and BGC' are similar. Note that these triangles are right triangles, so that Corollary is applicable. Hence $\angle GBC' = \angle BEF$. Since EF is parallel to DC', EB extended past B makes the same angle with DC' as with EF, and hence coincides with BG. Hence EBG is a straight line. That completes the proof.

6.3. **Verification of the other equal-triangles axioms.**

Lemma 6.2. *Congruent triangles are equal.*

Proof. Immediate from the definition of equal triangles, which is "up to congruence."

Lemma 6.3. *Triangle ABC is equal to triangles BCA, CAB, ACB, BAC, and CBA.*

Proof. We first prove that the triangles ABC and BAC are equal. The first circumscribed rectangles of ABC and BAC are congruent, since the bases AB and BA are equal, and the two altitudes are equal to the altitudes of triangles ABC and BAC, which are equal. Therefore ABC and BAC are equal, as claimed.

Since all the permutations on three letters are generated by (213) and (231), the conclusion follows from that case (which corresponds to permutation (213)) and Theorem 6.1 (which corresponds to (231) together with the transitivity of equal-triangles (Lemma 5.11). That completes the proof.

The only other equal-figure axioms that involve only triangles are the two de Zolt axioms. (These are the only axioms that assert non-equality. Without them, we could interpret all triangles as equal!) The next two lemmas prove the de Zolt axioms as theorems, using the defined notion of equal triangles.

Lemma 6.4. *Suppose $\mathbf{B}(b, e, d)$. Then dbc and ebc are not equal triangles.*

Proof. Suppose, for proof by contradiction, that they are equal. Then by the definition of equal triangles, the first circumscribed rectangles of dbc and ebc are equal. These rectangles have the same height (the perpendicular from c to the line containing d, b, and c), but their widths are respectively db and eb. By definition of less than for lines, $ed < bd$. But $bd = db$. Since "the part is not equal to the whole", this is impossible. In our formal development, Euclid's principle that the part is not equal to the whole is a theorem about congruence. By Lemma 5.5, two rectangles of the same height are equal if and only if they have the same width; hence we have reached a contradiction. That completes the proof.

Lemma 6.5. *Let abc be a triangle, and suppose* $\mathbf{B}(b, e, a)$ *and* $\mathbf{B}(b, f, c)$*. Then abc is not equal to* ebf*.*

Proof. It suffices to show that triangle bfe is not equal to bca. The first circumscribed rectangles of bfe and bca have bases respectively bf and bc, and $bf < bc$. The altitude of ebf is also less than the altitude of abc. Hence, by Lemma 5.6, they two rectangles are not equal. Hence, by definition of equal triangles, the triangle bfe is not equal to bca. That completes the proof.

7. Equal quadrilaterals

Equal quadrilaterals will be defined in this section. The notion will be defined only for convex quadrilaterals and quadrilaterals that are really triangles. We will define the notion of convex so that it implies the quadrilateral lies in a plane; that is convenient, since we wish to do plane geometry, but without a dimension axiom, following Euclid.

The restriction to convex quadrilaterals bears some discussion. There are two ways a quadrilateral might fail to be convex: either it is *strictly* non-convex, or possibly it is "really a triangle", i.e., one of its vertices is between the two adjacent vertices. It seems that we must consider quadrilaterals that are really triangles, as otherwise there is no connection between "equal triangles" and "equal quadrilaterals." That connection occurs in our former axiom (now to be a theorem) `paste4` (which we will explain in due course, but not here). That axiom in turn is used in just one place: Prop. I.45.

In Euclid Books I-III, one does not find any propositions that mention or require strictly non-convex quadrilaterals. Indeed if Euclid had wanted to include them, he would have had to be much more forthcoming and explicit about what counts as a "part" of a figure, as a line connecting two non-adjacent vertices can no longer be supposed to cut off a part. In the rest of the paper, "quadrilateral" means "convex quadrilateral" or "really a triangle."

Definition 7.1. *Quadrilateral* $JKLM$ *is* **convex** *if its diagonals meet. That is, there is a point* E *between* J *and* L *that is also between* M *and* K*.*

Definition 7.2. *Let* $JKLM$ *be a convex quadrilateral. A* **circumscribed rectangle** *of quadrilateral* $JKLM$ *is a rectangle* $ABCD$ *with two sides parallel to diagonal* KM*, or two sides parallel to diagonal* JL*, and each vertex of* $JKLM$ *lies on some side of* $ABCD$*.*

A **circumscribed rectangle** *of a quadrilateral that is really a triangle is a circumscribed rectangle of that triangle.*

Each convex rectangle thus has two circumscribed rectangles, with sides parallel to one or the other diagonal of the quadrilateral. A rectangle that is really a triangle has three circumscribed rectangles, all of which are equal rectangles by Theorem 6.1.

Definition 7.3. *Two quadrilaterals are* **equal quadrilaterals** *or* **equal figures** *if one of the circumscribed rectangles of one is equal to one of the circumscribed rectangles of the other.*

We turn now to the verification of the axioms for equal quadrilaterals. See the Appendix for formal statements of these axioms.

7.1. **Verification of the axioms `EFPermutation` and `EFsymmetric`.** The axiom `EFpermutation` says that a quadrilateral is equal to any quadrilateral obtained by a permutation of the vertices, provided the two quadrilaterals have the same (set of) diagonals. (That is, you can't just switch two adjacent vertices.) Since all such permuted quadrilaterals have the same two circumscribed rectangles, the verification of the axiom is immediate.

The axiom `EFsymmetric` follows immediately from the symmetry of the relation of "equal rectangles." The axiom `EFtransitive` similarly follows from the transitivity of "equal rectangles".

7.2. **Verification of the axiom `halvesofequals`.** This axiom says that if equal quadrilaterals are each divided along a diagonal into equal triangles, then all four triangles are equal. Euclid used this principle without stating it[10] , and none of his common notions seem relevant.

Lemma 7.4. *The line connecting midpoints of opposite sides of a rectangle divides it into two equal rectangles.*

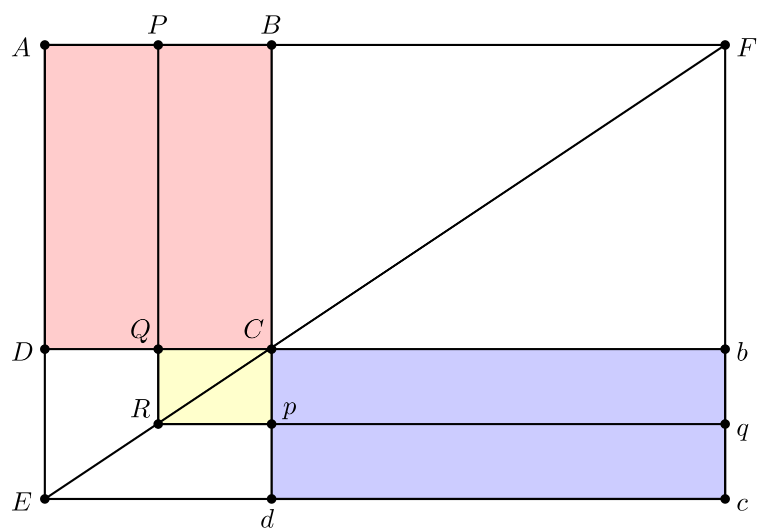

FIGURE 13. Equal rectangles $ABCD$ and $Cbcd$ are divided in half by PQ and pq. The resulting half-rectangles are equal since R lies on the diagonal.

Proof. Let $ABCD$ and $abcd$ be the equal rectangles, placed as shown in Fig. 13, so that C and a coincide and the lines dividing the two rectangles in half are PQ and pq as shown. Then since $ABCD$ and $abcd$ are equal rectangles, the sides can be extended to form a larger rectangle, as shown, whose diagonal EF passes through

[10]In Propositions I.37, I.38, and I.48

the point that is both C and a. Now let R be the point of intersection of PQ and pq. Then R is the intersection of the lines connecting the midpoints of opposite sides of rectangle $DCdE$. Then it is also the intersection of the diagonals of $DCdE$. Then R lies on the diagonal EC. But that is collinear with F, since rectangles $ABCD$ and $CbcD$ are equal. Then by definition of equal rectangles, rectangles $PBCQ$ and $Cbqp$ are equal. That completes the proof of the lemma.

Lemma 7.5. *If a quadrilateral $SPTQ$ is divided along its diagonal PQ into two equal triangles, then the circumscribed rectangle $ABCD$ with AD parallel to PQ is divided into equal rectangles by PQ. These rectangles are also congruent.*

Proof. By hypothesis, the triangles SPQ and TPQ are equal. (Since we have already verified axiom `ETpermutation`, the order of the vertices does not matter.) Then by definition, their circumscribed rectangles $APQB$ and $CPQD$ are equal. Since they have the same height PQ, they are congruent by Lemma 5.5. That completes the proof.

Lemma 7.6 (Halves of equals are equal)**.** *If equal quadrilaterals are each divided along a diagonal into equal triangles, then all four triangles are equal.*

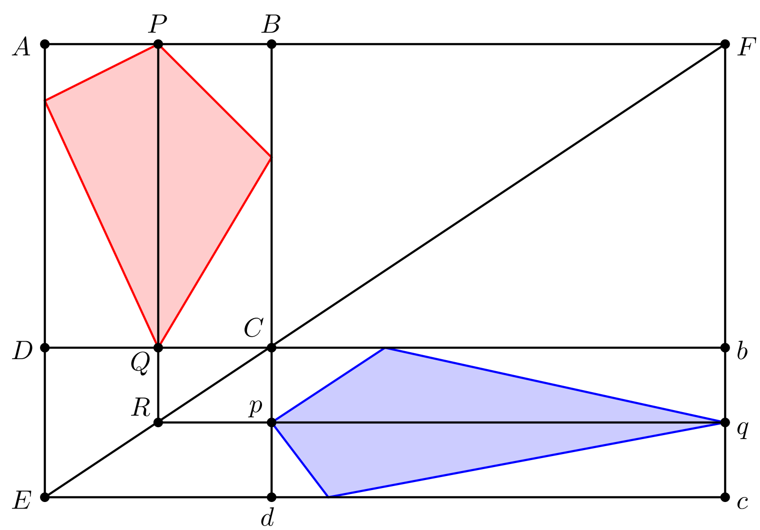

FIGURE 14. The two quadrilaterals are equal, and each is divided in two equal triangles. Then all the triangles are equal.

Proof. See Fig. 14. By Lemma 7.5, the circumscribed rectangles of the two quadrilaterals are divided into two equal rectangles by the diagonals that divide the quadrilaterals into two equal triangles. By Lemma 7.4, the half-rectangles are all equal. Then, by definition of equal triangles, the triangles whose circumscribed rectangles are those equal half-rectangles are equal. That completes the proof.

7.3. **Verification of `paste3` and `paste4`.** The axiom `paste3` is the case of "if equals be added to equals, the wholes are equals", when the equals being added are triangles with a common edge, and the wholes are quadrilaterals. See Fig. 15.

Lemma 7.7 (`paste3`)**.** *Suppose ABC and abc are equal triangles, and ABD and abd are equal triangles, and AC meets BD at M and ac meets bd at m. The cases*

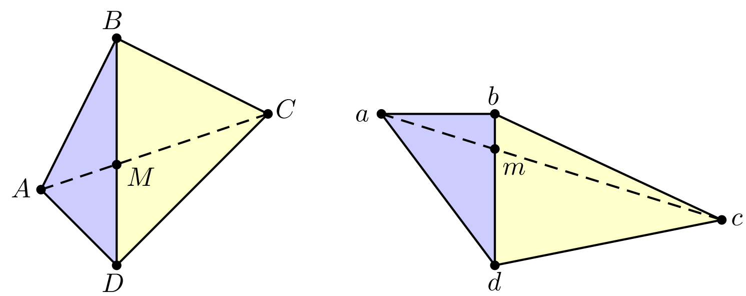

FIGURE 15. `paste3`: If ABD and abd are equal, and CBD and cbd are equal, then $ABCD$ and $abcd$ are equal.

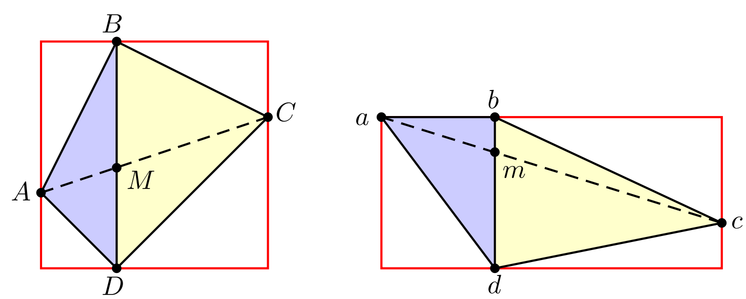

FIGURE 16. `paste3`: showing the circumscribing rectangles

when $M = A$ or $M = C$ or $m = a$ or $m = c$ are also allowed. Then $ABCD$ and $abcd$ are equal quadrilaterals.

Proof. Construct the circumscribed rectangles of $ABCD$ and $abcd$ (with sides parallel to BD and bd, respectively). Then these rectangles are each divided into two rectangles, which are the circumscribed rectangles of triangles ABD, BCD, abd, and bcd. Since the circumscribed rectangles of equal triangles are equal, the hypotheses of Lemma 5.8 are satisfied, and the conclusion of that lemma is that the circumscribed rectangles of $ABCD$ and $abcd$ are equal rectangles. Then by Definition 7.3, $ABCD$ is equal to $abcd$. That completes the proof.

Remark. The axiom `paste3` was used in [4] in the proofs of Propositions I.35a, I.42, I.47B, and in proving lemma `paste3`, which was in turn used in I.48. In the present approach, we prove I.42 using `doublesofequals` (which will be proved next) and the rest directly. However, now we use `paste3` in proving I.45.

Lemma 7.8 (Doubles of equals are equal)**.** *If two quadrilaterals $ABCD$ and $abcd$ are each divided along the diagonals AB and ab into two equal triangles, and triangles ABC and abc are equal, then the two quadrilaterals are equal.*

Remark. It is allowed that one or both quadrilaterals may be "really a triangle", with one end of the diagonal between the adjacent vertices. This is needed in proving I.44.

Proof. This lemma is immediate consequence of `paste3`. However, we like the following proof directly from the definition, that does not appeal to Lemma 5.8, so we present it as well. Refer to Fig. 14. The hypothesis is that all four triangles shown are equal. We have to show that rectangles ABC and $Cbcd$ are equal, which implies that the red quadrilateral and the green quadrilateral are equal. Because the red and green triangles are equal, point R lies on the (extended) diagonal FC. Because the two red triangles are equal, Q is the midpoint of DC. Because the green triangles are equal, p is the midpoint of Cd. The diagonals of a rectangle bisect each other (as Euclid could have easily proved after Prop. I.34 without using any equal figures axioms; we did so in [4]); the diagonals of $DCdE$ in particular meet at R. Hence C lies on EF. Hence rectangle $ABCD$ is equal to rectangle $Cbcd$, as claimed. That completes the proof.

The axiom `paste4` is very similar to `paste3`: like `paste3`, it is about pasting two triangles together to get a quadrilateral. But in `paste4`, the "triangles" are quadrilaterals that are "really triangles", in the sense that one vertex is between the two adjacent vertices. The picture is the same as Fig. 15, except that two more points are added along two sides of the triangles. The verification that `paste4` holds with the defined notion of equal quadrilaterals is the same as the verification of `paste3`, since the circumscribed rectangle of such a quadrilateral is the same as the circumscribed rectangle of the triangle.

Remarks. In our formalization [4], Axiom `paste3` was used to prove Propositions I.35 and Lemma `EFreflexive`, both of which we can now prove directly, and to prove Propositions I.42 and I.42B and I.47B where `doublesofequals` suffices. In fact in the proof of I.47, Euclid specifically states, in parentheses, "But the doubles of equals are equal to one another." This is clearly intended as a justification for the statement that follows that quotation, which however has no official justification, as Euclid has not proved that doubles of equals are equals.

7.4. **Proposition I.35.**

Lemma 7.9 (Proposition I.35)**.** *Parallelgrams which are on the same base and in the same parallels are equal to one another*

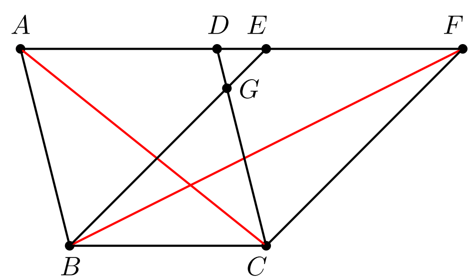

FIGURE 17. Euclid's figure for I.35

Remark. This is the first proposition in which Euclid needs (something like) the equal-figures axioms; in particular triangle DEG (see Fig. 17) needs to be equal to triangle EDG for Euclid's proof to work. But here we derive Prop. I.35 from I.37, which we have derived already, directly from the definition of "equal triangles."

Proof. Fig. 17 is Euclid's figure, except that we have added the (red) diagonals AC and BF. By I.37, triangles ABC and BCF are equal, since they are triangles with the same base and in the same parallels. By I.34, the diagonals divide the parallelograms into equal triangles. By Lemma 7.8, the parallelograms are equal. That completes the proof.

7.5. **Prop. I.42, I.43, and I.44.** We have already remarked that I.42 follows from Lemma 7.8. I.44 does not use any equal-figures axioms, other than transitivity, which we have proved. Therefore I.44 is provable. We will turn to I.43, but we need a lemma first.

Lemma 7.10. *Equal rectangles (in the sense of Definition 5.1) are equal quadrilaterals (in the sense of Definition 7.3). In other words, circumscribed rectangles of equal rectangles are equal.*

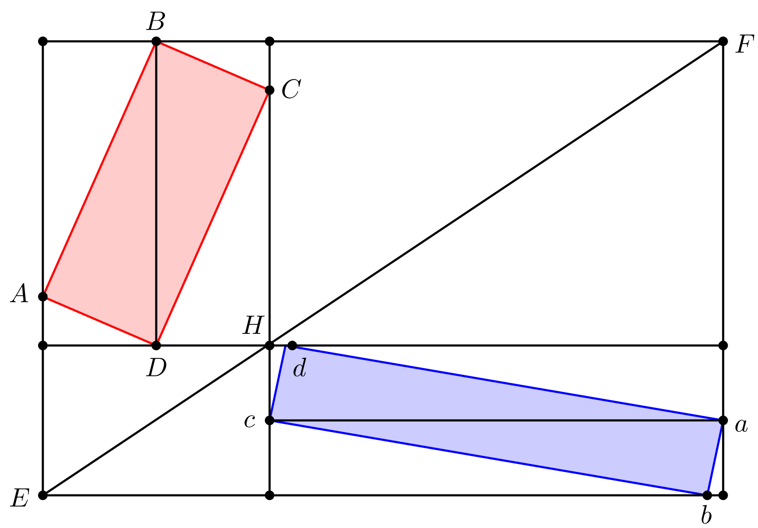

FIGURE 18. Equal rectangles are also equal quadrilaterals

Proof. Let $ABCD$ and $abcd$ be equal rectangles. Their diagonals divide each of them into two congruent triangles. By the definition of equal triangles, triangle ABD is equal to triangle abc, since they have equal circumscribing rectangles (and we have proved the order of vertices does not matter).

One circumscribed rectangle of triangle ABD is half of the circumscribed rectangle of $ABCD$, and one circumscribed rectangle of triangle abc is half of the circumscribed rectangle of $abcd$. Then by Lemma 7.8, quadrilaterals $ABCD$ and $abcd$ are equal. That completes the proof.

Lemma 7.11 (Prop. I.43)**.** *In any parallelogram the complements of the parallelograms about the diameter are equal to one another.*

Remark. In Euclid's proof, he needs to apply the common notion that equals subtracted from equals leave equal remainders. In formalizing that argument in [4], we used the axioms `cutoff1` and `cutoff2`. But here, we prove the theorem without common notions or cutoff lemmas, by reducing it to the case of a rectangle instead of a parallelogram, and then using the definition of equal rectangles.

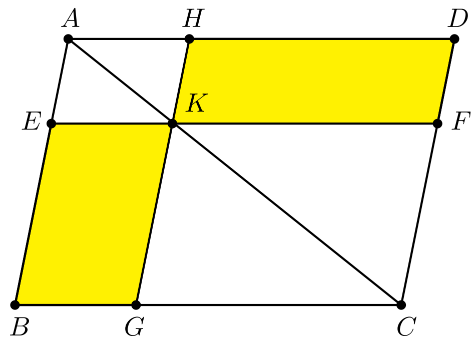

FIGURE 19. Euclid's figure for Prop. I.43. To prove: the yellow parallelograms $BGKE$ and $KFDH$ are equal.

Proof. Let A' be a point collinear with AD such that $A'B$ is perpendicular to BC. Let E' be the intersection of $A'D$ and the line containing EF. Similarly let $H'K'$ be perpendicular to BC at G and $D'F'$ perpendicular to BC at C, with $K'F'$ collinear with KF and $H'D'$ collinear with HD. Since opposite sides of a parallelogram are equal, and parallelograms with equal bases and in the same parallels are equal, the parallelograms with primes are equal to the parallelograms without primes. Therefore, it suffices to prove the lemma in the case when all the parallelograms are rectangles. But in that case, by the definition of equal rectangles, $H'D'K'F'$ is equal to $E'K'G'B'$, as rectangles, if and only if K' lies on $A'C$. Now by Lemma 7.10, they are also equal in the sense of "equal quadrilaterals." That completes the proof.

7.6. **Prop. I.45.** In the formalization of [4], Prop. I.45 is proved with the aid of the axiom `paste2` and the theorem `paste5`, which was originally an axiom, but can be proved from the other paste axioms. The paste axioms are used to formalize an inference that Euclid justifies by the common notion "if equals be added to equals, the wholes are equal." Since we have not verified `paste2`, we either have to do so, or prove I.45 directly. In this section we give a direct proof of the step of I.45 in question.

In Euclid's proof of I.45, we have a quadrilateral $ABCD$, whose two parts ABC and ABD are respectively equal to parallelograms $FGHK$ and $GLMH$, which share the common side GH and together form the "whole", that is, the larger parallelogram $FLMH$, as shown in Fig. 20.

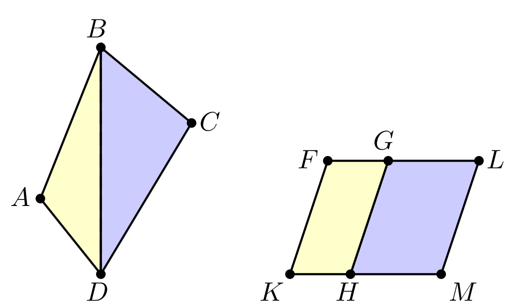

FIGURE 20. Euclid's figure for Prop. I.45 (color added)

By Euclid I.36, we could reduce to the case when the parallelograms are rectangles, but that does not make the proof immediate. Axiom `paste2` permitted, in our old formalization, “adding” triangles that share a common side, but here we have to “add” equals, where on one side we are adding parallelograms (or rectangles) with a common side.

That is the purpose of the theorem `paste5`, which we proved in [4] from the axiom `paste2`. More generally, `paste5` permits $FLMH$ to be any convex quadrilateral, not just a parallelogram, divided in two by a line GH from side FL to side KM. In this section, we only prove the parallelogram case, which suffices for I.45.

Lemma 7.12. *Every quadrilateral or triangle is equal to half its circumscribed rectangle, where “half” means that the rectangle is divided by connecting the midpoints of two opposite sides.*

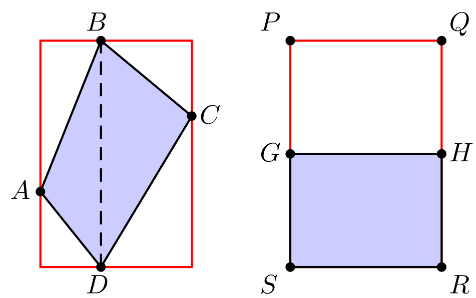

Figure 21. $ABCD$ is equal to half of $PQRS$, namely $GHRS$

Proof. By Euclid I.40 (triangles with the same base and in the same parallels are equal), we may without loss of generality assume that triangles BDA and BDC are isosceles, i.e., $AB = AD$ and $BC = CD$. Applying Euclid I.40 again, we may assume without loss of generality that B is in the upper left corner and D in the lower right corner. See Fig. 22.

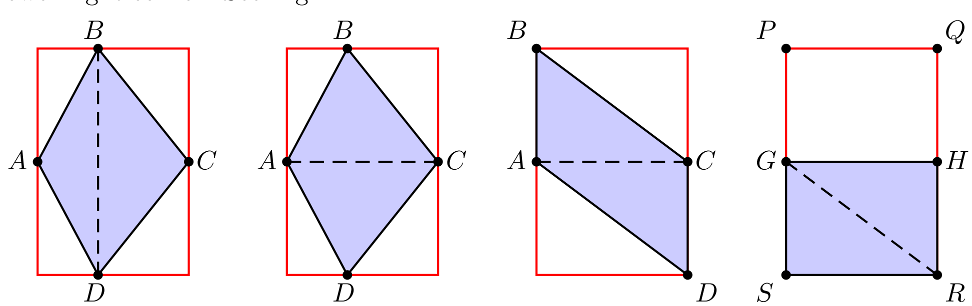

Figure 22. The blue (or shaded) quadrilaterals are all equal.

Now we are in a position to apply `paste3` (Lemma 7.7). Triangle ACD is equal to triangle GHS (since they are congruent). Triangle ACB is equal to triangle GSR (since they too are congruent). By Lemma 6.3, the order of the vertices does not matter. By `paste3`, quadrilaterals $ABCD$ and $GHSR$ are equal; again the order of the vertices does not matter, by the definition of equal quadrilaterals. That completes the proof.

Lemma 7.13. *Let $ABCD$ be a convex quadrilateral and $FLMK$ a parallelogram, composed of two parallelograms $FGHK$ and $GLMH$. Suppose triangle ADB is equal to $FGHK$ and triangle CBD is equal to $GLMH$. Then $ABCD$ is equal to $FLMK$. See Fig. 20.*

Remark. This is the part of Euclid I.45 that is justified using the common notion "if equals be added to equals, the wholes are equal." In our formalization [4], we used an equal-figures axiom `paste4`.

Technical remark. We did not formally define the notion of a triangle being equal to a quadrilateral. To state the lemma formally, we need to consider another point E between B and D somewhere on the boundary of each triangle mentioned, so triangles ABD and BCD are formalized as quadrilaterals $ABED$ and $CBED$.

Proof. By Euclid I.36 (which we have already proved), it suffices to consider the case in which the parallelograms are rectangles. The resulting situation is shown in Fig. 23.

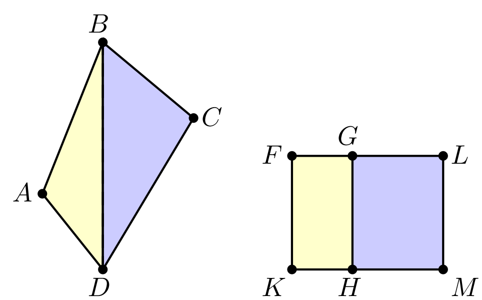

FIGURE 23. To prove: $ABCD$ is equal to $FLMK$.

We construct a larger rectangle, double the height of $FLMK$, by adding a new rectangle $kmLF$ above FL, the same size and shape as $FLMK$. Also define h on line GH and line km. Then G is the midpoint of hH and L is the midpoint of mM. See Fig. 24.

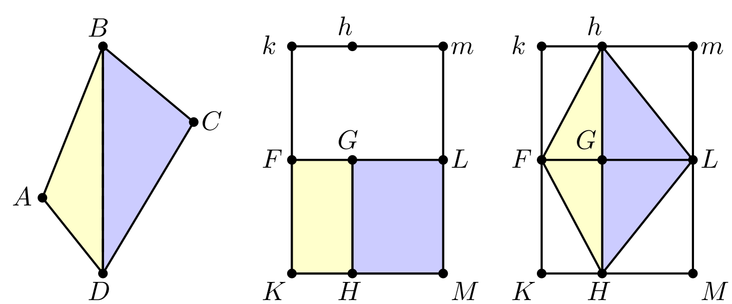

FIGURE 24. Some steps in the proof of Lemma 7.13

Triangle FGh is congruent to HKF (since the diagonals of a parallelogram divide into two congruent triangles), and also to triangle FGH, since right triangles with congruent legs are congruent. Then by Lemma 7.7 (`paste3`), $FGHK$ is equal to $hGHF$. Since $hGHF$ is really a triangle, $FGHK$ is equal to hHF. Therefore ABD is equal to hHF. Similarly, ABD is equal to hHL. Then by Lemma 7.7 (`paste3`), $ABCD$ is equal to $FhLH$. By Lemma 7.12, $FhLH$ is equal to half its circumscribed rectangle, which is $FLMK$. By the transitivity of "equal quadrilaterals", $ABCD$ is equal to $FLMH$. That completes the proof.

Corollary 7.14. *Euclid I.45 is provable, either abstractly from the other equal-figure axioms, but wihout* `paste2` *or any of the cutoff axioms, or using the definitions of "equal triangles" and "equal quadrilaterals" and not the equal-figure axioms.*

Remark. This corollary eliminates the need to prove `paste5` as part of the formal development of Euclid. It does not quite eliminate `paste2`, as that is used in the formal development to prove I.35 as well as `paste5`. Here we proved I.35 directly, using the defined notion of "equal figures", so `paste2` was not needed anywhere in Euclid Book I.

Proof. Lemma 7.13 provides the step in the proof of Prop. I.45 for which we needed `paste4`. That completes the proof.

8. A Euclidean theory of area

Prop. I.45 allows to construct a parallelogram with one angle specified that is equal to a given quadrilateral. The same method of proof, taken one step further, could have allowed Euclid to construct a parallelogram with one angle *and one side* specified, equal to a given quadrilateral. Perhaps Euclid thought that was so obvious that he did not need to spell it out. That is somewhat surprising, since it could be used to define "equal area." Namely, two figures have equal area, if they are both equivalent to the same rectangle. If one side of the rectangle is regarded as a linear measuring unit, the other side gives the area of the figure measured in square units. But Euclid did not take this step. We take it in the following definition:

Definition 8.1. *Let Δ be a convex quadrilateral or triangle, and let UV be any given segment (thought of as the "unit segment"). Then the* **area of Δ relative to unit** *UV is any rectangle equal to Δ and having one side equal to UV.*

Theorem 8.2. *Every convex quadrilateral or triangle has an area, and the area is unique (up to congruence).*

Proof. As remarked above, this is a minor extension of Euclid I.45.

By the "sum" of two rectangles with side UV we mean another such rectangle formed by placing the two end-to-end; of course that notion can be formulated precisely without mentioning motion. With UV fixed, the "area" of a triangle or quadrilateral Δ is by definition the rectangle with side UV that is equal to the circumscribed rectangle of Δ. That rectangle is unique, by Lemma 5.5. We can now prove that we can test two figures for equality by comparing their areas:

Theorem 8.3. *With area defined as above, relative to a fixed "unit segment", two quadrilaterals or triangles are equal if and only if they have the same area.*

Proof. Let Γ and Δ be equal figures. As remarked above, the method of Prop. I.45 and I. 38 can be used to show that each figure is equal to a rectangle with one side equal to the unit segment UV. By transitivity, the figures are equal if they have equal area. That is half the theorem. Now, suppose the figures Γ and Δ have equal area; we must prove they are equal. Since they have equal area, by the definition of area, there are rectangles R_1 and R_2, equal respectively to the circumscribed rectangles of Γ and Δ, such that R_1 and R_2 are equal rectangles. By Lemma 7.10, R_1 and R_2 are equal quadrilaterals (as well as equal rectangles). By the transitivity of "equal rectangles", the circumscribed rectangles of Γ and Δ are equal. Hence, by definition, Γ and Δ are equal figures. That completes the proof.

Remark. This is a kind of "completeness theorem" for the notion of equal figures, and also for the equal-figures axioms. If "figures" means convex quadrilaterals and triangles, then we have not omitted any axioms that would be needed to use "equal area" as an interpretation of "equal figures". But that is only one property we would wish "area" to have.

Theorem 8.4 (Additivity of area)**.** *Suppose figures Γ and Δ have a common line BD and together form another figure Ξ. Then the sum of the areas of Γ and Δ is the area of Ξ.*

Remarks. Remember that the area is a rectangle (with unit height) and "sum" refers to the "rectangle sum", resulting from placing two unit-sided rectangles end-to-end with unit sides together. Also remember that "figure" means (for the proof below at least) triangle or convex quadrilateral. This theorem amounts to the "mother of all paste axioms," in the sense that it represents more closely the intuition behind the paste axioms.

Proof. Let R_1 be the area of Γ . Let R_2 be the area of Δ. (Then R_1 and R_2 are rectangles.) Let us first consider the case when Γ and Δ are both triangles (or quadrilaterals that are "really triangles") sharing an edge. Since Ξ is a figure, it is either a convex quadrilateral or "really a triangle". Since R_1 is equal to Γ (by definition of area) and R_2 is equal to Δ, we can conclude by Lemma 7.13 that Ξ is equal to the rectangle sum $R_1 + R_2$, i.e., the rectangle obtained by pasting together R_1 and R_2 along their unit edges. But by Lemma 5.6, that rectangle must be the area of Ξ. That completes the proof in case Γ and Δ are triangles.

Before proceeding, we note that rectangle addition $R_1 + R_2$ is both commutative and associative. These properties follow from Lemma 5.5 and elementary theorems about congruence of line segments.

Now consider the case when Γ is a triangle and Δ is a convex quadrilateral. Since Ξ is a quadrilateral, Γ and Δ must have a vertex in common. The situation must be as shown in Fig. 25, in which Ξ is $ABDE$, Γ is ABC, and Δ is $ACDE$. By

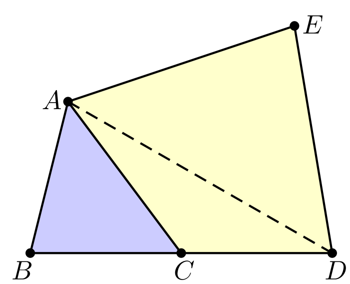

FIGURE 25. $area(ABDE) = area(ABC) + area(ACDE)$

hypothesis, Ξ is a figure, so it is convex, i.e., B meets AD. Then line AD divides

Δ into two triangles, and we have

$$\begin{aligned}
area(\Xi) &= area(AED) + area(ADB) \\
&= area(AED) + (area(AEC) + area(ACB)) \\
&= (area(AED) + area(AEC)) + area(ACB) \qquad \text{see below} \\
&= area(AEDC) + area(ACB) \qquad \text{by associative of rectangle addition} \\
&= area(\Delta) + area(\Gamma) \\
&= area(\Gamma) + area(\Delta)
\end{aligned}$$

That completes the proof in case Γ is a triangle and Δ is a convex quadrilateral. Since rectangle addition is commutative, that also takes care of the case when Δ is a triangle.

The only remaining case is when both Γ and Δ are quadrilaterals. Then the situation is as shown in Fig. 26.

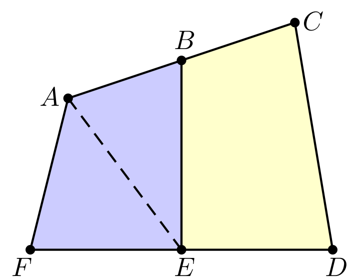

FIGURE 26. $area(ACDE) = area(ABEF) + area(BCDE)$

We apply the same method: adding a quadrilateral amounts to adding two triangles. Then

$$\begin{aligned}
area(\Xi) &= area(ACDF) \\
&= area(ACDE) + area(AFE) \qquad \text{as proved above} \\
&= (area(ABE) + area(BCDE)) + area(AFE) \\
&= (area(ABE) + area(AFE)) + area(BCDE) \\
&= area(ABFE) + area(BCDE) \\
&= area(\Gamma) + area(\Delta)
\end{aligned}$$

That completes the proof.

Remark. It is worth remembering that our treatment has equal-figure axioms only for triangles and convex quadrilaterals. While that seems to cover Euclid Books I to II, it is still unsatisfactorily far from capturing the intuitive notion of “figure”, which should at least include all (convex or not) connected polygons, and perhaps disconnected ones, and ones with curved boundaries. In particular, Book IV deals with polygons inscribed in circles, so it will be necessary to extend the notion of “figure” before we can formalize Book IV. These matters are discussed at length in [10] and [11], and are beyond the scope of this paper, whose aim is simply to remove the necessity for an undefined notion of “equal figures” in Euclid Books I and II (and III comes for free, since it does not mention figures at all).

8.1. **A conservative extension theorem.** Let us review. We have defined "equal triangles" and "equal quadrilaterals", and proved many of the equal-figure axioms, using those defined notions instead of a primitive notion. In particular we proved that those defined notions are equivalence relations, and that the order of listing the vertices does not matter, and we proved `paste3` and `paste4`, corresponding to pasting triangles along a common side.

We so far never needed the following axioms: `paste1`, `paste2`, `cutoff1`, and `cutoff2`. In fact, as you can see in the appendix:

- `paste1` was never used, even in the formalization in [4].
- `paste2` was used (only) to prove Prop. I.35, which we proved directly in Lemma 7.9, and to prove `paste5`, which in turn was used (only) in Prop. I.48.
- `cutoff1` and `cutoff2` were used (only) in Prop. I.35 and Prop. I.43, both of which we have proved directly in this paper.

We also were able to prove directly those propositions of Euclid in which he made use of "equal figures." Therefore, we have now achieved the aim of showing that Euclid did not actually need to take the notion of "equal figures" as primitive.[11]

But there is still a bit more to prove concerning the relation between the defined notion of "equal figures" and the axiomatic treatment in [4].

Theorem 8.5 (Conservative extension theorem)**.** *Any theorem not mentioning "equal figures" that can be proved with the aid of the equal-figures axioms, can also be proved without those axioms.*

Proof. More generally, in any formal proof we can replace the symbols `ET` and `EF` by the defined notions of "equal triangles" and "equal quadrilaterals", to get the "interpretation" of the proof. We claim that every step in the interpretation of a proof is a theorem provable without equal-figure axioms. This we prove by induction on the length of proofs. The only difficult part is the base case, in which ϕ is an equal-figures axiom. In this paper we have proved the interpretations of all the equal-figures axioms except `paste1` and `paste2`. We must now take care of this loose end.

Fortunately, we are spared from the need to prove `paste1` and `paste2` directly from the definitions of "equal triangle" and "equal quadrilateral". Instead, they can be derived from the additivity of area. Here is how. Both those axioms have the form that some figure Ξ is composed of two figures Γ and Δ where Γ is a triangle and Δ is a quadrilateral. In `paste1`, the two are combined as in Fig. 25, and in `paste2`, as in Fig. 26. The hypothesis is that there are figures Γ' and Δ', equal respectively to Γ and Δ, and combining to make a figure Ξ'. The desired conclusion is that Γ' and Δ' are equal figures. According to Theorem **??**, it suffices to prove that $area(\Xi)area(\Xi')$. That follows from the additivity of area. Explicitly, we have

$$\begin{aligned} area(\Xi) &= area(\Gamma) + area(\Delta) \\ &= area(\Gamma') + area(\Delta') \qquad \text{by Theorem 8.3} \\ &= area(\Xi') \qquad \text{by the additivity of area} \end{aligned}$$

That completes the proof of the theorem.

[11] Except of course, that we still need to show how to develop the theory of proportion with techniques from Book I.

9. The early theory of proportionality

The question of "early development of the theory of proportions" (that is, its development without the Axiom of Archimedes, from principles in the first part of Euclid Book I) was already considered by German geometers in the 1890s.[12] This subject has been worked on by Kupffer, Schur, Dehn, Hessenberg, Hilbert, and Bernays. Above we showed that the heart of the matter is just two theorems, which we call (following Bernays) the interchange theorem and the fundamental theorem of proportion. In this section, we discuss the beautiful proofs of Karl Kupffer, who found them in 1893, as described in [14], and those of Bernays, both of which answer to our need for a development by the means available to Euclid.[13]

9.1. **Proportionality in Euclid Book VI and in Hilbert.** Euclid Book VI presents a theory of proportionality based on the theory of "magnitudes", due to Eudoxes and using the axiom of Archimedes, which occupies Book V. The interchange theorem does occur in Book VI, but Book VI makes use of Prop. I.38, which makes use of I. 35, which requires the use of the equal-figures axiom that ABC is equal to BCA. It would therefore be circular to make I.35 rely on the Eudoxian theory of proportions. Perhaps this is the reason that Euclid turned to applying the common notions to "equal figures."

The theory of proportion can be derived from two fundamental theorems of projective geometry, Desargues's theorem and Pascal's theorem. Hilbert [11], §14, sketches a proof of Pascal's theorem based on theorems about circles occurring in Euclid Book III. To use Hilbert's approach, we would have to prove those theorems about circles and cyclic quadrilaterals, without using the equal-figure axioms of Euclid. That is possible, but it would require inserting a few theorems from Book III before Prop. I.35. However, it is not at all certain that Desargues's theorem can be proved with the tools at hand before I.35. Hilbert only proves it in §24, after defining segment arithmetic. Thus, we do not find direct support in Hilbert for the claim that Euclid could have developed proportionality in Book I.

9.2. **Bernay's Supplement II.** The eighth edition (1956) of Hilbert's *Foundations of Geometry* [11] contained Supplement II by Paul Bernays (which is still in the cited tenth edition). The Supplement is entitled, *A simplified development of the theory of proportion.* Bernay's definition of proportion, though phrased in terms of congruent angles instead of parallel lines, is easily seen to be equivalent to our Definition 4.1. Bernay sketches a proof of the interchange theorem.

Bernays next considers what he calls the "fundamental theorem of proportion", namely, if two parallels delineate the segments a, a' on one side of an angle and b, b' on the other side of the angle, then $a : a' = b : b'$. Bernays proves the fundamental theorem (on pp. 204-205). The proof is both clear and easily accessible, so we do not repeat it. It depends on the theorem that the three bisectors of the angles of a triangle intersect in one point (the incenter). Since we wish to claim that Bernays's proof of the fundamental theorem uses only methods from the part of Book I not

[12]Thus "early" refers to the logical status of the work, i.e., not relying on the later parts of Euclid that need Archimedes, and not to the chronology, since this work was done in the nineteenth century.

[13]Already in 1810, Bolzano [5] called Euclid's use of Book V to reach the theory of similar triangles an "atrocious detour". Baldwin [2] calls this "Bolzano's challenge", and compares the treatments of similarity in Euclid, Descartes, and Hilbert.

using equal figures, we need to check that the incenter theorem can be proved by those means. That theorem occurs in Euclid, implicitly rather than explicitly, as Prop. IV.4, which is stated, *In a given triangle to inscribe a circle.* But the first half of Euclid's proof of IV.4 does not mention circles and proves the incenter theorem using only congruent triangles and perpendiculars.

Bernays also proves the existence and uniqueness of the fourth proportional. Hence Bernays's 1956 Supplement II provides almost what we need in this paper: a development of the theory of proportions based on the methods of Euclid Book I. It falls short of that requirement only by needing a couple of simple theorems from Book III about circles, which do not use "equal figures" for their proofs. Nevertheless, we shall show (in §9.3 below) how to overcome even this small defect, using a proof due to Kupffer.

In summary: Bernays's Supplement II proves all the theorems listed in §4 as "Theorems", using techniques to be discussed in more detail below, but generally acceptable for our purposes. In the next section, we will prove the other results (lemmas and corollaries) from §4.

9.3. **Kupffer's development of proportionality.** As it turns out, Bernays was by no means the first ones to consider the possibility of developing the theory of proportions without using the theory of magnitudes (and Archimedes's axiom) as in Euclid Book IV. This was done by Karl Kupffer possibly among others, as early as 1893. We cite [14], but that 1902 letter just calls attention to Kupffer's 1893 lecture, whose audience included Schur. Kupffer gave *two* elementary proofs of Theorem 4.7, both of which Euclid "could have given."[14]

Since Schur's paper is in German, and since we want a detailed proof to serve as a basis for formalization, we give both proofs here. The first proof uses some theorems which, as Euclid stated them, mention circles, but are easily stated and proved without mentioning circles, by elementary means from Book I. Namely, Prop. III.21 (chords subtending the same arc are equal), Prop. IV.5 (three points determine a circle), and the following lemma, which is itself proved from III.21. Prop. III.21 uses III.20, which uses I.5 and I.32; the point is that the use of equal figures starts with I.34, so III.21 could be reached in two propositions after I.32, without using equal figures. The proof of Prop. IV.5 references only I.10 and I.4. So these theorems can all be proved without much of a detour from Book I, as we checked carefully.[15]

Lemma 9.1 (Cyclic quadrilateral theorem)**.** *Let $ABCD$ be a convex quadrilateral whose diagonals meet at O. Suppose angles OAB and ODC are equal. Then the four vertices lie on a circle.*

---

[14]Kupffur gave two proofs of the interchange theorem in 1893. Bernays gave a proof in 1956, identical to Kupffur's first proof, that Bernays attributed to Federigo Enriques's 1911 book [6]. The relevant material is in a chapter written by someone else, namely Giovanni Vailati. On p. 239, Vailati gives Kupffur's *second* proof, with credit and citation, and mentions his first proof, but then says that the first proof is actually due to Weierstrass, who (Vailati says) was the first to develop proportion theory without the axiom of Archimedes. But Vailati gives no citation to support this claim, and I could not pick up the trail.

[15] Euclid III.20 and III.21 need repairs both in statement and proof. Some of these problems have been known for centuries (see Heath's commentary [7]). But Heath does not remark on the step in Euclid's proof of III.20 for which Euclid gives no justification, but which is difficult to prove formally, and the final "Therefore etc." obscures the difficult proof that the two cases Euclid presents (and the one he does not present) are actually exhaustive.

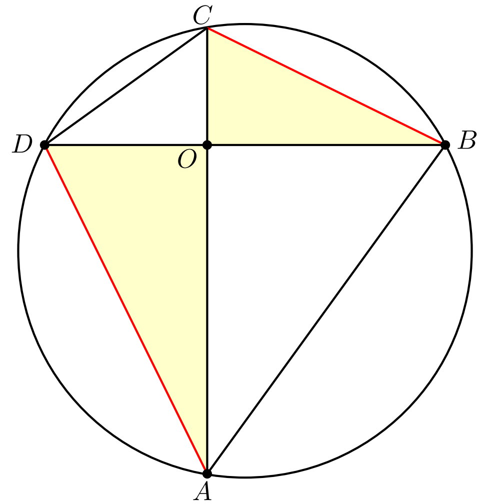

Figure 27. The cyclic quadrilateral theorem

Proof. By Prop. IV.5, any three non-collinear points lie on a circle, so let K be a circle containing A, C, and D. Then point O is inside K, since it lies between A and C. Hence, by the line-circle axiom, line DO meets circle K in two points; one of these points is D. Call the other one E. Then angle ODC and angle OAE subtend the same chord EC. Hence by III.21, they are equal. But angle ODC is equal to angle OAB by hypothesis. Hence angles OAB and OAE are equal. Hence B lies on line AE. But B also lies on line OD. Hence B is the intersection point of DO and AE. But that intersection point is E. Hence $B = E$. Hence B lies on circle K. That completes the proof.

9.4. **Kupffer's first proof of the interchange theorem.** Recall that the interchange theorem (Theorem 4.7) is: if $r : s = p : q$, then $r : p = s : q$.

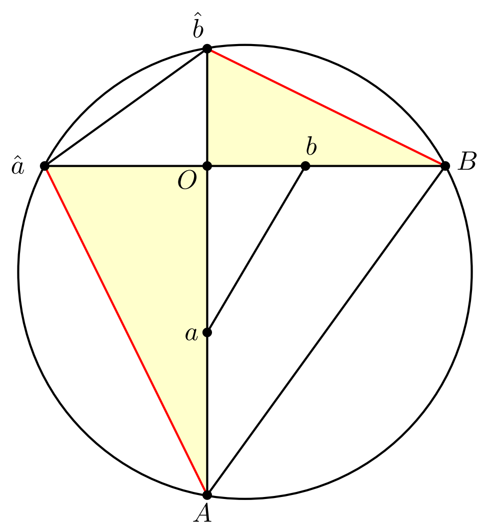

Figure 28. Kupffer's first proof of the interchange theorem

Proof. Suppose $r : s = p : q$. Then by Definition 4.1, there is a right angle AOB with $AO = r$ and $BO = s$, and point a on ray OA and b on ray OB such that $Oa = p$ and $Ob = q$, and $AB \mid\mid ab$. Without loss of generality we may assume $a < A$. Let $\hat{a}$ and $\hat{b}$ be points on the other side of O from b and a, respectively, such that $O\hat{a} = Oa$ and $O\hat{b} = Ob$. Angle aOb is equal to angle $\hat{a}O\hat{b}$, since they are vertical angles. Then triangle aOb is congruent to triangle $\hat{a}o\hat{b}$, by SAS. Since $AB \mid\mid ab$, angle OAB is equal to angle Oab, and hence also to angle $O\hat{a}\hat{b}$. Then $A, B, \hat{a}, \hat{b}$ form a cyclic quadrilateral, i.e., all four lie on a circle, by Lemma 9.1. Then $OA\hat{a}$ and $OB\hat{b}$ have corresponding angles equal, since their angles at O are vertical angles, and their angles at A and B both subtend the same arc $\hat{a}\hat{b}$, so they are equal by Euclid III.21, and their angles at $\hat{a}$ and $\hat{b}$ subtend the same arc AB, so they are also equal by Euclid III.21.

By Corollary 4.10, $OA : O\hat{a} = OB : O\hat{b}$. But $O\hat{a} = Oa$ and $O\hat{b} = Ob$. Therefore $AO : Oa = OB : Ob$; that is $r : p = s : q$. That completes Kupfer's proof.

9.5. **Kupffer's second proof of the interchange theorem.** This does not use any theorems about circles. It depends only on the fact that the three altitudes of a triangle meet in a point (the "orthocenter"). That theorem was known to Archimedes, but it does not occur in Euclid. It does, however, have a short proof using the methods of Book I, without using equal figures, and hence certainly could have been proved by Euclid. Such a proof can be found, for example, in [10], p. 54.

Pascal's theorem concerns three rays in a plane, meeting at a common point O. Kupffer's insight was that the special case when the three rays form two right angles is enough for the theory of proportionality, and that special case can be proved using existence of the orthocenter.

Theorem 9.2 (Pascal, Kupffer's version)**.** *Let ABC be three distinct points on line OA perpendicular at O to the line containing three distinct points $A'B'C'$, all on the same side of O and distinct from O. Suppose that $AB' \mid\mid BA'$ and $BC' \mid\mid CB'$. Then $AC' \mid\mid CA'$.*

Remark. Pascal's theorem differs from this theorem in not requiring angle AOC' to be a right angle. This theorem implies Pascal's theorem easily with the aid of Desargues's theorem. On the other hand, it is known that Desargues's theorem can be proved with three applications of Pascal's theorem (but not this special case). At any rate, this version can be proved without Desargues.

Proof. We assume known that the three altitudes of a triangle meet in a point, the *orthocenter* of the triangle. We just translate Schur's proof from German in [14], filling in no additional steps. Construct the perpendicular line from B to CA'. Let it meet line OD in point D'. Then C is the orthocenter of the triangle $BA'D$. Thus $CD' \perp BA'$ and therefore $CD' \perp AB'$. Therefore C is the orthocenter of triangle $AB'D'$. Therefore $AD' \perp CB'$ and also $AD' \perp BC'$. Finally, B is the orthocenter of the triangle $AC'D'$. Hence $AC' \perp BD'$. Therefore $AC' || CA'$, which is what was to be proved.

Corollary 9.3. *If $a : b = p : q$, then $a : p = b : q$.*

Remark. This is a proof of Theorem 4.7 by the methods of Euclid Book I, without even a slight detour into Book III.

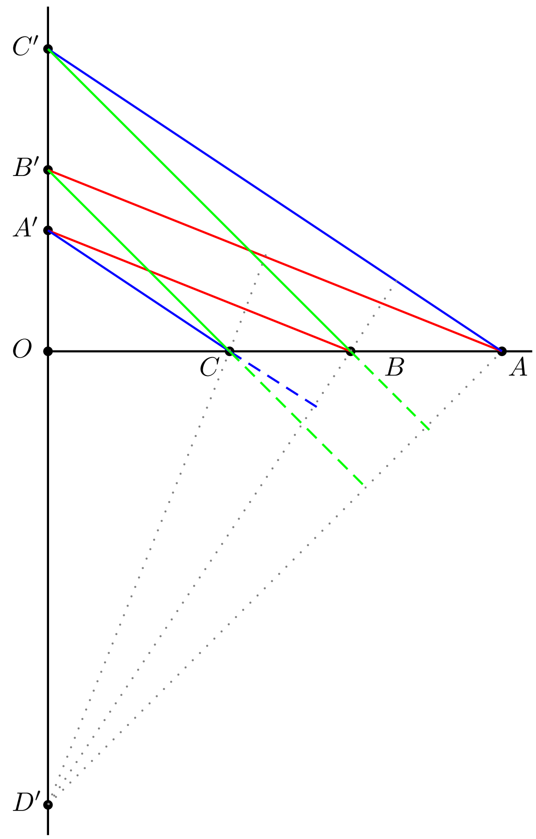

FIGURE 29. Kupffer's second proof. Given two pairs of parallel lines (red and green), prove that the blue lines are parallel too.

Proof. Given $a : b = p : q$, by Definition 4.1, there is a right angle with points B', A, A',B as in Fig. 29, with the red lines $B'A$ and $A'B$ parallel, and $a = OA$, $b = OB'$, $p = OB$, and $q = OA'$. Now construct C on ray OB so that $OC = OB' = b$, and construct C' on ray OA' such that $OC' = OB = p$. Then by Theorem 9.2, the blue lines $C'A$ and $A'C$ are parallel. Then by Definition 4.1, $OA : OC' = OC : OA'$. That is, $a : p = b : q$. That completes the proof.

10. CONCLUSIONS

We have given a definition of "equal figures" in the spirit of Euclid, using a diagram similar to the diagram for Prop. I.44. The fundamental properties of this defined notion seem to require (parts of) the theory of proportion. Our work then fell into two parts:

- Using methods like those in Euclid Book I, as well as the elementary theory of proportions, we proved all the theorems of Book I, and all the equal-figures axioms used in [4], using the defined notion of "equal figures."

- We then showed, using theorems of Kupffer and Bernays, that the required theory of proportions can also be developed using the methods of Euclid Book I. In particular Desargues's theorem is not needed.
- The final result is that it is possible to use the new definition of "equal figures" to justify Euclid's use of that notion without appealing to the Common Notions, or to the "equal figures" axioms used in [10] or [4].
- And Euclid could have done it.

Appendix: Listing of the equal-figures axioms

This formal listing is intended for reference. Only ASCII symbols are used, to facilitate cut-and-paste to computer-readable files. Polish notation can be found in [4].

Congruent triangles are equal.

```
axiom-congruentequal
forall A B C a b c, TC(A,B,C,a,b,c) ==> ET(A,B,C,a,b,c)
```

Triangles with the same vertices are equal.

```
axiom-ETpermutation
forall A B C a b c, ET(A,B,C,a,b,c) ==> ET(A,B,C,b,c,a) /\
    ET(A,B,C,a,c,b) /\ ET(A,B,C,b,a,c) /\ ET(A,B,C,c,b,a) /\
    ET(A,B,C,c,a,b)
```

Triangle equality is a symmetric relation

```
axiom-ETsymmetric
forall A B C a b c, ET(A,B,C,a,b,c) ==> ET(a,b,c,A,B,C)
```

Quadrilaterals with the same vertices are equal

```
axiom-EFpermutation
forall A B C D a b c d, EF(A,B,C,D,a,b,c,d) ==> EF(A,B,C,D,b,c,d,a)
    /\ EF(A,B,C,D,d,c,b,a) /\ EF(A,B,C,D,c,d,a,b) /\
    EF(A,B,C,D,b,a,d,c) /\ EF(A,B,C,D,d,a,b,c) /\ EF(A,B,C,D,c,b,a,d)
    /\ EF(A,B,C,D,a,d,c,b)
```

Halves of equals are equal

```
axiom-halvesofequals
forall A B C D a b c d, ET(A,B,C,B,C,D) /\ OS(A,B,C,D) /\
    ET(a,b,c,b,c,d) /\ OS(a,b,c,d) /\ EF(A,B,D,C,a,b,d,c) ==>
    ET(A,B,C,a,b,c)
```

Equal quadrilaterals is a symmetric relation

```
axiom-EFsymmetric
forall A B C D a b c d, EF(A,B,C,D,a,b,c,d) ==> EF(a,b,c,d,A,B,C,D)
```

Equal quadrilaterals is a transitive relation

```
axiom-EFtransitive
forall A B C D P Q R S a b c d, EF(A,B,C,D,a,b,c,d) /\
    EF(a,b,c,d,P,Q,R,S) ==> EF(A,B,C,D,P,Q,R,S)
```

Equal triangles is a transitive relation

```
axiom-ETtransitive
forall A B C P Q R a b c, ET(A,B,C,a,b,c) /\ ET(a,b,c,P,Q,R) ==>
    ET(A,B,C,P,Q,R)
```

Cutting off equal triangles from equal triangles yields equal quadrilaterals

```
axiom-cutoff1
forall A B C D E a b c d e, BE(A,B,C) /\ BE(a,b,c) /\ BE(E,D,C) /\
    BE(e,d,c) /\ ET(B,C,D,b,c,d) /\ ET(A,C,E,a,c,e) ==>
    EF(A,B,D,E,a,b,d,e)
```

Cutting off equal triangles from equal quadrilaterals yields equal quadrilaterals

```
axiom-cutoff2
forall A B C D E a b c d e, BE(B,C,D) /\ BE(b,c,d) /\ ET(C,D,E,c,d,e)
    /\ EF(A,B,D,E,a,b,d,e) ==> EF(A,B,C,E,a,b,c,e)
```

Pasting equal triangles yields equal triangles

```
axiom-paste1
forall A B C D E a b c d e, BE(A,B,C) /\ BE(a,b,c) /\ BE(E,D,C) /\
    BE(e,d,c) /\ ET(B,C,D,b,c,d) /\ EF(A,B,D,E,a,b,d,e) ==>
    ET(A,C,E,a,c,e)
```

Cutting off a triangle makes an unequal triangle

```
axiom-deZolt1
forall B C D E, BE(B,E,D) ==> ~ET(D,B,C,E,B,C)
```

Cutting off a quadrilateral makes an unequal triangle

```
axiom-deZolt2
forall A B C E F, TR(A,B,C) /\ BE(B,E,A) /\ BE(B,F,C) ==>
    ~ET(A,B,C,E,B,F)
```

Pasting equal triangles to equal quadrilaterals yields equal quadrilaterals

```
axiom-paste2
forall A B C D E M a b c d e m, BE(B,C,D) /\ BE(b,c,d) /\
    ET(C,D,E,c,d,e) /\ EF(A,B,C,E,a,b,c,e) /\ BE(A,M,D) /\ BE(B,M,E)
    /\ BE(a,m,d) /\ BE(b,m,e) ==> EF(A,B,D,E,a,b,d,e)
```

Pasting equal triangles to equal triangles yields equal quadrilaterals

```
axiom-paste3
forall A B C D M a b c d m, ET(A,B,C,a,b,c) /\ ET(A,B,D,a,b,d) /\
    BE(C,M,D) /\ BE(A,M,B) \/ EQ(A,M) \/ EQ(M,B) /\ BE(c,m,d) /\
    BE(a,m,b) \/ EQ(a,m) \/ EQ(m,b) ==> EF(A,C,B,D,a,c,b,d)
```

Pasting equal quadrilaterals yields equal quadrilaterals

```
axiom-paste4
forall A B C D F G H J K L M P e m, EF(A,B,m,D,F,K,H,G) /\
    EF(D,B,e,C,G,H,M,L) /\ BE(A,P,C) /\ BE(B,P,D) /\ BE(K,H,M) /\
    BE(F,G,L) /\ BE(B,m,D) /\ BE(B,e,C) /\ BE(F,J,M) /\ BE(K,J,L) ==>
    EF(A,B,C,D,F,K,M,L)
```

Appendix: Where the equal-figures axioms are used

The following listing shows all the lines in the formal development of [4] that are justified by the equal-figure axioms other than the axioms `ETpermutation`, `EFpermutation`, and the axioms asserting that `ET` and `EF` are equivalence relations. The middle entry in each line is the statement justified; in most cases, the reader will be able to identify the corresponding line in Euclid's own proof, which will either be justified by a common notion, or not justified at all. To decode the statements: for example in `EFADGBFEGC`, the initial `EF` means "equal figures" and the statement means that $ADGB$ and $FEGC$ are equal quadrilaterals.

```
Prop35A.prf:       EFADGBFEGC     axiom:cutoff1
Prop43.prf:        EFAKGBAKFD     axiom:cutoff1
Prop43.prf:        EFGBEKFDHK     axiom:cutoff2
EFreflexive.prf:   EFabcdabcd     axiom:paste3
Prop35A.prf:       EFADCBFEBC     axiom:paste2
Prop35A.prf:       EFCDABBEFC     axiom:paste2
Prop35A.prf:       EFBAECCFDB     axiom:paste3
Prop42.prf:        EFABECFECG     axiom:paste3
Prop42B.prf:       EFABECabec     axiom:paste3
Prop45.prf:        EFABCDFKML     axiom:paste4
Prop47B.prf:       EFFBAGDBML     axiom:paste3
Prop48.prf:        EFBCEDBced     lemma:paste5
paste5.prf:        EFDBCLdbcl     axiom:paste2
paste5.prf:        EFBDECbdec     axiom:paste2
squaresequal.prf:  EFBADCbadc     axiom:paste3
Prop48.prf:        EFACKHAckh     lemma:squaresequal
Prop48.prf:        EFABFGABfg     lemma:squaresequal
Prop39A.prf:       NOETDBCEBC     axiom:deZolt1
Prop39A.prf:       NOETEBCDBC     axiom:deZolt1
Prop48A.prf:       NOETDABFAE     axiom:deZolt2
Prop48A.prf:       NOETdabfae     axiom:deZolt2
Prop37.prf:        ETCBACBD       axiom:halvesofequals
Prop38.prf:        ETEFDCBA       axiom:halvesofequals
Prop48A.prf:       ETABDabd       axiom:halvesofequals
paste5.prf:        ETMCLmcl       axiom:halvesofequals
paste5.prf:        ETECLecl       axiom:halvesofequals
```

References

[1] Hero Alexandrinus. *Opera Quae Supersunt Omnia*, volume iii. Teubner, Leipzig, 1903. Editor: Hermann Schöne. English translation by Henry Mendell available on the Web.

[2] John Baldwin. Axiomatizing changing conceptions of the geometric continuum I: Euclid-Hilbert. *Philosophia Mathematica*, 3:346–374, 2018.

[3] Michael Beeson. Euclid after computer proof-checking. *American Mathematical Monthly*, 2022.

[4] Michael Beeson, Julien Narboux, and Freek Wiedijk. Proof-checking Euclid. *Annals of Mathematics and Artificial Intelligence*, 85(2):213–257, January 2019.

[5] Bernhard Bolzano. *Beiträge zu einer begründeteren Darstellung der Mathematik.* Caspar Widtmann, Prague, 1810. English translation in [8], pp. 174-224.

[6] Federigo Enriques. *Fragen der Elementargeometrie, I. Teil.* Hermann Thieme, 1911.

[7] Euclid. *The Thirteen Books of The Elements.* Dover, New York, 1956. Three volumes. Includes commentary by the translator, Sir Thomas L. Heath.

[8] W. Ewald. *Philosophy of Logic.* Elsevier, Amsterdam, 1996.

[9] Robin Hartshorne. Francois Viète–life. `https://math.berkeley.edu/~robin/Viete/work.html`.

[10] Robin Hartshorne. *Geometry: Euclid and Beyond.* Springer, New York, 2000.

[11] David Hilbert. *Foundations of Geometry (Grundlagen der Geometrie).* Open Court, La Salle, Illinois, 1960. Second English edition, translated from the tenth German edition by Leo Unger. Original publication date, 1899.

[12] Karl Kupffer. Die Darstellung einiger Kapitel der Elementarmathematik. *Sitzungsberichte der Dorpater Naturforscherges*, page 373 ff., 1893.

[13] Moritz Pasch. *Vorlesung über Neuere Geometrie.* Teubner, Leipzig, 1882.

[14] Friedrich Schur. Zur Proportionslehre. *Mathematische Annalen*, 1902.

[15] Francisci Vieta. *Opera Mathematica*. Lugduni Batavorum, 1646.

Michael Beeson, San José State University (emeritus), profbeeson@gmail.com